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STRONGLY CONVEX SETS WITH VARIABLE RADII

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Abstract. We introduce in a general Hilbert space the class of $\rho(\cdot)$-strongly convex sets. Various characterizations and properties of such sets involving the farthest distance function and the farthest points are provided.

1. **Introduction.** Motivated by some convexity-type properties of reachable sets of nonlinear control systems, H. Frankowska and C. Olech proved in Theorem 3.1 of their 1980 paper [15] the R-convexity of the integral of a multimapping $M : [0,1] \rightrightarrows \mathbb{R}^n$ under certain conditions. Such a study of convexity properties of reachable sets of nonlinear control systems seems to be started in 1975 with A. Pliś [28] who demonstrated, under some assumptions, the local R-convexity of reachable sets. The result in [15] on the R-convexity of the integral a multimapping $M : [0,1] \rightrightarrows \mathbb{R}^n$ was a significant extension of a previous 1979 contribution of St. Lojaciewicz [24].

A closed set C in $\mathbb{R}^n$ is called *R-convex* in [15] for a real $R > 0$ when it is the intersection of a collection of closed balls in $\mathbb{R}^n$ with radius R. The proof of H. Frankowska and C. Olech in [15] of Theorem 3.1 is based for a large part on their Proposition 3.1 that we promote as a theorem in the following form:

Theorem 1.1. *Let C be a nonempty closed convex set in $\mathbb{R}^n$ endowed with its canonical Euclidean norm $\|\cdot\|$ and let $R > 0$. The following are equivalent:*
(a) The set C is R-convex;
(b) for any $x, y \in C$ with $\|x - y\| \le 2R$, every arc of circle of radius R which joins x and y and whose length is not greater that πR is contained in C;
(c) for any point x in the boundary of C and any normal vector v to C at x with $\|v\| = 1$, one has

$$\langle v, y - x\rangle \le -\frac{1}{2R}\|y - x\|^2 \quad \text{for all } y \in C;$$

(d) for any x_i in the boundary of C with $i = 1, 2$ and any normal vector v_i to C at x_i with $\|v_i\| = 1$, one has

$$\|x_1 - x_2\| \le R\|v_1 - v_2\|.$$

Closed sets in $\mathbb{R}^n$ (in the plane) satisfying the property (b) in Theorem 1.1 was earlier considered in 1935 by A. E. Mayer [25] and developed in the more

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general setting of Minkowski norm in [26] under the name "*Überkonvexen Mengen*" (in German). Subsequently to [26], sets with this arc property or similar other properties were analyzed in [5, 7, 8, 39]. The formulation as intersection of closed balls with common radius R probably first appeared in the two-dimensional case of the plane in "Théorème fondamental" of E. Blanc [5, p. 219] as a characterization therein of "ensembles surconvexes" (in French). Given a real $R > 0$, a set C in the plane is declared by Blanc [5] "R-surconvexe" if for two points A, B in the plane with $d(A, B) \leq 2R$ the *lens* $L(A, B; R)$ is contained in C. The lens $L(A, B; R)$ in the plane is defined in [5] as the closed bounded convex set whose boundary is the union of the two arcs of circle with radius R and length not greater than πR joining A and B. Regarding the property (b) in Theorem 1.1 again, it is worth noticing (as said in [5, p. 215]) that lenses were utilized in 1921 by H. Lebesgue [22, p. 77,79] in a study of "orbiform" curves. It was also the study [25] of Mayer on "orbiform sets" which led him to the property (b) in Theorem 1.1 above and hence to analyze in the same paper [25] some aspects of this property for their own sake. Sets which are intersections of closed balls with common radius R were utilized in 1966 by B. T. Polyak [32] and E. S. Levintin and B. T. Polyak [23] for the convergence of certain optimization algorithms, and by J. J. Moreau [27] in 1975 for an asymptotic analysis of sweeping processes (see also [10]). They are nowadays called *R-strongly convex sets* or *strongly convex sets with radius R*, and a large development of their properties in Hilbert spaces began in 1982 with J.-P. Vial [37, 38]. Recent other important developments have been provided by M. V. Balashov, G. E. Ivanov, E. S. Polovinkin, and others (see, e.g., [2, 18, 27, 36]).

A convex set C which is R-strongly convex (presented as R-convex set ahead Theorem 1.1) is known to have nonempty interior (when it is not reduced to a singleton), hence for any x in the boundary $\operatorname{bdry} C$ of the set C one can choose a unit outward normal vector v to C at x, and for such a vector v the characterization (c) in Theorem 1.1 is equivalent (as it can be easily verified) to the inclusion $C \subset B[x - Rv, R]$. As a particular consequence of this latter inclusion for $x \in \operatorname{bdry} C$, one could formally say that the curvature of $\operatorname{bdry} C$ at x is not less than the curvature $1/R$ of $B[x - Rv, R]$ at x; see, e.g., (12) for two-dimensional analytical arguments of this feature.

The aim of the present paper is to analyze a class of convex sets for which at each point x of the boundary the curvature is bounded from below by $1/\rho(x)$ (see (12)) where $\rho(\cdot)$ is a given suitable positive function. As we will see, such sets, that we call *$\rho(\cdot)$-strongly convex* or *strongly convex with variable radius $\rho(\cdot)$*, can be characterized by requiring that the property (c) in Theorem 1.1 be satisfied with $\rho(x)$ in place of the real constant R. The subject here has then to be seen as subsequent to that of *$\rho(\cdot)$-prox-regular* sets characterized by the property that for any pair (x, v) with x in the boundary of C and v in the proximal normal cone (see Section 2 for the definition) to C at x with $\|v\| = 1$ one has

$$\langle v, y - x \rangle \leq \frac{1}{2\rho(x)} \|y - x\|^2 \quad \text{for all } y \in C.$$

The $\rho(\cdot)$-prox-regularity of sets has been considered by A. Canino [9] under a different name (see also [12, 36]). It is a variant of the r-prox-regularity of sets where the function $\rho(\cdot)$ is constant with value r, as mainly initiated in [13] in finite dimensions and developed in infinite dimensions in [3, 4, 11, 12, 17, 19, 34, 36, 38] and references in those papers. Our analysis of $\rho(\cdot)$-strongly convex sets will be carried out in the setting of a general Hilbert space. Section 2 recalls some notions and results. The concept and properties of $\rho(\cdot)$-strongly convex sets are developed in Section 3.

2. **Farthest distance and farthest points.** Throughout the paper, $\mathbb{N}$ denotes the set of positive integers $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{R}$ stands for the set of real numbers. Let $(X, \|\cdot\|)$ be a (real) normed space. The identity mapping of X is denoted Id_X. The closed (resp. open) ball in X with center $x \in X$ and radius $r > 0$ is denoted as usual by $B[x, r]$ (resp. $B(x, r)$). We will use the letter $\mathbb{B}$ (resp. $\mathbb{U}$) for the closed (resp. open) unit ball in X, i.e., $\mathbb{B} := B[0_X, 1]$ (resp. $\mathbb{U} := B(0_X, 1)$). We will also put $\mathbb{S} := \{x \in X : \|x\| = 1\}$, i.e., $\mathbb{S}$ is the unit sphere of X. The interior (resp. closure, boundary) of a subset C of $(X, \|\cdot\|)$ is denoted by $\operatorname{int} C$ (resp. $\operatorname{cl} C$, $\operatorname{bdry} C$). If Q is a nonempty subset of $\mathcal{H}$ containing C, we will set $\operatorname{int}_Q C$ (resp. $\operatorname{cl}_Q C$, $\operatorname{bdry}_Q C$) for the interior (resp. closure, boundary) of C in Q equipped with the induced topology. The diameter of C is the extended real

$$\operatorname{diam} C := \sup\{\|x - x'\| : x, x' \in C\}.$$

Given a multimapping $M : E \rightrightarrows F$ between two sets E and F, its (effective) domain is given by $\operatorname{Dom} M := \{x \in E : M(x) \neq \emptyset\}$ and its inverse multimapping $M^{-1} : F \rightrightarrows E$ is defined by

$$M^{-1}(y) := \{x \in E : y \in M(x)\} \quad \text{for all } y \in F.$$

The graph of the multimapping M is the set $\operatorname{gph} M := \{(x, y) \in E \times F : y \in M(x)\}$ while the image of a set $A \subset E$ is defined as

$$M(A) := \bigcup_{x \in A} M(x).$$

The (standard) *distance function* d_C and the *farthest distance function* dfar_C from/to the set C are defined for every $x \in X$ by

$$d_C(x) := d(x, C) := \inf_{y \in C} \|x - y\|$$

and

$$\operatorname{dfar}_C(x) := \operatorname{dfar}(x, C) := \sup_{y \in C} \|x - y\|,$$

with the (usual) convention that the latter supremum is 0 if C is empty. The Lipschitz continuity of the distance function d_C (for $C \neq \emptyset$) is classical with $|d_C(x) - d_C(x')| \leq \|x - x'\|$ for all $x, x' \in X$. A similar (less classical) result also holds for dfar_C. Indeed, given $x, x' \in X$ and writing $\|x - y\| \leq \|x - x'\| + \|x' - y\|$ we see (by taking the supremum of both sides over $y \in C$) that

$$\sup_{y \in C} \|x - y\| \leq \|x - x'\| + \sup_{y \in C} \|x' - y\|, \text{ i.e. } \operatorname{dfar}_C(x) \leq \|x - x'\| + \operatorname{dfar}_C(x'),$$

so dfar_C is continuous on X, and if C is unbounded it follows that dfar_C is finite valued and Lipschitz continuous on X with

$$|\operatorname{dfar}_C(x) - \operatorname{dfar}_C(x')| \leq \|x - x'\|. \tag{1}$$

With the (standard) distance function d_C and the farthest distance function dfar_C one generally associates the multimappings $\operatorname{Proj}_C : X \rightrightarrows X$ of nearest points in C and $\operatorname{Far}_C : X \rightrightarrows X$ of farthest points in C given for every $x \in X$ by

$$\operatorname{Proj}_C(x) := \operatorname{Proj}(C, x) := \{y \in C : \|x - y\| = d_C(x)\}$$

and

$$\operatorname{Far}_C(x) := \operatorname{Far}(C, x) := \{y \in C : \|x - y\| = \operatorname{dfar}_C(x)\}.$$

It is an exercise to check that

$$\operatorname{Proj}_C(x) \subset \operatorname{bdry} C \quad \text{and} \quad \operatorname{Far}_C(x) \subset \operatorname{bdry} C. \tag{2}$$

When for some $\overline{x} \in X$ the set $\operatorname{Proj}_C(\overline{x})$ (resp. $\operatorname{Far}_C(\overline{x})$) is reduced to a singleton, i.e., $\operatorname{Proj}_C(\overline{x}) = \{\overline{y}\}$ (resp. $\operatorname{Far}_C(\overline{x}) = \{\overline{y}\}$) it will be convenient to denote this vector $\overline{y} \in C$ by $\operatorname{proj}_C(\overline{x})$ (resp. $\operatorname{far}_C(\overline{x})$). Every sequence $(y_n)_n$ in C satisfying $\|\overline{x} - y_n\| \to \operatorname{dfar}_C(\overline{x})$ is called a maximizing sequence of/for $\operatorname{dfar}_C(\overline{x})$. One says that $\operatorname{dfar}_C(\overline{x})$ is *strongly attained* if all its maximizing sequences converge in C. In such a case, one can easily check that $v := \operatorname{far}_C(\overline{x})$ is well defined along with the convergence of every maximizing sequence for $\operatorname{dfar}_C(\overline{x})$ to v. Putting for each real $\eta > 0$

$$\operatorname{Far}_{C,\eta}(\overline{x}) := \{y \in C : \|\overline{x} - y\| + \eta \geq \operatorname{dfar}_C(\overline{x})\}, \tag{3}$$

it is not difficult to see, when the normed space $(X, \|\cdot\|)$ is complete and C is closed in X, that $\operatorname{dfar}_C(\overline{x})$ is strongly attained if and only if

$$\lim_{\eta \downarrow 0} \operatorname{diam} \operatorname{Far}_{C,\eta}(\overline{x}) = 0. \tag{4}$$

Notice also that both graphs $\operatorname{gph} \operatorname{Proj}_C$ and $\operatorname{gph} \operatorname{Far}_C$ are closed in $X \times X$ whenever the set C is closed.

In the rest of this section and in the next ones, $\mathcal{H}$ is a (real) Hilbert space *not reduced to zero* equipped with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$ given by $\|x\|^2 = \langle x, x \rangle$ for all $x \in \mathcal{H}$. Take now C as a nonempty subset of $\mathcal{H}$. The *proximal normal cone* of/to the set C at $x \in C$, denoted by $N^P(C; x)$, is defined as (see, e.g., [35]) the set of $v \in \mathcal{H}$ for which there is a real $\sigma > 0$ (depending on v) such that x is a nearest point of $x + \sigma v$, i.e., $x \in \operatorname{Proj}_C(x + \sigma v)$. This can be rewritten as

$$N^P(C; x) = \left\{v \in \mathcal{H} : \exists \sigma \geq 0, \forall x' \in C, \langle v, x' - x \rangle \leq \sigma \|x' - x\|^2\right\}.$$

When C is convex, the proximal normal cone $N^P(C; x)$ is known to coincide with the (standard) normal cone *in the sense of convex analysis*, i.e.,

$$N^P(C; x) = \{v \in \mathcal{H} : \langle v, x' - x \rangle \leq 0, \forall x' \in C\} =: N(C; x).$$

More generally, given a point $a \in \mathcal{H}$, a closed vector subspace E in $\mathcal{H}$ and a nonempty set S in the affine subspace $a + E$, i.e, $S \subset a + E$, the proximal normal cone $N^{P,E}(S; x)$ of/to S relative to E (or in E) at $x \in S$ is

$$N^{P,E}(S; x) := \{v \in E : \exists \sigma > 0,\ x \in \operatorname{Proj}_S(x + \sigma v)\},$$

or equivalently

$$N^{P,E}(S; x) = \left\{v \in E : \exists \sigma \geq 0, \forall x' \in S, \langle v, x' - x \rangle \leq \sigma \|x' - x\|^2\right\}.$$

According to the equivalences valid for any $x, y \in \mathcal{H}$

$$y \in \operatorname{Proj}_C(x) \Leftrightarrow y \in C \quad \text{and} \quad \langle x - y, c - y \rangle \leq \frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in C$$

and

$$y \in \operatorname{Far}_C(x) \Leftrightarrow y \in C \quad \text{and} \quad \langle y - x, c - y \rangle \leq -\frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in C, \tag{5}$$

it is clear that

$$x - y \in N^P(C; y) \quad \text{for all } (x, y) \in \operatorname{gph} \operatorname{Proj}_C$$

and

$$y - x \in N^P(C; y) \quad \text{for all } (x, y) \in \operatorname{gph} \operatorname{Far}_C, \tag{6}$$

where $\operatorname{gph}\operatorname{Proj}_C$ and $\operatorname{gph}\operatorname{Far}_C$ denote the graphs (as recalled above) of the multimappings Proj_C and Far_C. It is worth pointing out that for any $(x,y) \in \operatorname{gph}\operatorname{Far}_C$

$$y = \operatorname{far}_C\big(x + s(x-y)\big) \quad \text{for all } s > 0. \tag{7}$$

We proceed now to some properties of farthest/nearest distance function and farthest/nearest points that we will need later. First, we recall two theorems of K. S. Lau and of S. Fitzpatrick on the genericity of points with nearest points and of points with farthest points respectively. For their proofs we refer to [20] and [14] respectively.

Recall that the norm of a normed space $(X, \|\!|\cdot\|\!|)$ possesses the *(sequential) Kadec-Klee property* provided that for any sequence $(x_n)_n$ in X one has

$$\big(\,\|\!|x_n - x\|\!| \to 0\,\big) \Longleftrightarrow \big(\,x_n \to x \text{ weakly and } \|\!|x_n\|\!| \to \|\!|x\|\!|\big).$$

It is well known (and not difficult to check) that Hilbert spaces enjoy the sequential Kadec-Klee property.

Theorem 2.1 (Lau theorem for nearest points)**.** *Let X be a reflexive Banach space endowed with a norm $\|\!|\cdot\|\!|$ satisfying the sequential Kadec-Klee property and let C be a nonempty closed subset of X. Then, the set of points of $X \setminus C$ admitting nearest points in C contains a dense G_δ set of $X \setminus C$.*

If in addition the sequential Kadec-Klee norm $\|\!|\cdot\|\!|$ is strictly convex, then there exists a dense G_δ set of points in $X \setminus C$ with unique nearest point in C.

Note that, under its above assumptions with $C \neq X$, the theorem entails in particular that

$$\{x \in \operatorname{bdry} C : \exists u \in X \setminus C,\, x \in \operatorname{Proj}_C(u)\} \quad \text{is dense in } \operatorname{bdry} C. \tag{8}$$

Indeed, take any $\overline{x} \in \operatorname{bdry} C$ and any real $\varepsilon > 0$. By the above theorem there exist $u \in B(\overline{x}, \varepsilon/2) \setminus C$ and $x \in \operatorname{Proj}_C(u)$ (so in particular $x \in \operatorname{bdry} C$). Further, the inequalities

$$\|\!|x - \overline{x}\|\!| \le \|\!|x - u\|\!| + \|\!|u - \overline{x}\|\!| < d_C(u) + \frac{\varepsilon}{2} \le \varepsilon$$

ensure that $x \in (\operatorname{bdry} C) \cap B(\overline{x}, \varepsilon)$ as desired.

Theorem 2.2 (Fitzpatrick theorem for farthest points)**.** *Let $(X, \|\!|\cdot\|\!|)$ be a reflexive Banach space whose norm is strictly convex and possesses the (sequential) Kadec-Klee property. Let C be a nonempty closed bounded subset of X. Then there exists a dense G_δ set of points in X with unique farthest point in C.*

The results in the next theorem are also due to S. Fitspatrick [14]. The theorem summarizes diverse basic characterizations of the Fréchet differentiability of the farthest distance function. Before stating the theorem, recall that a real-valued function is $C^{1,1}$ on an open set U of a normed space when it is differentiable on U and its derivative is locally Lipschitz on U.

Theorem 2.3. *Let C be a nonempty closed bounded subset of the Hilbert space $\mathcal{H}$ not reduced to a singleton and let U be a nonempty open subset of $\mathcal{H}$. The following assertions are equivalent:*

(a) The function dfar_C is $C^{1,1}$ on U;
(b) the function dfar_C is Fréchet differentiable on U;
(c) dfar_C is Gâteaux differentiable on U and $\|D_G \operatorname{dfar}_C(x)\| = 1$ for every $x \in U$;
(d) dfar_C is Gâteaux differentiable on U and $\operatorname{Far}_C(x) \neq \emptyset$ for every $x \in U$;
(e) the mapping $\operatorname{far}_C : U \to X$ is well defined on U and locally Lipschitz therein;

(f) the mapping $\operatorname{far}_C : U \to X$ *is well defined on* U *and norm-to-norm continuous therein;*
(g) the mapping $\operatorname{far}_C : U \to X$ *is well defined on* U *and norm-to-weak sequentially continuous therein;*
(h) the supremum $\operatorname{dfar}_C(x)$ *is strongly attained for every* $x \in U$*.*

Under anyone of the latter assumptions, one has

$$\nabla \operatorname{dfar}_C(x) = \frac{x - \operatorname{far}_C(x)}{\operatorname{dfar}_C(x)} \quad \text{for all } x \in U.$$

We end this section by stating a lemma of G. E. Ivanov [18, Lemma 7] concerning a behavior of the farthest point mapping far_C under Fréchet differentiability of the farthest distance function. It is the counterpart for the farthest distance function dfar_C of Lemma 3.3 in [29] (see also, [12, 36]) related to a similar behavior of proj_C under the condition of Fréchet differentiability of the standard distance function d_C.

Lemma 2.4 (Ivanov)**.** *Let* C *be a nonempty closed bounded subset of a locally uniformly convex reflexive Banach space* $(X, \|\!|\cdot|\!\|)$*. If* $\operatorname{dfar}_C(\cdot)$ *is Fréchet differentiable on a neighborhood of* $\overline{x} \in X$*, then there exist a real* $\delta > 0$ *such that for each* $x \in B(\overline{x}, \delta)$ *one has that* $\operatorname{far}_C(x)$ *is well defined along with the existence of a real* $\tau \in]0, 1[$ *(depending on* x*) for which*

$$\operatorname{far}_C\big(x + t(\operatorname{far}_C(x) - x)\big) = \operatorname{far}_C(x) \quad \text{for all } t \le \tau.$$

3. $\rho(\cdot)$-strongly convex sets.

We pass now to the presentation and development of $\rho(\cdot)$-strongly convex sets. As said in the introduction, an R-strongly convex set C for a constant real $R > 0$ is nowadays generally defined in the Hilbert setting as an intersection of a collection of closed balls with the common radius R, so the curvature of the boundary of C is bounded from below by the constant $1/R$ (as formally explained in the introduction). Such R-strongly convex sets with constant real R are largely studied in the literature (see, e.g., the aforementioned references in the introduction). Sets which are $\rho(\cdot)$-strongly convex will be defined here in such a way that we have the refinement of boundedness from below of the curvature by the function $1/\rho(\cdot)$.

3.1. Definition and characterizations with normals.

Consider a nonempty closed subset C of the (real) Hilbert space $\mathcal{H}$ with $C \neq \mathcal{H}$. Thanks to Lau's theorem relative to nearest points (see Theorem 2.1) we can choose some $\overline{x} \in \mathcal{H} \setminus C$ such that $\operatorname{proj}_C(\overline{x}) \in \operatorname{bdry} C$ is well defined and this leads to

$$\frac{\overline{x} - \operatorname{proj}_C(\overline{x})}{d_C(\overline{x})} \in N^P\big(C; \operatorname{proj}_C(\overline{x})\big) \cap \mathbb{S}.$$

This ensures the following non-vacuity property

$$\Lambda^P(C) := \big\{(x, v) \in \mathcal{H}^2 : x \in \operatorname{bdry} C, v \in N^P(C; x) \cap \mathbb{S}\big\} \neq \emptyset. \tag{9}$$

Definition 3.1. Let C be a nonempty closed subset of the Hilbert space $\mathcal{H}$ with $C \neq \mathcal{H}$ and let $\rho : \operatorname{bdry} C \to]0, +\infty[$ be a positive function. We say that C is $\rho(\cdot)$-*strongly convex* whenever for any $(x, v) \in \mathcal{H}^2$ with $x \in \operatorname{bdry} C$ and $v \in N^P(C; x) \cap \mathbb{S}$, we have

$$x \in \operatorname{Far}_C\big(x - \rho(x)v\big).$$

Whenever $\rho(\cdot) \equiv R$, it is usually said that C is an R-*strongly convex* set (or *uniformly strongly convex* set of constant R). Similarly, if the closed set C is included in a closed affine subspace $a+E$ of $\mathcal{H}$ and $\rho(\cdot)$ is defined on $\operatorname{bdry}_{a+E} C$, where E is a closed vector subspace of $\mathcal{H}$, we say that C is $\rho(\cdot)$-strongly convex in $a+E$ (or relative to $a+E$) provided the inclusion $x \in \operatorname{Far}_C\big(x-\rho(x)v\big)$ is satisfied when $x \in \operatorname{bdry}_{a+E}(C)$ and $v \in N^{P,E}(C;x) \cap \mathbb{S}$.

Given a nonempty closed subset C of $\mathcal{H}$, a function $\rho : \operatorname{bdry} C \to]0,+\infty[$ and a closed affine subspace L of $\mathcal{H}$ with $C_L := C \cap L \neq \emptyset$, we easily note that $B_L := \operatorname{bdry}_L C_L \subset L \cap \operatorname{bdry} C$. So, we will say (by convenience) that C_L is $\rho(\cdot)$-*strongly convex in* L when it is $\rho_{|_{B_L}}(\cdot)$-strongly convex in L for the restriction $\rho_{|_{B_L}}$ of the function ρ to the set B_L.

Remark 3.2. Let C be a nonempty closed (not necessarily bounded) subset of $\mathcal{H}$ with $C \neq \mathcal{H}$ and let also $\rho : \operatorname{bdry} C \to]0,+\infty[$ be a positive function. Through the equivalence in (5), it is readily seen that the set C is $\rho(\cdot)$-strongly convex if and only if

$$\langle v, x'-x\rangle \le -\frac{1}{2\rho(x)}\|x'-x\|^2 \quad \text{for all } x' \in C \text{ and all } (x,v) \in \Lambda^P(C). \tag{10}$$

We point out that there is no need to assume that C is bounded in the above equivalence (10). Indeed, it directly follows from the above definition that a $\rho(\cdot)$-strongly convex set is bounded. Conversely, if the latter estimate (10) holds, fixing any (x_0, v_0) with $x_0 \in \operatorname{bdry} C$ and $v_0 \in N^P(C;x) \cap \mathbb{S}$ (see (9)) and applying the Cauchy-Schwarz inequality guarantee that

$$\|c - x_0\| \le 2\rho(x_0) \quad \text{for all } c \in C, \tag{11}$$

and this ensures the boundedness of the set C. In fact, noticing that the set $G := \{x \in \operatorname{bdry} C : N^P(C;x) \neq \{0\}\}$ is dense in $\operatorname{bdry} C$ (as follows from (8)), we can refine the inequality (11) as

$$\operatorname{diam} C = \sup_{x \in G, x' \in G} \|x'-x\| \le 2 \sup_G \rho(\cdot) \le 2 \sup_{\operatorname{bdry} C} \rho(\cdot).$$

If the set C is $\rho(\cdot)$-strongly convex for some function $\rho(\cdot)$ satisfying the inequality $s := \sup_{\operatorname{bdry} C} \rho(\cdot) < +\infty$ (which always holds true whenever $\dim X < \infty$ and $\rho(\cdot)$ is upper semicontinuous), then C is s-strongly convex (that is, $\rho_0(\cdot)$-strongly convex with $\rho_0(\cdot) \equiv s$) since

$$\langle v, x'-x\rangle \le -\frac{1}{2\rho(x)}\|x'-x\|^2 \le -\frac{1}{2s}\|x'-x\|^2$$

for all $x' \in C$ and all (x,v) with $x \in \operatorname{bdry} C$ and $v \in N^P(C;x) \cap \mathbb{S}$. □

Before starting with the analysis of $\rho(\cdot)$-strongly convex sets, let us consider two basic two-dimensional situations. Given a $\rho(\cdot)$-strongly convex set in $\mathbb{R}^2$ whose boundary Γ is sufficiently regular, the first situation is devoted to show analytically that the curvature of Γ at every $x \in \Gamma$ is bounded from below by $1/\rho(x)$.

Let C be a (non-singleton) $\rho(\cdot)$-strongly convex set in $\mathbb{R}^2$ endowed with its canonical inner product. Let Γ be the boundary of C, that is, $\Gamma := \operatorname{bdry} C$. Assume that Γ is of class C^2 near $x_0 \in \Gamma$ with non null first and second derivatives of parametric representation of an arc of Γ with x_0 in its relative interior (that is, in the interior of the arc with respect to the topology induced on Γ). Consider such a parametrization $g : I \to \mathbb{R}^2$ of this arc with an open interval I and the arclength s as parameter in I. Let $s_0 \in I$ be such that $g(s_0) = x_0$. We know (see any kinematic book) that $\vec{\tau}(s_0) := \frac{dg}{ds}(s_0)$ is the unit vector tangent to Γ at x_0 (in the sense of increasing s)

and $\frac{d^2g}{ds^2}(s_0) = \gamma(s_0)\vec{\nu}(s_0)$ where $\vec{\nu}(s_0)$ is the inward unit vector normal to C at x_0 and $\gamma(s_0)$ is the curvature of Γ at $g(s_0) = x_0$. Putting $\vec{\tau}_0 := \vec{\tau}(s_0)$, $\vec{\nu}_0 := \vec{\nu}(s_0)$ and $\gamma_0 := \gamma(s_0)$, we then have for $s \in I$

$$g(s) - g(s_0) = (s - s_0)\left(\vec{\tau}_0 + \frac{(s - s_0)}{2}\gamma_0\vec{\nu}_0 + (s - s_0)\vec{\varepsilon}(s)\right),$$

for some function $\vec{\varepsilon}(\cdot)$ defined on a neighborhood of s_0 with values in $\mathbb{R}^2$ such that $\lim_{s\to s_0}\vec{\varepsilon}(s) = 0_{\mathbb{R}^2}$. Since $-\vec{\nu}_0$ is unit outward normal to C at $x_0 = g(s_0)$, the inequality (10) yields for every $s \in I$

$$\begin{aligned}&\left\langle -\vec{\nu}_0, (s - s_0)\left(\vec{\tau}_0 + \frac{(s - s_0)}{2}\gamma_0\vec{\nu}_0 + (s - s_0)\vec{\varepsilon}(s)\right)\right\rangle\\ &\le -\frac{1}{2\rho(x_0)}(s - s_0)^2\left\|\vec{\tau}_0 + \frac{(s - s_0)}{2}\gamma_0\vec{\nu}_0 + (s - s_0)\vec{\varepsilon}(s)\right\|^2,\end{aligned}$$

so the equality $\langle\vec{\tau}_0, \vec{\nu}_0\rangle = 0$ gives

$$\begin{aligned}&\frac{(s - s_0)^2}{2}\gamma_0 + (s - s_0)^2\langle\vec{\nu}_0, \vec{\varepsilon}(s)\rangle\\ &\ge \frac{1}{2\rho(x_0)}(s - s_0)^2\left\|\vec{\tau}_0 + \frac{(s - s_0)}{2}\gamma_0\vec{\nu}_0 + (s - s_0)\vec{\varepsilon}(s)\right\|^2,\end{aligned}$$

and hence for some function $\vec{\varepsilon}_1(\cdot)$ defined on a neighborhood of s_0 with values in $\mathbb{R}^2$ such that $\lim_{s\to s_0}\vec{\varepsilon}_1(s) = 0_{\mathbb{R}^2}$ one has

$$\gamma_0 + 2\langle\vec{\nu}_0, \vec{\varepsilon}(s)\rangle \ge \frac{1}{\rho(x_0)}\|\vec{\tau}_0 + \vec{\varepsilon}_1(s)\|^2.$$

It follows as $s \to s_0$ that

$$\gamma_0 \ge \frac{1}{\rho(x_0)}. \tag{12}$$

This confirms in some way that the curvature of $\operatorname{bdry} C$ at $x \in \operatorname{bdry} C$ is bounded from below by $1/\rho(x)$.

The second situation aims, with a two-dimensional simple example, to make clear the idea and interest of considering the function $\rho(\cdot)$ when dealing with strongly convex sets. Fix three reals R_1, R_2 and r with $0 < r < R_1 < R_2$. Set $a := (-r, 0)$ and $b := (r, 0)$. Let Γ_1 (resp. Γ_2) be the closed short arc of circle with center c_1 (resp. c_2) in $\mathbb{R}^2$ and radius R_1 (resp. R_2) joining a and b and whose points has non-positive (resp. non-negative) second components. By short arc of circle with radius R_1 we mean that its length is not greater than πR_1. Let C be the convex hull of $\Gamma := \Gamma_1 \cup \Gamma_2$. Pick any real $\varepsilon \in]0, \ell/2[$ where ℓ is the length of Γ_1. Let $\Gamma_{a,\varepsilon}$ and $\Gamma_{b,\varepsilon}$ be the two arcs of Γ_1 of length ε containing a and b respectively (see Figure 1). Let $\rho : \Gamma \to \mathbb{R}$ be a continuous function with $R_1 \le \rho(x) \le R_2$ for all $x \in \Gamma$ and such that $\rho(x) = R_1$ for $x \in \Gamma_1 \setminus (\Gamma_{a,\varepsilon} \cup \Gamma_{b,\varepsilon})$ and $\rho(x) = R_2$ for $x \in \Gamma_2$. Since C is R_2-strongly convex, for any pair (x, v) with $x \in \Gamma_2$ and $v \in N(C; x) \cap \mathbb{S}$ one has by (10)

$$\langle v, y - x\rangle \le -\frac{1}{R_2}\|y - x\|^2 = -\frac{1}{\rho(x)}\|y - x\|^2 \quad \text{for all } y \in C.$$

On the other hand, the inclusion $C \subset B[c_1, R_1]$ gives for any pair (x, v) with $x \in \Gamma\setminus\Gamma_2$ and $v \in N(C; x)\cap\mathbb{S}$ that (since $x \in \operatorname{bdry} B[c_1, R_1]$ and $v \in N(B[c_1, R_1]; x)\cap\mathbb{S}$)

$$\langle v, y - x\rangle \le -\frac{1}{R_1}\|y - x\|^2 \le -\frac{1}{\rho(x)}\|y - x\|^2 \quad \text{for all } y \in C.$$

It ensues that for any pair (x, v) with $x \in \Gamma$ and $v \in N(C;x) \cap \mathbb{S}$ one has

$$\langle v, y - x \rangle \leq -\frac{1}{\rho(x)} \|y - x\|^2 \quad \text{for all } y \in C,$$

which tells us by (10) again that C is $\rho(\cdot)$-strongly convex.

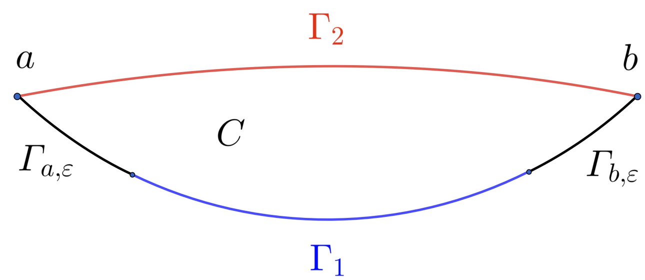

FIGURE 1. The $\rho(\cdot)$-strongly convex set C

We now state and prove several characterizations of $\rho(\cdot)$-strongly convex sets.

Theorem 3.3 (normal properties of $\rho(\cdot)$-strongly convex sets)**.** *Let C be a nonempty closed subset of the Hilbert space $\mathcal{H}$ with $C \neq \mathcal{H}$ and let $\rho : \operatorname{bdry} C \to]0, +\infty[$ be a function. The following are equivalent:*
(a) The set C is $\rho(\cdot)$-strongly convex;
(b) for any $(x, v) \in \mathcal{H}^2$ with $x \in \operatorname{bdry} C$ and $v \in N^P(C;x) \cap \mathbb{S}$ and for any real $t > \rho(x)$, one has

$$x = \operatorname{far}_C(x - tv);$$

(c) one has the equality

$$C = \bigcap_{x \in \operatorname{bdry} C, v \in N^P(C;x) \cap \mathbb{S}} B[x - \rho(x)v, \rho(x)];$$

(d) one has the inclusion

$$C \subset \bigcap_{x \in \operatorname{bdry} C, v \in N^P(C;x) \cap \mathbb{S}} B[x - \rho(x)v, \rho(x)];$$

(e) for all $x' \in C$ and for all $(x, v) \in \mathcal{H}^2$ with $x \in \operatorname{bdry} C$ and $v \in N^P(C;x) \cap \mathbb{S}$, one has

$$\langle v, x' - x \rangle \leq -\frac{1}{2\rho(x)} \|x' - x\|^2;$$

(f) for all $x' \in C$ and for all $(x, v) \in \mathcal{H}^2$ with $x \in \operatorname{bdry} C$ and $v \in N^P(C;x)$, one has

$$\langle v, x' - x \rangle \leq -\frac{\|v\|}{2\rho(x)} \|x' - x\|^2;$$

(g) for all $x_1, x_2 \in \operatorname{bdry} C$, for all $v_1 \in N^P(C;x_1)$, all $v_2 \in N^P(C;x_2)$, one has

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq \frac{1}{2}\left(\frac{\|v_1\|}{\rho(x_1)} + \frac{\|v_2\|}{\rho(x_2)}\right) \|x_1 - x_2\|^2;$$

(h) for all $x_i \in \mathcal{H}$, all $y_i \in [\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot)]^{-1}(x_i) \cap \operatorname{bdry} C$, $i = 1, 2$,

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq \left(1 - \frac{\|y_1 - x_1\|}{2\rho(y_1)} - \frac{\|y_2 - x_2\|}{2\rho(y_2)}\right) \|y_1 - y_2\|^2 .$$

Proof. The equivalence $(a) \Leftrightarrow (b)$ directly follows from the equality (7) while the equivalence (10) translates $(a) \Leftrightarrow (e)$. Further, it is readily seen that $(e) \Leftrightarrow (f)$. The implications $(c) \Rightarrow (d) \Rightarrow (a)$ being evident, let us show that $(a) \Rightarrow (c)$. Assume that C is $\rho(\cdot)$-strongly convex. Observe first that the implication $(a) \Rightarrow (f)$ easily entails that the multimapping $N^P(C;\cdot)$ is monotone on $\mathcal{H}$ (since $N^P(C;x) = \{0\}$ if $x \in \operatorname{int} C$), hence the closed set C is convex (see, e.g., [35, Corollary 6.69]). On the other hand, (9) says that

$$\Lambda^P(C) := \big\{(x,v) \in \mathcal{H}^2 : x \in \operatorname{bdry} C, v \in N(C;x) \cap \mathbb{S}\big\} \neq \emptyset,$$

and then using the $\rho(\cdot)$-strong convexity of C furnishes in a straightforward way that

$$C \subset \bigcap_{(x,v)\in\Lambda^P(C)} B[x - \rho(x)v, \rho(x)] =: \mathcal{I}.$$

By contradiction, suppose that $\mathcal{I} \not\subset C$. Doing so, we can find some $y \in \mathcal{I}$ such that $y \notin C$. Set $d := d_C(y) > 0$, $p := \operatorname{proj}_C(y)$ (which is well defined since the set C is nonempty, closed and convex by what precedes) and set also $w := \frac{y-p}{d} \in N(C;p) \cap \mathbb{S}$. Since $B[p - \rho(p)w, \rho(p)] \supset \mathcal{I}$, we have

$$\|y - (p - \rho(p)w)\|^2 \leq \rho(p)^2,$$

or equivalently,

$$\big(1 + \frac{\rho(p)}{d}\big)^2 \|y - p\|^2 \leq \rho(p)^2.$$

Consequently, we get $(d+\rho(p))^2 \leq \rho(p)^2$, and this cannot hold true. The implication $(a) \Rightarrow (c)$ is then proved.

It remains to observe that the equivalences $(e) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h)$ are evident to complete the proof. □

The next proposition is concerned with the intersection of a $\rho(\cdot)$-strongly convex set with a closed affine subspace.

Proposition 3.4. *Assume that the dimension of the Hilbert space $\mathcal{H}$ is greater than 2. Let C be a closed convex subset of $\mathcal{H}$ with nonempty interior and with $C \neq \mathcal{H}$, and let $\rho : \operatorname{bdry} C \to]0, +\infty[$ be a function. The following hold.*
(a) If for any two-dimensional affine subspace L of $\mathcal{H}$ intersecting $\operatorname{int} C$, the set $C \cap L$ is $\rho(\cdot)$-strongly convex in L, then the set C is $\rho(\cdot)$-strongly convex (in $\mathcal{H}$).
(b) If C is $\rho(\cdot)$-strongly convex, then for any closed affine subspace L of $\mathcal{H}$ intersecting $\operatorname{int} C$ the set $C \cap L$ is $\rho(\cdot)$-strongly convex in L.
(c) The set C is $\rho(\cdot)$-strongly convex if and only if, for any two-dimensional affine subspace L of $\mathcal{H}$ intersecting $\operatorname{int} C$, the set $C \cap L$ is $\rho(\cdot)$-strongly convex in L.

Proof. (a) Assume that the property in (a) is satisfied. Let $(x,v) \in \mathcal{H}^2$ with $x \in \operatorname{bdry} C$ and $v \in N^P(C;x)$ with $\|v\| = 1$. Take any $y \in \operatorname{int} C$ and put $u := x + v$. Let E be a two-dimensional vector subspace of $\mathcal{H}$ such that $x + E$ contains both u and y, that is, E contains v and $y - x$. By assumption the set $C_E := C \cap (x + E)$ is $\rho(\cdot)$-strongly convex in $x + E$. Observe that $v \in N^{P,E}(C_E;x)$, so $x \in \operatorname{bdry}_{x+E}(C_E)$ since

$\|v\| = 1$. By the $\rho(\cdot)$-strong convexity of C_E in $x+E$ we have $x \in \mathrm{Far}_{C_E}(x-\rho(x)v)$. Using the inclusion $y \in C_E$ we deduce that

$$\langle v, y-x\rangle \leq -\frac{1}{2\rho(x)}\|y-x\|^2.$$

Since $\operatorname{int} C$ is dense in the convex set C, the latter inequality still holds true for every $y \in C$. This entails the $\rho(\cdot)$-strong convexity of C (in $\mathcal{H}$) by (e) in Theorem 3.3.

(b) Assume that C is ρ-strongly convex. Let E be any closed vector subspace of $\mathcal{H}$ and a be any point in $\operatorname{int} C$, and let $C_E := C \cap (a+E)$. Let $(x,v) \in (a+E)\times E$ with $x \in \mathrm{bdry}_{a+E}(C_E)$ and $v \in N^{P,E}(C_E;x)$ with $\|v\| = 1$. Observe that $x \in \mathrm{bdry}\, C$ since it is not difficult to see that $\mathrm{bdry}_{a+E}(C_E) \subset (a+E)\cap \mathrm{bdry}\, C$ (as already said after Definition 3.1). Take any $y \in C_E$. Note that $N^{P,E}(C_E;x) = N(C_E;x)\cap E$ as easily seen. Further, since $(a+E)\cap \operatorname{int} C \neq \emptyset$ we have (see, e.g., [35, Corollary 3.177(b)])

$$N(C_E;x) = N(C\cap(a+E);x) = N(C;x) + N(a+E;x) = N(C;x) + E^{\perp},$$

where as usual $E^{\perp}$ denotes the orthogonal of the set E. Hence, there are $v_C \in N(C;x)$ and $v_E \in E^{\perp}$ such that $v = v_C + v_E$. Since $\langle v_E, v\rangle = 0$, we have $\|v_C\|^2 = \|v\|^2 + \|v_E\|^2$, and hence $1 = \|v\|^2 \leq \|v_C\|^2$. Then we can write by (f) in Theorem 3.3

$$\begin{aligned}
\langle v, y-x\rangle &= \langle v_C, y-x\rangle + \langle v_E, y-x\rangle \\
&= \langle v_C, y-x\rangle \\
&\leq -\frac{\|v_C\|}{2\rho(x)}\|y-x\|^2 \\
&\leq -\frac{1}{2\rho(x)}\|y-x\|^2.
\end{aligned}$$

The inequality $\langle v, y-x\rangle \leq -\frac{1}{2\rho(x)}\|y-x\|^2$ for all $y \in C_E$ means (see (5)) $x \in \mathrm{Far}_{C_E}(x-\rho(x)v)$. It follows that C_E is $\rho(\cdot)$-strongly convex in $(a+E)$, justifying (b).

(c) The equivalence in (c) follows from (a) and (b). □

Now we derive from Theorem 3.3 the following important property on the support function of a $\rho(\cdot)$-strongly convex set.

Proposition 3.5. *Let C be a $\rho(\cdot)$-strongly convex set in the Hilbert space $\mathcal{H}$ for some function $\rho : \mathrm{bdry}\, C \to]0,+\infty[$. Then, for any $\zeta \in \mathcal{H}\setminus\{0\}$, there exists one and only one $c_\zeta \in C$ such that*

$$\sigma(\zeta, C) = \langle \zeta, c_\zeta\rangle,$$

or equivalently such that

$$\zeta \in N(C; c_\zeta).$$

Proof. Take any nonzero $\zeta \in \mathcal{H}$. According to the weak compactness of C, we know that we can find some $c_\zeta \in C$ such that

$$\sigma(\zeta, C) := \sup_{x\in C}\langle \zeta, x\rangle = \langle \zeta, c_\zeta\rangle.$$

Consider any $c_1, c_2 \in C$ such that

$$\sigma(\zeta, C) = \langle \zeta, c_1\rangle = \langle \zeta, c_2\rangle.$$

Applying the implication $(a) \Rightarrow (g)$ in Theorem 3.3 (using the obvious inclusion $\zeta \in N(C; c_i)$ with $i = 1, 2$) we see that for $\kappa := \|\zeta\| \left(\frac{1}{2\rho(c_1)} + \frac{1}{2\rho(c_2)} \right)$

$$0 = \langle \zeta - \zeta, c_1 - c_2 \rangle \geq \kappa \|c_1 - c_2\|^2,$$

and this entails that $c_1 = c_2$. The proof is complete. □

We establish now a lemma in preparation of the next proposition.

Lemma 3.6. *Let C be a nonempty subset of the Hilbert space $\mathcal{H}$ and let $x, y \in C$ and $t \in [0, 1]$. Assume that there exists a real $\sigma \geq 0$ such that*

$$\mathrm{dfar}_C\big((1-t)x + ty\big) - \sigma \mathrm{dfar}_C^2\big((1-t)x + ty\big) \geq \sigma t(1-t)\|x - y\|^2. \tag{13}$$

Then, one has

$$\mathrm{dfar}_C\big((1-t)x + ty\big) \geq \sigma \max\{t, 1-t\}\|x - y\|^2.$$

Proof. Set $z := (1-t)x + ty$ and $\kappa := \|x - y\|^2$. Using the inclusions $x, y \in C$ we have

$$\mathrm{dfar}_C^2(z) \geq \|z - x\|^2 = t^2\kappa \quad \text{and} \quad \mathrm{dfar}_C^2(z) \geq \|z - y\|^2 = (1-t)^2\kappa.$$

Put $\delta := \mathrm{dfar}_C^2(z) + t(1-t)\kappa$. We easily derive from the latter inequalities that

$$\delta \geq \max\{t\kappa, (1-t)\kappa\}. \tag{14}$$

On the other hand, note that (13) can be rewritten as $\sigma^{-1}\mathrm{dfar}_C(z) \geq \delta$. This and (14) furnish

$$\mathrm{dfar}_C(z) \geq \sigma\delta \geq \sigma\kappa \max\{t, (1-t)\}.$$

The proof is then complete. □

By means of the above lemma we can prove a lower estimate property for the value at convex combination of the farthest distance function to $\rho(\cdot)$-strongly convex sets. It complements upper estimates in the same line in [12, Proposition 9] (see also [36, Proposition 15.16]) for the (standard) distance function d_C to prox-regular sets C. Before stating the lower estimate property, recall by Theorem 2.2 that $\mathrm{Dom}\,\mathrm{Far}_C$ is dense in $\mathcal{H}$ for any nonempty closed bounded set C in $\mathcal{H}$.

Proposition 3.7. *Let C be a $\rho(\cdot)$-strongly convex subset of the Hilbert space $\mathcal{H}$ for some function $\rho : \mathrm{bdry}\, C \to]0, +\infty[$. The following hold.*
(a) For any $x_1, \ldots, x_n \in C$ and $t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $z := \sum_{i=1}^n t_i x_i \in \mathrm{Dom}\,\mathrm{Far}_C$, one has

$$\mathrm{dfar}_C(z) \geq \frac{1}{2\rho(v)}\mathrm{dfar}_C^2(z) + \frac{1}{4\rho(v)} \sum_{1 \leq i,j \leq n} t_i t_j \|x_i - x_j\|^2 \quad \text{for all } v \in \mathrm{Far}_C(z).$$

(b) For any $x, y \in C$ and $t \in [0, 1]$ such that $z := tx + (1-t)y \in \mathrm{Dom}\,\mathrm{Far}_C$

$$\mathrm{dfar}_C(z) - \frac{1}{2\rho(v)}\mathrm{dfar}_C^2(z) \geq \frac{1}{2\rho(v)} t(1-t)\|x - y\|^2 \quad \text{for all } v \in \mathrm{Far}_C(z).$$

(c) For any $x, y \in C$ and $t \in [0, 1]$ such that $z := tx + (1-t)y \in \mathrm{Dom}\,\mathrm{Far}_C$

$$\mathrm{dfar}_C(z) \geq \frac{1}{2\rho(v)} \max\{t, 1-t\}\|x - y\|^2 \quad \text{for all } v \in \mathrm{Far}_C(z).$$

Proof. (a) Let $x_1, \ldots, x_n \in C$ and $t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $z := \sum_{i=1}^n t_i x_i \in \operatorname{Dom} \operatorname{Far}_C$. Fix any $v \in \operatorname{Far}_C(z)$. By (6) we have $z - v \in -N(C; v)$. Using the $\rho(\cdot)$-strong convexity of the set C, we can then write (by (f) in Theorem 3.3) for each $i \in \{1, \ldots, n\}$

$$\langle z - v, x_i - v \rangle \geq \frac{1}{2\rho(v)} \|z - v\| \|x_i - v\|^2,$$

which obviously implies

$$\left\langle z - v, \sum_{i=1}^n t_i x_i - v \right\rangle \geq \frac{1}{2\rho(v)} \|z - v\| \sum_{i=1}^n t_i \|x_i - v\|^2.$$

This and the definition of z easily give

$$\operatorname{dfar}_C(z) = \|z - v\| \geq \frac{1}{2\rho(v)} \sum_{i=1}^n t_i \|x_i - v\|^2. \tag{15}$$

Set $\alpha := \sum_{i=1}^n t_i \|x_i - z\|^2$ and fix any $u \in \mathcal{H}$. Observe first that

$$\sum_{i=1}^n t_i \|x_i - u\|^2 = \sum_{i=1}^n t_i \big(\|x_i - z\|^2 + \|z - u\|^2 + 2 \langle x_i - z, z - u \rangle \big) = \alpha + \|z - u\|^2. \tag{16}$$

Therefore, we have

$$\sum_{1 \leq i,j \leq n} t_j t_i \|x_i - x_j\|^2 = \sum_{j=1}^n t_j \|z - x_j\|^2 + \sum_{j=1}^n t_j \alpha.$$

Putting together the latter equality and the fact that $\sum_{j=1}^n t_j = 1$, we arrive to

$$\alpha = \frac{1}{2} \sum_{1 \leq i,j \leq n} t_j t_i \|x_i - x_j\|^2.$$

Combining this equality with the second equality in (16) then ensures

$$\sum_{i=1}^n t_i \|x_i - v\|^2 = \frac{1}{2} \sum_{1 \leq i,j \leq n} t_j t_i \|x_i - x_j\|^2 + \|z - v\|^2.$$

Coming back to (15), we obtain

$$\operatorname{dfar}_C(z) \geq \frac{1}{4\rho(v)} \sum_{1 \leq i,j \leq n} t_j t_i \|x_i - x_j\|^2 + \frac{1}{2\rho(v)} \|z - v\|^2,$$

which translates the inequality in (a).
(b) It is a direct consequence of (a) above.
(c) It suffices to combine (b) above and Lemma 3.6. □

3.2. **Main properties of $\rho(\cdot)$-strongly convex sets.** Consider again a function $\rho : \operatorname{bdry} C \to]0, +\infty[$ defined on a nonempty closed bounded subset C of the Hilbert space $\mathcal{H}$. By Theorem 2.2 we know that $\operatorname{Dom} \operatorname{far}_C$ contains a dense G_δ set in $\mathcal{H}$. Accordingly, take any $\overline{u} \in \operatorname{Dom} \operatorname{Far}_C$ and $\overline{y} \in \operatorname{Far}_C(\overline{u})$. Putting $u_t := \overline{u} + t(\overline{u} - \overline{y})$ we know (see (7)) that for each real $t > 0$ we have $\overline{y} = \operatorname{far}_C(u_t)$, and hence $\operatorname{dfar}_C(u_t) = (1 + t)\|\overline{u} - \overline{y}\|$. Therefore, for every real $\gamma \geq 1$ we can choose some real $t > 0$ such that $\overline{y} \in \operatorname{Far}_C(u_t)$ and $\operatorname{dfar}_C(u_t) > \gamma \rho(\overline{y})$. This ensures in particular that

$$\{u \in \mathcal{H} : \exists y \in \operatorname{Far}_C(u),\ \operatorname{dfar}_C(u) > \gamma \rho(y)\} \neq \emptyset \quad \text{for every } \gamma \geq 1. \tag{17}$$

On the other hand, given any $u \in \mathcal{H}$ there exists by (Theorem 2.2) a sequence $(u_n, y_n)_{n\in\mathbb{N}}$ in $\operatorname{gph}\operatorname{Far}_C$ such that $u_n \to u$. Writing for every integer $n \geq 1$

$$\|y_n - u_n\| - \|u_n - u\| \leq \|y_n - u\| \leq \|y_n - u_n\| + \|u_n - u\|$$

and using the convergence $\|y_n - u_n\| = \operatorname{dfar}_C(u_n) \to \operatorname{dfar}_C(u)$, yield $\|y_n - u\| \to \operatorname{dfar}_C(u)$, hence

$$y_n \in \operatorname{bdry} C \quad \text{and} \quad \|y_n - u\| \to \operatorname{dfar}_C(u). \tag{18}$$

This says that for every $u \in \mathcal{H}$ the lower limit

$$\liminf_{\substack{\|x-u\|\to \operatorname{dfar}_C(u)\\ x\in \operatorname{bdry} C}} \frac{\|x-u\|}{\rho(x)}$$

is well defined. Accordingly, in addition to (17) our analysis below will begin, for $\gamma \geq 1$, with the similar set $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ given by

$$\left\{u\in\mathcal{H} : \exists\gamma' > \gamma, \exists\eta \in]0, \operatorname{dfar}_C(u)[, \forall x\in(\operatorname{bdry} C)\setminus B\big(u, \operatorname{dfar}_C(u)-\eta\big), \frac{\|x-u\|}{\rho(x)} > \gamma'\right\}.$$

It is clear that

$$\mathcal{D}_{\rho(\cdot)}(C) := \mathcal{D}^{1}_{\rho(\cdot)}(C) = \bigcup_{\gamma>1} \mathcal{D}^{\gamma}_{\rho(\cdot)}(C),$$

along with

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) = \left\{u \in \mathcal{H} : \liminf_{\substack{\|x-u\|\to \operatorname{dfar}_C(u)\\ x\in \operatorname{bdry} C}} \frac{\|x-u\|}{\rho(x)} > \gamma\right\} \quad \text{for every } \ \gamma \geq 1. \tag{19}$$

Obviously, for any real $\gamma \geq 1$ the inclusion

$$\{u \in \mathcal{H} : \exists y \in \operatorname{Far}_C(u),\ \operatorname{dfar}_C(u) > \gamma\rho(y)\} \subset \mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \tag{20}$$

always holds, and it makes a first link between $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ and the set involved in (17). In fact, we will see later (in Theorem 3.10) that the latter inclusion is an equality whenever the set C is $\rho(\cdot)$-strongly convex.

When C is a singleton, say $C = \{c\}$, it is an exercise to check that

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) = \mathcal{H} \setminus B[c, \gamma\rho(c)] \quad \text{for every } \gamma \geq 1.$$

In particular, we note that

$$\mathcal{D}_{\rho(\cdot)}(C) \cap C = \emptyset \quad \text{whenever } C \text{ is a singleton.}$$

The next proposition shows that the set $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ is always open in $\mathcal{H}$.

Proposition 3.8. *Let C be a nonempty closed bounded subset of the Hilbert space $\mathcal{H}$ and let $\rho : \operatorname{bdry} C \to]0, +\infty[$ be a given function. For any real $\gamma \geq 1$ the set $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ is open.*

Proof. We may suppose that C is not a singleton, since otherwise the result is trivial. Fix any real $\gamma \geq 1$ and any $u_0 \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. For each $u \in \mathcal{H}$ define

$$\ell(u) := \liminf_{\substack{\|x-u\|\to \operatorname{dfar}_C(u)\\ x\in \operatorname{bdry} C}} \frac{\|x-u\|}{\rho(x)},$$

so $\ell(u_0) > \gamma$ by (19). Choose a real $\gamma' > \gamma$ with $\ell(u_0) > \gamma'$ and a real $\lambda > 1$ such that $\ell(u_0) > \lambda\gamma'$. By definition of ℓ there exists a positive real $\delta_0 < \mathrm{dfar}_C(u_0)$ such that

$$\big(x \in \mathrm{bdry}\, C,\ \mathrm{dfar}_C(u_0) - \|x - u_0\| < \delta_0\big) \Longrightarrow \frac{\|x - u_0\|}{\rho(x)} > \lambda\gamma'.$$

Choose a positive real $\delta < \min\big\{\delta_0/3, \big(1 - \lambda^{-1}\big)\,(\mathrm{dfar}_C(u_0) - \delta_0)\big\}$. Fix any $u \in B(u_0, \delta)$. By (18) take any $x \in \mathrm{bdry}\, C$ satisfying $\mathrm{dfar}_C(u) - \|x - u\| < \delta$. This and (1) ensure that

$$\mathrm{dfar}_C(u_0) - \|x - u_0\| \le \mathrm{dfar}_C(u) - \|x - u\| + 2\|u - u_0\| < 3\delta < \delta_0, \tag{21}$$

which entails

$$\frac{\|x - u_0\|}{\rho(x)} > \lambda\gamma'. \tag{22}$$

On the other hand, from the inequality $\delta < (1-\lambda^{-1})(\mathrm{dfar}_C(u_0) - \delta_0)$ and from (21) we have $\delta < (1 - \lambda^{-1})\|x - u_0\|$, which allows us to write

$$\|x - u\| \ge \|x - u_0\| - \|u - u_0\| > \|x - u_0\| - \delta > \frac{1}{\lambda}\|x - u_0\|.$$

Combinining this with (22) gives $\frac{\|x-u\|}{\rho(x)} > \gamma'$. Taking the $\liminf$ as $\|x - u\| \to \mathrm{dfar}_C(u)$ with $x \in \mathrm{bdry}\, C$ we obtain $\ell(u) \ge \gamma' > \gamma$, hence $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. This being true for every $u \in B(u_0, \delta)$, we conclude that $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ is open. □

With the above open sets $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ at hands we can state and prove the following theorem on $\rho(\cdot)$-strongly convex sets.

Theorem 3.9. *Let C be a nonempty closed bounded subset of the Hilbert space $\mathcal{H}$ and let $\rho : \mathrm{bdry}\, C \to]0, +\infty[$ be a function which is lower semicontinuous relative to $\mathrm{bdry}\, C$. Consider the following assertions.*
(h) For all $x_i \in \mathcal{H}$, all $y_i \in [\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot)]^{-1}(x_i) \cap \mathrm{bdry}\, C$, $i = 1, 2$,

$$\langle y_1 - y_2, x_1 - x_2\rangle \le \left(1 - \frac{\|y_1 - x_1\|}{2\rho(y_1)} - \frac{\|y_2 - x_2\|}{2\rho(y_2)}\right) \|y_1 - y_2\|^2 .$$

(i) For any real $\gamma > 1$ and any $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$, the set $\mathrm{Far}_C(u)$ is a singleton, that is, $\mathrm{far}_C(u)$ exists, one has $\mathrm{dfar}_C(u) > \gamma\rho\big(\mathrm{far}_C(u)\big)$, the equality

$$\{\mathrm{far}_C(u)\} = \big(\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot) \cap (\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B})\big)^{-1}(u)$$

holds, and the mapping far_C is Lipschitz continuous on the open set $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ with $(\gamma - 1)^{-1}$ as a Lipschitz constant, that is,

$$\|\mathrm{far}_C(u_1) - \mathrm{far}_C(u_2)\| \le (\gamma - 1)^{-1}\,\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C).$$

(j) For any real $\gamma > 1$ and any $u_1, u_2 \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$, one has that $\mathrm{far}_C(u_1)$ and $\mathrm{far}_C(u_2)$ are well defined and satisfy

$$\|(\mathrm{far}_C - \mathrm{Id}_{\mathcal{H}})(u_1) - (\mathrm{far}_C - \mathrm{Id}_{\mathcal{H}})(u_2)\|^2 \ge \|u_1 - u_2\|^2 + (2\gamma - 1)\|\mathrm{far}_C(u_1) - \mathrm{far}_C(u_2)\|^2.$$

(k) The mapping far_C is well defined on the open set $\mathcal{D}_{\rho(\cdot)}(C)$ and for all $u_1, u_2 \in \mathcal{D}_{\rho(\cdot)}(C)$, one has

$$\|\mathrm{far}_C(u_1) - \mathrm{far}_C(u_2)\| \le \Big(\frac{\mathrm{dfar}_C(u_1)}{2\rho(\mathrm{far}_C(u_1))} + \frac{\mathrm{dfar}_C(u_2)}{2\rho(\mathrm{far}_C(u_2))} - 1\Big)^{-1}\|u_1 - u_2\|.$$

Then, the $\rho(\cdot)$-strong convexity of the set C is equivalent to the assertion (h) which itself implies each one of the assertions (i), (j) and (k).

Proof. Proposition 3.8 says that $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ is open for every real $\gamma \geq 1$, and we have already seen in Theorem 3.3 that the $\rho(\cdot)$-strong convexity of C is equivalent to the assertion (h).
$(h) \Rightarrow (i)$. If C is a singleton, say $C = \{c\}$, we obviously have all the desired properties since $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) = \mathcal{H} \setminus B[c, \gamma\rho(c)]$. Then, we may assume that C is not a singleton. Fix any real $\gamma > 1$. Let us consider the multimapping $M : \mathcal{H} \rightrightarrows \mathcal{H}$ defined by

$$M(y) := y - N^P(C; y) \cap \big(\mathcal{H} \setminus \gamma\rho(y)\mathbb{B}\big) \quad \text{for all } y \in \mathcal{H}.$$

Note that $\operatorname{Dom} M = \{y \in \operatorname{bdry} C : N^P(C; y) \neq \{0\}\} \neq \emptyset$. Fix any $x_1, x_2 \in \operatorname{Dom} M^{-1}$ and let $y_1 \in M^{-1}(x_1)$ and $y_2 \in M^{-1}(x_2)$. Since $y_i \in [\mathrm{Id}_{\mathcal{H}} - N^P(C; \cdot)]^{-1}(x_i) \cap \operatorname{bdry} C$ for each $i \in \{1, 2\}$, we can apply our assumption given by the property (h) to get

$$\langle y_1 - y_2, x_2 - x_1\rangle \geq \left(\frac{\|y_1 - x_1\|}{2\rho(y_1)} + \frac{\|y_2 - x_2\|}{2\rho(y_2)} - 1\right) \|y_1 - y_2\|^2.$$

Thanks to the fact that $y_1 - x_1 \notin \gamma\rho(y_1)\mathbb{B}$ and $y_2 - x_2 \notin \gamma\rho(y_2)\mathbb{B}$, we see that

$$\|y_1 - y_2\| \, \|x_2 - x_1\| \geq (\gamma - 1)\|y_1 - y_2\|^2.$$

Consequently, the multimapping $M^{-1}(\cdot) = \Big(\mathrm{Id}_{\mathcal{H}} - N^P(C; \cdot) \cap \big(\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B}\big)\Big)^{-1}$ is single-valued and Lipschitz continuous on its domain with $(\gamma - 1)^{-1}$ as a Lipschitz constant.

Let $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. We can apply Fitzpatrick's theorem relative to farthest points (see Theorem 2.2) to get a sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $u_n \to u$ such that $\operatorname{far}_C(u_n)$ is well-defined for every integer $n \geq 1$. Thanks to the continuity of $\operatorname{dfar}_C(\cdot)$, it is clear that $\|u_n - \operatorname{far}_C(u_n)\| = \operatorname{dfar}_C(u_n) \to \operatorname{dfar}_C(u)$. This along with the convergence $u_n \to u$ entail $\|u - \operatorname{far}_C(u_n)\| \to \operatorname{dfar}_C(u)$. Then, the inclusion $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ gives the estimate

$$\liminf_{n\to\infty} \frac{\|u - \operatorname{far}_C(u_n)\|}{\rho(\operatorname{far}_C(u_n))} > \gamma.$$

Combining this and the convergence $\frac{\|u - \operatorname{far}_C(u_n)\|}{\|u_n - \operatorname{far}_C(u_n)\|} \to 1$, we obtain

$$l := \liminf_{n\to\infty} \frac{\|u_n - \operatorname{far}_C(u_n)\|}{\rho(\operatorname{far}_C(u_n))} > \gamma.$$

Fix any real $\gamma' \in]\gamma, l[$. Hence, we may suppose

$$\|u_n - \operatorname{far}_C(u_n)\| > \gamma'\rho(\operatorname{far}_C(u_n)) \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

This and (6) furnish

$$\operatorname{far}_C(u_n) \in M^{-1}(u_n) = \Big(\mathrm{Id}_{\mathcal{H}} - N^P(C; \cdot) \cap \big(\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B}\big)\Big)^{-1}(u_n) \quad \text{for all } n \in \mathbb{N}.$$

We are then in a position to use the $(\gamma - 1)^{-1}$-Lipschitz property established above to get

$$\|\operatorname{far}_C(u_p) - \operatorname{far}_C(u_q)\| \leq (\gamma - 1)^{-1}\|u_p - u_q\| \quad \text{for all } p, q \in \mathbb{N},$$

in particular the sequence $\big(\operatorname{far}_C(u_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathcal{H}$. Keeping in mind that $\operatorname{bdry} C$ is a closed subset of the Hilbert space $\mathcal{H}$, the latter sequence (strongly) converges to some vector $v \in \operatorname{bdry} C$, that is,

$$v_n := \operatorname{far}_C(u_n) \to v \in \operatorname{bdry} C.$$

Further, we see that the continuity of dfar_C and the equality $\|u_n - v_n\| = \mathrm{dfar}_C(u_n)$ (valid for every integer $n \in \mathbb{N}$) entail that $\|u - v\| = \mathrm{dfar}_C(u)$, i.e., $v \in \mathrm{Far}_C(u)$. Coming back to the inequality (23), it follows by the lower semicontinuity of $\rho(\cdot)$ relative to $\mathrm{bdry}\, C$ that

$$\mathrm{dfar}_C(u) = \|u - v\| \geq \gamma' \rho(v) > \gamma\rho(v).$$

Putting together the latter estimate, the inclusion $v \in \mathrm{Far}_C(u)$ and (6) yield

$$v \in \Big(\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot) \cap \big(\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B}\big)\Big)^{-1}(u) = M^{-1}(u).$$

Consequently, we have established that

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \subset \mathrm{Dom}\,\mathrm{Far}_C \quad \text{and} \quad \mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \subset \mathrm{Dom}\, M^{-1}.$$

Since $M^{-1} = \Big(\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot) \cap \big(\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B}\big)\Big)^{-1}$ is single-valued on its domain and $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$, we arrive to the existence of $\mathrm{far}_C(u)$ along with

$$\mathrm{far}_C(u) = M^{-1}(u) = \Big(\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot) \cap \big(\mathcal{H} \setminus \gamma\rho(\cdot)\mathbb{B}\big)\Big)^{-1}(u).$$

This and the $(\gamma-1)^{-1}$-Lipschitz continuity of M^{-1} on its domain combined with the above inclusion $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \subset \mathrm{Dom}\, M^{-1}$ assures us that

$$\|\mathrm{far}_C(u_1) - \mathrm{far}_C(u_2)\| \leq (\gamma - 1)^{-1}\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C),$$

which translates the desired implication $(h) \Rightarrow (i)$.

$(h) \Rightarrow (j)$. Fix any $u_1, u_2 \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. According to the implication $(h) \Rightarrow (i)$, we know that for each $i \in \{1, 2\}$, the vector $y_i := \mathrm{far}_C(u_i)$ is well defined along with $\|u_i - y_i\| > \gamma\rho(y_i)$. Noting also that $y_i \in [\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot)]^{-1}(u_i) \cap \mathrm{bdry}\, C$ for each $i \in \{1, 2\}$ (see (2) and (6)), the estimate provided by (h) yields

$$\langle y_1 - y_2, u_1 - u_2\rangle \leq (1 - \gamma)\|y_1 - y_2\|^2.$$

It remains to combine the latter estimate with the elementary equality

$$\|(y_1 - u_1) - (y_2 - u_2)\|^2 = \|u_1 - u_2\|^2 + \|y_1 - y_2\|^2 - 2\,\langle y_1 - y_2, u_1 - u_2\rangle$$

to obtain the desired inequality in (j).

$(h) \Rightarrow (k)$. As above the implication $(h) \Rightarrow (i)$ tells us that the mapping $\mathrm{far}_C(\cdot)$ is well defined on $\mathcal{D}_{\rho(\cdot)}(C)$. Fix any $u_1, u_2 \in \mathcal{D}_{\rho(\cdot)}(C)$. Set $y_1 := \mathrm{far}_C(u_1)$ and $y_2 \in \mathrm{far}_C(u_2)$. Thanks to the inclusion $y_i \in [\mathrm{Id}_{\mathcal{H}} - N^P(C;\cdot)]^{-1}(u_i) \cap \mathrm{bdry}\, C$ (due to (2) and (6)) valid for each $i \in \{1, 2\}$, we can use the inequality provided by (h) to get

$$\langle y_1 - y_2, u_2 - u_1\rangle \geq \left(\frac{\|y_1 - u_1\|}{2\rho(y_1)} + \frac{\|y_2 - u_2\|}{2\rho(y_2)} - 1\right)\|y_1 - y_2\|^2.$$

It remains to apply the Cauchy-Schwarz inequality and to observe that $\|y_i - u_i\| = \mathrm{dfar}_C(u_i)$ for any $i \in \{1, 2\}$ to obtain the desired estimate in (k). The proof of the theorem is then complete. □

Let us continue with a function $\rho : \mathrm{bdry}\, C \to]0, +\infty[$ defined on the boundary a nonempty closed bounded subset C of the Hilbert space $\mathcal{H}$. Let us consider for every real $\gamma \geq 1$ (in addition to the set $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$) the set $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ given by

$$\left\{u \in \mathcal{H} : \exists \gamma' > \gamma, \forall \eta > 0, \exists (u', y) \in \mathrm{gph}\,\mathrm{Far}_C \cap \big(B(u, \eta) \times \mathcal{H}\big), \frac{\|y - u'\|}{\rho(y)} > \gamma'\right\}$$

which can be rewritten as

$$\mathcal{E}^{\gamma}_{\rho(\cdot)}(C) = \left\{ u \in \mathcal{H} : \limsup_{\substack{\mathrm{Dom\,Far}_C \ni u' \to u \\ y \in \mathrm{Far}_C(u')}} \frac{\|y - u'\|}{\rho(y)} > \gamma \right\}.$$

Notice that the right members of the former and latter equalities for $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ make sense according to Fitzpatrick's theorem relative to farthest points (see Theorem 2.2). It can be checked in a straightforward way that

$$\{u \in \mathcal{H} : \exists y \in \mathrm{Far}_C(u), \mathrm{dfar}_C(u) > \gamma\rho(y)\} \subset \mathcal{E}^{\gamma}_{\rho(\cdot)}(C) \quad \text{for all } \gamma \geq 1 \tag{24}$$

along with

$$\mathcal{E}_{\rho(\cdot)}(C) := \mathcal{E}^{1}_{\rho(\cdot)}(C) = \bigcup_{\gamma > 1} \mathcal{E}^{\gamma}_{\rho(\cdot)}(C).$$

In addition to the inclusion $\{u \in \mathcal{H} : \exists y \in \mathrm{Far}_C(u), \mathrm{dfar}_C(u) > \gamma\rho(y)\} \subset \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ in (20), we have also observed above the inclusion of the left-hand side into $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$. In fact, we claim that one has the stronger inclusion

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \subset \mathcal{E}^{\gamma}_{\rho(\cdot)}(C) \quad \text{for all } \gamma \geq 1. \tag{25}$$

Indeed, take any $u \in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. Choose sequences $(u_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $y_n \in \mathrm{Far}_C(u_n)$ and $\mathrm{Dom\,Far}_C \ni u_n \to u$ and such that

$$\frac{\|y_n - u_n\|}{\rho(y_n)} \to \limsup_{\substack{\mathrm{Dom\,Far}_C \ni u' \to u \\ y \in \mathrm{Far}_C(u')}} \frac{\|y - u'\|}{\rho(y)} =: \ell.$$

The convergence $u_n \to u$ and the continuity of dfar_C ensure that

$$\|y_n - u_n\| = \mathrm{dfar}_C(u_n) \to \mathrm{dfar}_C(u),$$

hence $\|y_n - u\| \to \mathrm{dfar}_C(u)$. This and the inclusion $y_n \in \mathrm{bdry}\, C$ give

$$\ell = \lim_{n\to\infty} \frac{\|y_n - u_n\|}{\rho(y_n)} = \lim_{n\to\infty} \frac{\|y_n - u\|}{\rho(y_n)} \geq \liminf_{\substack{\|y-u\| \to \mathrm{dfar}_C(u) \\ y \in \mathrm{bdry}\, C}} \frac{\|y - u\|}{\rho(y)} > \gamma.$$

This justifies the inclusion (25).

If C is a singleton, i.e., $C = \{c\}$ for some vector $c \in \mathcal{H}$, it is readily observed that

$$\mathcal{E}^{\gamma}_{\rho(\cdot)}(C) = \mathcal{H} \setminus B[c, \gamma\rho(c)].$$

In particular, we note that

$$\mathcal{E}_{\rho(\cdot)}(C) \cap C = \emptyset \quad \text{whenever } C \text{ is a singleton.} \tag{26}$$

Under the $\rho(\cdot)$-strong convexity of C, the next theorem shows in particular the coincidence of the sets $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ and $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$.

Theorem 3.10. *Let C be a $\rho(\cdot)$-strongly convex subset of the Hilbert space $\mathcal{H}$ for some function $\rho : \mathrm{bdry}\, C \to]0, +\infty[$ which is continuous relative to* bdry C*. The following hold.*
(a) For every real $\gamma \geq 1$, one has

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) = \{u \in \mathcal{H} : \exists y \in \mathrm{Far}_C(u), \mathrm{dfar}_C(u) > \gamma\rho(y)\} = \mathcal{E}^{\gamma}_{\rho(\cdot)}(C).$$

(b) The mapping $\mathrm{far}_C(\cdot)$ *is well defined on the open set* $\mathcal{D}_{\rho(\cdot)}(C)$ *and locally Lipschitz on this open set.*

Proof. (a) Fix any real $\gamma \geq 1$. It follows from (24) and from (i) in Theorem 3.9 that

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) \subset \{u \in \mathcal{H} : \exists y \in \mathrm{Far}_C(u), \mathrm{dfar}_C(u) > \gamma\rho(y)\} \subset \mathcal{E}^{\gamma}_{\rho(\cdot)}(C). \tag{27}$$

Fix any $u \in \mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$. There are a real γ' and two sequences $(u_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $u_n \to u$ and $y_n \in \mathrm{Far}_C(u_n) \subset \mathrm{bdry}\, C$ for every integer $n \geq 1$ (see (2) for the second inclusion) such that

$$\lim_{n\to\infty} \frac{\|u_n - y_n\|}{\rho(y_n)} = \limsup_{\substack{\mathrm{Dom}\,\mathrm{Far}_C \ni u' \to u \\ q \in \mathrm{Far}_C(u')}} \frac{\|q - u'\|}{\rho(q)} > \gamma' > \gamma.$$

Without loss of generality, we may suppose that

$$\frac{\|u_n - y_n\|}{\rho(y_n)} > \gamma' \quad \text{for all } n \in \mathbb{N}. \tag{28}$$

Let $u' \in \mathrm{Dom}\,\mathrm{Far}_C$, $y' \in \mathrm{Far}_C(u')$ and $z \in C$. Through the inclusion $y' - u' \in N^P(C; y')$ (see (6)) and the elementary equality

$$\|u' - z\|^2 = \|u' - y'\|^2 + \|y' - z\|^2 + 2\langle y' - u', z - y'\rangle$$

we see that the $\rho(\cdot)$-strong convexity of C gives by the equivalence $(a) \Leftrightarrow (f)$ in Theorem 3.3 that

$$\|u' - z\|^2 \leq \|u' - y'\|^2 + \|y' - z\|^2 - \frac{\|y' - u'\|}{\rho(y')}\|z - y'\|^2,$$

or equivalently,

$$\Big(\frac{\|y' - u'\|}{\rho(y')} - 1\Big)\|z - y'\|^2 \leq \|u' - y'\|^2 - \|u' - z\|^2. \tag{29}$$

Then, the latter inequality and (28) ensure that we obtain for all integers $m, n \in \mathbb{N}$

$$\begin{aligned}
(\gamma' - 1)\|y_m - y_n\|^2 &\leq \Big(\frac{\|y_n - u_n\|}{\rho(y_n)} - 1\Big)\|y_m - y_n\|^2 \\
&\leq \|y_n - u_n\|^2 - \|u_n - y_m\|^2 \\
&\leq \mathrm{dfar}^2_C(u_n) - \big(\|u_m - y_m\| - \|u_n - u_m\|\big)^2 \\
&= \mathrm{dfar}^2_C(u_n) - \big(\mathrm{dfar}_C(u_m) - \|u_n - u_m\|\big)^2.
\end{aligned}$$

Therefore, it is readily seen that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathcal{H}$ which converges to some $y \in \mathcal{H}$. This and the equality $\|u_n - y_n\| = \mathrm{dfar}_C(u_n)$ ensure that $\|u - y\| = \mathrm{dfar}_C(u)$. Further, the closedness of C implies that $y \in C$, hence $y \in \mathrm{Far}_C(u) \subset \mathrm{bdry}\, C$. Keeping in mind (28) and using the fact that the function $\rho(\cdot)$ is continuous relative to $\mathrm{bdry}\, C$ and takes positive values, we can write

$$\frac{\mathrm{dfar}_C(u)}{\rho(y)} = \lim_{n\to\infty} \frac{\|u_n - y_n\|}{\rho(y_n)} = \lim_{n\to\infty} \frac{\mathrm{dfar}_C(u_n)}{\rho(y_n)} \geq \gamma' > \gamma. \tag{30}$$

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{bdry}\, C$ with $\|c_n - u\| \to \mathrm{dfar}_C(u)$ such that

$$\lim_{n\to\infty} \frac{\|c_n - u\|}{\rho(c_n)} = \liminf_{\substack{\|q-u\| \to \mathrm{dfar}_C(u) \\ q \in \mathrm{bdry}\, C}} \frac{\|q - u\|}{\rho(q)}.$$

In view of (29), we have

$$\Big(\frac{\|y-u\|}{\rho(y)}-1\Big)\|c_n-y\|^2 \le \|y-u\|^2-\|u-c_n\|^2,$$

hence by (30)

$$(\gamma'-1)\|c_n-y\|^2 \le \mathrm{dfar}_C^2(u)-\|u-c_n\|^2.$$

Then, we see that the sequence $(c_n)_{n\in\mathbb{N}}$ converges to y (since $\|c_n-u\|\to \mathrm{dfar}_C(u)$), in particular (by the continuity of the function $\rho(\cdot)$)

$$\liminf_{\substack{\|q-u\|\to \mathrm{dfar}_C(u)\\ q\in \mathrm{bdry}\, C}} \frac{\|q-u\|}{\rho(q)} = \liminf_{n\to\infty}\frac{\|c_n-u\|}{\rho(c_n)} = \frac{\|y-u\|}{\rho(y)} = \frac{\mathrm{dfar}_C(u)}{\rho(y)} \ge \gamma' > \gamma,$$

where the first inequality is due to (30) again. This gives $u\in \mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$. Then, it is established that the inclusions in (27) are in fact equalities, that is,

$$\mathcal{D}^{\gamma}_{\rho(\cdot)}(C) = \{u\in\mathcal{H} : \exists y\in \mathrm{Far}_C(u), \mathrm{dfar}_C(u) > \gamma\rho(y)\} = \mathcal{E}^{\gamma}_{\rho(\cdot)}(C). \tag{31}$$

(b) We conclude the proof by noting that (b) is a direct consequence of the openness property given in Proposition 3.8 and of the implications (h) ⇒ (i) in Theorem 3.9 and $(a) \Rightarrow (h)$ in Theorem 3.3. □

Given a nonempty closed bounded convex subset C of the Hilbert space $\mathcal{H}$, it is well-known that $\mathbb{S} = \left\{\frac{u-\mathrm{proj}_C(u)}{d_C(u)} : u\in\mathcal{H}\setminus C\right\}$. We now show that a similar equality holds whenever C is assumed in addition to be $\rho(\cdot)$-strongly convex, where the nearest point $\mathrm{proj}_C(u)$ is replaced by the farthest point $\mathrm{far}_C(u)$. We need first the following lemma which is in the line of [1, Proposition 3.3].

Lemma 3.11. *Let $\rho(\cdot)$ be a function (resp. lower semicontinuous function) from* $\mathrm{bdry}\, C$ *into* $]0,+\infty[$ *and let C be a $\rho(\cdot)$-strongly convex set in the Hilbert space $\mathcal{H}$. Let $x,x'\in\mathcal{H}$ with $x'-x\in N(C;x')$ and $\|x-x'\|\ge\rho(x')$ (resp. $\|x-x'\|>\rho(x')$). Then, one has $x'\in \mathrm{Far}_C(x)$ (resp. $x'=\mathrm{far}_C(x)$).*

Proof. First, note that in both cases $x'\in \mathrm{bdry}\, C$. Consider any $y\in C$. According to Theorem 3.3 we have

$$\langle x'-x, y-x'\rangle \le -\frac{1}{2\rho(x')}\|x'-x\|\|y-x'\|^2.$$

If $\rho(x')\le\|x-x'\|$, then

$$\langle x'-x, y-x'\rangle \le -\frac{1}{2}\|y-x'\|^2,$$

and this translates (see (5)) the inclusion $x'\in\mathrm{Far}_C(x)$.

Assume now that $\rho(x')<\|x-x'\|$, so by what precedes $x'\in \mathrm{Far}_C(x)$. We then have

$$\mathrm{dfar}_C(x) = \|x-x'\| > \rho(x')$$

and this guarantees (20) the inclusion $x'\in\mathcal{D}_{\rho(\cdot)}(C)$. It remains to apply Theorem 3.9 to conclude that $\mathrm{far}_C(x)$ is well defined, that is, $x'=\mathrm{far}_C(x)$. □

We are now in a position to establish the description of the unit sphere through farthest points as stated above.

Proposition 3.12. *Let C be a subset of the Hilbert space $\mathcal{H}$ which is $\rho(\cdot)$-strongly convex for some lower semicontinuous function $\rho(\cdot) : \operatorname{bdry} C \to]0, +\infty[$ and which is not a singleton. Let also $(x', v) \in \mathcal{H}^2$ with $x' \in \operatorname{bdry} C$ and $v \in N(C; x') \cap \mathbb{S}$. For every real $\alpha > \rho(x')$, one has with $x := x' - \alpha v$*

$$\operatorname{dfar}_C(x) = \alpha \quad \textit{and} \quad v = \frac{\operatorname{far}_C(x) - x}{\operatorname{dfar}_C(x)}.$$

Further, one has

$$\mathbb{S} = \left\{ \frac{\operatorname{far}_C(u) - u}{\operatorname{dfar}_C(u)} : u \in \mathcal{H} \right\}.$$

Proof. We have $x' - x \in N(C; x')$ and $\|x' - x\| = \alpha > \rho(x')$, so Lemma 3.11 tells us that $x' = \operatorname{far}_C(x)$. This ensures the first two desired equalities.

Let $v \in \mathbb{S}$. According to Proposition 3.5, there is $c \in C$ such that $v \in N(C; c)$. Choosing some real $\alpha > \rho(c)$ and setting $u := c - \alpha v$, we arrive by what precedes to

$$v = \frac{\operatorname{far}_C(u) - u}{\operatorname{dfar}_C(u)}.$$

The proof is then complete. □

Our aim is now to provide besides Theorem 3.3 several characterizations of $\rho(\cdot)$-strongly convex sets through either the regularity of the farthest distance function or the existence of farthest points. We start with the lemma below which is an adaptation of Lemma 3.4 of M.V. Balashov and G.E. Ivanov [3] (see also Lemma 10 in [18]). The context here of Hilbert spaces allow us to obtain more accurate estimates.

Lemma 3.13. *Let $r > 0$ be a positive real and let a, b, x in the Hilbert space $\mathcal{H}$ be such that*

$$\|a - x\| \le r \le \|b - x\|.$$

Then, one has with $z := a + \frac{r}{\|a-b\|}(a - b)$ and $m := \min\{r, \|a - b\|\}$

$$2m \left(1 - \sqrt{1 - \frac{\|z - x\|^2}{4r^2}} \right) \le \|a - b\| + r - \|b - x\|, \tag{32}$$

in particular

$$m \frac{\|z - x\|^2}{4r^2} \le \|a - b\| + r - \|b - x\|. \tag{33}$$

Proof. First, we note that $\|z - x\| \le 2r$ (since $\|a - x\| \le r$), so (33) is a direct consequence of (32) thanks to the elementary inequality

$$2\sqrt{1 - t} \le 2 - t$$

valid for every real $t \le 1$.

Now, let us establish (32). Set $u := \frac{a-x}{r} \in \mathbb{B}$ and $v := \frac{b-a}{\|b-a\|} \in \mathbb{S}$. It is clear that

$$\|u + v\|^2 + \|v - u\|^2 = 2(\|u\|^2 + \|v\|^2) \le 4,$$

or equivalently,

$$\|u + v\|^2 \le 4 - \|v - u\|^2. \tag{34}$$

We also easily observe that

$$u - v = \frac{1}{r}(a - x + z - a) = \frac{z - x}{r}.$$

Coming back to (34), we then get

$$\|u+v\| \leq \sqrt{4 - \frac{\|z-x\|^2}{r^2}} =: \delta.$$

Let us distinguish two cases:

Case 1. $m = \|a-b\|$. It is then easy to check

$$b - x = m(u+v) + \frac{r - \|a-b\|}{r}(a-x)$$

from which we derive (keeping in mind that $m \leq r$)

$$\|b-x\| \leq m\delta + r - \|a-b\|,$$

and this can be rewritten as

$$m(2-\delta) \leq 2m + r - \|a-b\| - \|b-x\| = \|a-b\| + r - \|b-x\|.$$

Case 2. $m = r$. We then have

$$b - x = r(u+v) + \frac{\|a-b\| - r}{\|a-b\|}(b-a),$$

hence (using $\|a-b\| \geq r$)

$$\|b-x\| \leq r\delta + \|a-b\| - r.$$

We then arrive to

$$r(2-\delta) \leq \|a-b\| + r - \|b-x\|.$$

We conclude that the desired inequality (32) holds in both cases. The proof of the lemma is complete. □

The next proposition provides a crucial estimate on the diameter of the set $\mathrm{Far}_{C,\eta}(\overline{x})$ (see (3)) for a $\rho(\cdot)$-strongly convex set C. We refer to Ivanov [18, Lemma 11] for the constant case $\rho(\cdot) \equiv R$. It will be convenient for the statement of the proposition to denote

$$L_{C,\rho(\cdot)}(u) := \limsup_{\substack{\mathrm{Dom}\,\mathrm{Far}_C \ni u' \to u \\ y \in \mathrm{Far}_C(u')}} \frac{\|y-u'\|}{\rho(y)},$$

so $u \in \mathcal{E}_{\rho(\cdot)}(C)$ means $L_{C,\rho(\cdot)}(u) > 1$.

Proposition 3.14. *Let C be a nonempty bounded subset not reduced to a singleton of the Hilbert space $\mathcal{H}$ for which there exists a function $\rho : \mathrm{bdry}\, C \to]0,+\infty[$ satisfying*

$$y \in \mathrm{Far}_C\left(y + \frac{\rho(y)}{\|x-y\|}(x-y)\right) \quad \text{for all } (x,y) \in \mathrm{gph}\,\mathrm{Far}_C \text{ with } x \in \mathcal{E}_{\rho(\cdot)}(C). \quad (35)$$

Assume that the set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ along with $\kappa := \inf_{\mathrm{Far}_C(\mathcal{H})} \rho > 0$.

Then, given any $\overline{x} \in \mathcal{E}_{\rho(\cdot)}(C)$, i.e. $L_{C,\rho(\cdot)}(\overline{x}) > 1$, there exists a real $r \in]0, \mathrm{dfar}_C(\overline{x})[$ such that for every real $\eta > 0$ with $1+2\eta/\kappa < L_{C,\rho(\cdot)}(\overline{x})$ (or equivalently $\overline{x} \in \mathcal{E}^{1+2\eta/\kappa}_{\rho(\cdot)}(C)$), one has

$$\mathrm{diam}\,\mathrm{Far}_{C,\eta}(\overline{x}) \leq \frac{4r}{\sqrt{\min\{r, \mathrm{dfar}_C(\overline{x}) - r\}}}\sqrt{\eta}.$$

Proof. Fix any $\overline{x} \in \mathcal{E}_{\rho(\cdot)}(C)$. Thanks to the definition and to the openness of the set $U := \mathcal{E}_{\rho(\cdot)}(C)$, we can find some sequence $((u_n, q_n))_{n\in\mathbb{N}}$ of $\operatorname{gph} \operatorname{Far}_C$ with $U \ni u_n \to \overline{x}$ and

$$L_{C,\rho(\cdot)}(\overline{x}) := \limsup_{\substack{\operatorname{Dom} \operatorname{Far}_C \ni u' \to \overline{x} \\ q \in \operatorname{Far}_C(u')}} \frac{\|q-u'\|}{\rho(q)} = \lim_{n\to\infty} \frac{\|q_n - u_n\|}{\rho(q_n)} > 1. \tag{36}$$

Extracting a subsequence if necessary we may suppose that $(q_n)_{n\in\mathbb{N}}$ weakly converges to some $\widehat{q} \in \mathcal{H}$ (denoted $q_n \stackrel{w}{\to} \widehat{q}$) along with the convergence $\rho(q_n) \to r$ for some real $r \in [\kappa, +\infty[$, where the inequality $r < \infty$ is due to (36) and to the boundedness of C. In view of (36), consider any real $\eta > 0$ such that $L_{C,\rho(\cdot)}(\overline{x}) > 1+2\eta/\kappa$. There is no loss of generality by writing

$$\operatorname{dfar}_C(u_n) = \|q_n - u_n\| > \rho(q_n)(1 + \frac{2\eta}{\kappa}) \quad \text{and} \quad \sigma_n := \|\overline{x} - u_n\| < \frac{\eta}{2} \tag{37}$$

for every integer $n \geq 1$. Passing to the limit as $n \to \infty$ in the first inequality gives

$$\operatorname{dfar}_C(\overline{x}) - r \geq 2r\eta/\kappa > 0. \tag{38}$$

Let $c \in \operatorname{Far}_{C,\eta}(\overline{x})$. Note that the inequalities in (38) obviously ensure that $\operatorname{dfar}_C(\overline{x}) > r$. Fix for a moment an integer $n \in \mathbb{N}$. Using the 1-Lipschitz property of dfar_C (see (1)), the first inequality in (37) and the definitions of κ and σ_n, we obtain

$$\operatorname{dfar}_C(\overline{x}) \geq \operatorname{dfar}_C(u_n) - \sigma_n \geq (1 + \frac{2\eta}{\kappa})\rho(q_n) - \sigma_n \geq \rho(q_n) + 2\eta - \sigma_n. \tag{39}$$

Keeping in mind the inclusion $c \in \operatorname{Far}_{C,\eta}(\overline{x})$, it follows from this and the second inequality in (37) that

$$\|c-u_n\| \geq \|c-\overline{x}\| - \|\overline{x}-u_n\| \geq \operatorname{dfar}_C(\overline{x}) - \eta - \sigma_n \geq \rho(q_n) + \eta - 2\sigma_n \geq \rho(q_n). \tag{40}$$

On the other hand, applying the assumptions (35) gives

$$q_n \in \operatorname{Far}_C(x_n) \quad \text{with} \quad x_n := q_n + \frac{\rho(q_n)}{\|q_n - u_n\|}(u_n - q_n),$$

and this obviously entails

$$\|x_n - c\| \leq \operatorname{dfar}_C(x_n) = \rho(q_n). \tag{41}$$

By definition of x_n and by the first inequality in (37) we also have

$$\|x_n - u_n\| = |\|q_n - u_n\| - \rho(q_n)| = \|q_n - u_n\| - \rho(q_n) = \operatorname{dfar}_C(u_n) - \rho(q_n). \tag{42}$$

This and the 1-Lipschitz property of dfar_C ensure that

$$\|x_n - u_n\| \leq \operatorname{dfar}_C(\overline{x}) + \|\overline{x} - u_n\| - \rho(q_n) = \operatorname{dfar}_C(\overline{x}) + \sigma_n - \rho(q_n). \tag{43}$$

Further, letting $n \to \infty$ in (42) furnishes $\|x_n - u_n\| \to \operatorname{dfar}_C(\overline{x}) - r$, thus

$$m_n := \min\{\rho(q_n), \|x_n - u_n\|\} \to \min\{r, \operatorname{dfar}_C(\overline{x}) - r\} > 0, \tag{44}$$

where the latter inequality is due to (38). From the definition of x_n again, it is also easily seen that

$$q_n = x_n + \frac{\rho(q_n)}{\|x_n - u_n\|}(x_n - u_n).$$

The latter equality combined with (40) and (41) allows us to apply Lemma 3.13 with $m_n = \min\{\rho(q_n), \|x_n - u_n\|\}$ to get

$$\frac{m_n\|q_n - c\|^2}{4\rho(q_n)^2} \leq \|x_n - u_n\| + \rho(q_n) - \|u_n - c\| =: \alpha_n. \tag{45}$$

Further, from (43) and (40) we have

$$\alpha_n \le \mathrm{dfar}_C(\overline{x}) + \sigma_n - \rho(q_n) + \rho(q_n) - \mathrm{dfar}_C(\overline{x}) + \eta + \sigma_n = 2\sigma_n + \eta. \tag{46}$$

Coming back to (39), we also have

$$\mathrm{dfar}_C(\overline{x}) - \rho(q_n) - \sigma_n \ge 2(\eta - \sigma_n) > 0.$$

According to this and to the fact that

$$\mathrm{dfar}_C(\overline{x}) - \|q_n - u_n\| = \mathrm{dfar}_C(\overline{x}) - \mathrm{dfar}_C(u_n) \le \|\overline{x} - u_n\| = \sigma_n,$$

it follows that

$$0 < 2(\eta - \sigma_n) \le \mathrm{dfar}_C(\overline{x}) - \rho(q_n) - \sigma_n \le \|q_n - u_n\| - \rho(q_n) = \|x_n - u_n\|,$$

where the latter equality is due to (42). We note by (45) and (46) that

$$\|q_n - c\| \le 2\rho(q_n)\sqrt{\frac{2\sigma_n + \eta}{m_n}} =: s_n.$$

Using the weak lower semicontinuity of $\|\cdot\|$ and the convergences $q_n \overset{w}{\to} \widehat{q}$ and $\rho(q_n) \to r$, we arrive by (44) to

$$\|\widehat{q} - c\| \le \liminf_{n\to+\infty} s_n = 2r\sqrt{\frac{\eta}{\min\{r, \mathrm{dfar}_C(\overline{x}) - r\}}} =: \beta.$$

Therefore, we have for all $c_1, c_2 \in \mathrm{Far}_{C,\eta}(\overline{x})$

$$\|c_1 - c_2\| \le \|c_1 - \widehat{q}\| + \|c_2 - \widehat{q}\| \le 2\beta.$$

This finishes the proof of the proposition. □

The condition $\kappa > 0$ in the above proposition holds true whenever C is $\rho(\cdot)$-strongly convex as shown below.

Proposition 3.15. *If a nonempty closed bounded subset C of the Hilbert space $\mathcal{H}$ is $\rho(\cdot)$-strongly convex for some lower semicontinuous function $\rho : \mathrm{bdry}\, C \to]0, +\infty[$, then $\inf_{\mathrm{bdry}\, C} \rho > 0$.*

Proof. By contradiction, suppose that $\inf_{\mathrm{bdry}\, C} \rho = 0$. Let $(c_n)_{n\ge1}$ be a sequence in $\mathrm{bdry}\, C$ with $\lim_{n\to\infty} \rho(c_n) = 0$. Fix some pair (x, v) such that $x \in \mathrm{bdry}\, C$ and $v \in N(C; x) \cap \mathbb{S}$. Since $\rho(\cdot)$ is lower semicontinuous and $\rho(x) > 0$, we must have $c_n \not\to x$, so there exist a real $\alpha > 0$ and a subsequence $(c_{s(n)})_{n\ge1}$ such that

$$\|c_{s(n)} - x\| \ge \alpha \quad \text{for all } n \in \mathbb{N}.$$

We deduce for each integer $n \in \mathbb{N}$, by the equivalence (a) $\Leftrightarrow$ (e) in Theorem 3.3, that

$$\left\langle v, \frac{c_{s(n)} - x}{\|c_{s(n)} - x\|} \right\rangle \le -\frac{1}{2\rho(c_{s(n)})}\|c_{s(n)} - x\|,$$

and this leads to the desired contradiction since

$$\left|\left\langle v, \frac{c_{s(n)} - x}{\|c_{s(n)} - x\|} \right\rangle\right| \le 1$$

while $-\frac{1}{2\rho(c_{s(n)})}\|c_{s(n)} - x\| \le -\frac{\alpha}{2\rho(c_{s(n)})} \to -\infty$. □

We can now prove the following theorem providing a series of properties of $\rho(\cdot)$-strongly convex sets through the differentiability of farthest distance function.

Theorem 3.16. *Let C be a nonempty closed bounded subset not reduced to a singleton of the Hilbert space $\mathcal{H}$ and let $\rho(\cdot) : \operatorname{bdry} C \to]0, +\infty[$ be a function which is continuous relative to $\operatorname{bdry} C$. Let also*

$$\mathcal{I}^{\gamma}_{\rho(\cdot)}(C) := \{u \in \mathcal{H} : \exists y \in \operatorname{Far}_C(u),\ \operatorname{dfar}_C(u) > \gamma\rho(y)\}.$$

Consider the following assertions:
(a) The set C is $\rho(\cdot)$-strongly convex.
(b) For any real $\gamma > 1$, the set $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the mapping far_C is well defined on $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ and $(\gamma-1)^{-1}$-Lipschitz continuous therein, that is,

$$\|\operatorname{far}_C(u_1) - \operatorname{far}_C(u_2)\| \le (\gamma-1)^{-1}\|u_1 - u_2\| \quad \textit{for all}\, u_1, u_2 \in \mathcal{E}^{\gamma}_{\rho(\cdot)}(C).$$

(c) For any real $\gamma > 1$, the set $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$ coincides with both $\mathcal{I}^{\gamma}_{\rho(\cdot)}(C)$ and $\mathcal{D}^{\gamma}_{\rho(\cdot)}(C)$ and the mapping far_C is well defined on $\mathcal{E}_{\rho(\cdot)}(C)$ along with

$$\|\operatorname{far}_C(u_1) - \operatorname{far}_C(u_2)\| \le \Big(\frac{\operatorname{dfar}_C(u_1)}{2\rho(\operatorname{far}_C(u_1))} + \frac{\operatorname{dfar}_C(u_2)}{2\rho(\operatorname{far}_C(u_2))} - 1\Big)^{-1}\|u_1 - u_2\|,$$

for all $u_1, u_2 \in \mathcal{E}_{\rho(\cdot)}(C)$.
(d) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the mapping far_C is well defined on $\mathcal{E}_{\rho(\cdot)}(C)$ and locally Lipschitz continuous on $\mathcal{E}_{\rho(\cdot)}(C)$.
(e) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the mapping far_C is well defined on $\mathcal{E}_{\rho(\cdot)}(C)$ and norm-to-norm continuous on $\mathcal{E}_{\rho(\cdot)}(C)$.
(f) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the mapping far_C is well defined on $\mathcal{E}_{\rho(\cdot)}(C)$ and norm-to-weak continuous on $\mathcal{E}_{\rho(\cdot)}(C)$.
(g) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the function dfar_C is of class $C^{1,1}$ on $\mathcal{E}_{\rho(\cdot)}(C)$.
(h) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and dfar_C is Fréchet differentiable on $\mathcal{E}_{\rho(\cdot)}(C)$.
(i) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the function dfar_C is Gâteaux differentiable on $\mathcal{E}_{\rho(\cdot)}(C)$ and $\operatorname{Far}_C(u) \neq \emptyset$ for all $u \in \mathcal{E}_{\rho(\cdot)}(C)$.
(j) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and the function dfar_C is Gâteaux differentiable on $\mathcal{E}_{\rho(\cdot)}(C)$ and $\|\nabla^G \operatorname{dfar}_C(u)\| = 1$ for all $u \in \mathcal{E}_{\rho(\cdot)}(C)$.
(k) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and $\operatorname{dfar}_C(u)$ is strongly attained for all $u \in \mathcal{E}_{\rho(\cdot)}(C)$.
(l) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and for any $u \in \mathcal{E}_{\rho(\cdot)}(C)$ such that the element $y := \operatorname{far}_C(u)$ is well defined one has

$$\operatorname{far}_C(u) = \operatorname{far}_C\big(y - t\frac{y-u}{\|y-u\|}\big) \quad \textit{for all}\, t \in]\rho(y), +\infty[.$$

(m) The set $\mathcal{E}_{\rho(\cdot)}(C)$ is open in $\mathcal{H}$ and for any $(u, y) \in \operatorname{gph} \operatorname{Far}_C$ with $u \in \mathcal{E}_{\rho(\cdot)}(C)$ one has

$$y \in \operatorname{Far}_C\big(y - \frac{\rho(y)}{\|y-u\|}(y-u)\big).$$

Then, the implications $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (d)$ hold, the assertion (d) is equivalent to anyone of $(e) - (k)$, and the implications $(h) \Rightarrow (l) \Rightarrow (m)$ also hold. If in addition, $\inf_{c \in \operatorname{Far}_C(\mathcal{H})} \rho(c) > 0$, then $(m) \Rightarrow (k)$.

Proof. First observe that Theorem 2.3 guarantees that $(d)-(k)$ are pairewise equivalent.

The implication $(a) \Rightarrow (c)$ directly follows from Theorem 3.10 and from the fact that the $\rho(\cdot)$-strong convexity of C implies (k) in Theorem 3.9. On the other hand, we obviously have $(c) \Rightarrow (b)$ and $(b) \Rightarrow (d)$.

To show that (h) implies (l) suppose that $\mathcal{E}_{\rho(\cdot)}(C)$ is open and $\mathrm{dfar}_C(\cdot)$ is Fréchet differentiable on $\mathcal{E}_{\rho(\cdot)}(C)$. According to Theorem 2.3, we know that the mapping $\mathrm{far}_C(\cdot)$ is well defined on the open set $\mathcal{E}_{\rho(\cdot)}(C)$ and continuous therein. Fix any $u \in \mathcal{E}_{\rho(\cdot)}(C)$. Choose a real $\gamma > 1$ such that $u \in \mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$. By definition of $\mathcal{E}^{\gamma}_{\rho(\cdot)}(C)$, we can find two sequences $(u'_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $y_n \in \mathrm{Far}_C(u'_n)$ for every integer $n \in \mathbb{N}$ such that

$$u'_n \to u \quad \text{and} \quad \frac{\|y_n - u'_n\|}{\rho(y_n)} > \gamma \quad \text{for all } n \geq 1.$$

Taking into account the latter convergence and the openness of the set $\mathcal{E}_{\rho(\cdot)}(C)$, we may suppose without loss of generality that $u'_n \in \mathcal{E}_{\rho(\cdot)}(C)$ for every integer $n \in \mathbb{N}$, and this allows us to write

$$y_n = \mathrm{far}_C(u'_n) \to \mathrm{far}_C(u) =: y \quad \text{and} \quad \frac{\|y - u\|}{\rho(y)} \geq \gamma.$$

Applying Lemma 2.4 provides some real $\tau_0 \in]0, 1[$ such that

$$\mathrm{far}_C(u) = \mathrm{far}_C\big(u + s(\mathrm{far}_C(u) - u)\big) \quad \text{for all } s \leq \tau_0.$$

Put $u_s := u + s\frac{\mathrm{far}_C(u)-u}{\|y-u\|}$ for every $s \in \mathbb{R}$ (keeping in mind that $x \neq u$ since C is not reduced to a singleton). Fix any $t \in]\rho(x), +\infty[$. If $t > \mathrm{dfar}_C(u)$, we have with $\tau := \frac{t}{\mathrm{dfar}_C(u)} > 1$ (see (7))

$$y = \mathrm{far}_C\big(\mathrm{far}_C(u) - \tau(\mathrm{far}_C(u) - u)\big) = \mathrm{far}_C\big(\mathrm{far}_C(u) - t\frac{y-u}{\|y-u\|}\big). \tag{47}$$

Consider now the case when $t \leq \mathrm{dfar}_C(u)$. Define the set

$$Q_t := \{s \in]-\infty, \mathrm{dfar}_C(u) - t] : u_s \in \mathcal{E}_{\rho(\cdot)}(C), \mathrm{far}_C(u) = \mathrm{far}_C(u_s)\}$$

which is nonempty since $0 \in Q_t$. Set $\kappa := \sup Q_t \leq \mathrm{dfar}_C(u) - t$ and choose a sequence $(s_n)_{n\in\mathbb{N}}$ of Q_t such that $s_n \to \kappa$. For every integer $n \in \mathbb{N}$ we can write

$$\begin{aligned}
\mathrm{dfar}_C(u_{s_n}) &= \|\mathrm{far}_C(u_{s_n}) - u_{s_n}\| \\
&= \left\|\mathrm{far}_C(u) - \big(u + s_n\frac{\mathrm{far}_C(u) - u}{\|y-u\|}\big)\right\| \\
&= \left|1 - \frac{s_n}{\|y-u\|}\right| \|\mathrm{far}_C(u) - u\| \\
&= |\|y-u\| - s_n| \\
&= \mathrm{dfar}_C(u) - s_n
\end{aligned} \tag{48}$$

and

$$\|u_{s_n} - y\| = \|u_{s_n} - \mathrm{far}_C(u)\| = \|u_{s_n} - \mathrm{far}_C(u_{s_n})\| = \mathrm{dfar}_C(u_{s_n}). \tag{49}$$

Putting together (48), (49) and the convergence $u_{s_n} \to u_\kappa$ we get

$$\|u_\kappa - y\| = \mathrm{dfar}_C(u_\kappa) = \mathrm{dfar}_C(u) - \kappa \geq t > \rho(y),$$

hence $y \in \mathrm{Far}_C(u_\kappa)$ and

$$\limsup_{\substack{\mathrm{Dom}\,\mathrm{Far}_C \ni u' \to u_\kappa \\ z \in \mathrm{Far}_C(u')}} \frac{\|z - u'\|}{\rho(z)} \geq \frac{\|y - u_\kappa\|}{\rho(y)} > 1.$$

This says that $u_\kappa \in \mathcal{E}_{\rho(\cdot)}(C)$. Then, we know that $\mathrm{dfar}_C(\cdot)$ is Fréchet differentiable at u_κ along with

$$y = \mathrm{far}_C(u_\kappa).$$

Now, we are going to show that $\kappa = \mathrm{dfar}_C(u) - t$. By contradiction, suppose that $\kappa \neq \mathrm{dfar}_C(u) - t$, that is, $\kappa < \mathrm{dfar}_C(u) - t$. Thanks to Lemma 2.4, we can find some $\theta \in]0,1[$ such that

$$\mathrm{far}_C(u_\kappa) = \mathrm{far}_C\big(u_\kappa + \theta'(\mathrm{far}_C(u_\kappa) - u_\kappa)\big) = \mathrm{far}_C\big(u_\kappa + \theta'(\mathrm{far}_C(u) - u_\kappa)\big), \tag{50}$$

for every real $\theta' \leq \theta$. Let us choose some real $\eta \in]0,\theta]$ small enough such that

$$\kappa + \eta(\mathrm{dfar}_C(u) - \kappa) < \mathrm{dfar}_C(u) - t \quad \text{and} \quad u_\kappa + \eta(\mathrm{far}_C(u) - u_\kappa) \in \mathcal{E}_{\rho(\cdot)}(C). \tag{51}$$

Putting $\alpha := \kappa + \eta(\mathrm{dfar}_C(u) - \kappa) > \kappa$, simple calculations show that

$$u_\kappa + \eta(\mathrm{far}_C(u) - u_\kappa) = u_\alpha,$$

which gives by (50) and (51) that $\alpha \in Q_t$. The latter inclusion is a contradiction since $\alpha > \kappa = \sup Q_t$. Then, it is established that

$$\kappa = \mathrm{dfar}_C(u) - t.$$

Combining this and the equality $\mathrm{far}_C(u) = y$, we obtain

$$u_\kappa = u + (\mathrm{dfar}_C(u) - t)\frac{y-u}{\|y-u\|} = y - t\frac{y-u}{\|y-u\|}.$$

According to this and to (47), we have for any $t \in]\rho(y), +\infty[$

$$y = \mathrm{far}_C(u_\kappa) = \mathrm{far}_C(y - t\frac{y-u}{\|y-u\|}),$$

so the implication $(h) \Rightarrow (l)$ is established.

Let us show $(l) \Rightarrow (m)$. Take any $(u,y) \in \mathrm{gph}\,\mathrm{Far}_C$ with $u \in \mathcal{E}_{\rho(\cdot)}(C)$ and any real $t > \rho(y)$. For every real $\theta > 0$ put $u_\theta := u + \theta(u-y)$, so $y = \mathrm{far}_C(u_\theta)$. Fix any real $\theta > 0$. By (l) we have

$$y = \mathrm{far}_C\left(y - t\frac{y-u_\theta}{\|y-u_\theta\|}\right) = \mathrm{far}_C\left(y - t\frac{y-u}{\|y-u\|}\right),$$

hence $\mathrm{dfar}_C\left(y - t\frac{y-u}{\|y-u\|}\right) = t$. Then, the limit as $t \downarrow \rho(y)$ ensures that we have $\mathrm{dfar}_C(y - \rho(y)\frac{y-u}{\|y-u\|}) = \rho(y)$. This means $y \in \mathrm{Far}_C\left(y - \rho(y)\frac{y-u}{\|y-u\|}\right)$, which justifies (m).

Assume in addition that $\inf_{\mathrm{Far}_C(\mathcal{H})} \rho > 0$. Let us establish the implication $(m) \Rightarrow (k)$. Fix any $\overline{u} \in \mathcal{E}_{\rho(\cdot)}(C)$. According to Proposition 3.14, under (m) we have

$$\lim_{\eta \downarrow 0} \mathrm{diam}\,\mathrm{Far}_{C,\eta}(\overline{u}) = 0,$$

and this ensures (see (4)) that $\mathrm{dfar}_C(\overline{u})$ is strongly attained. The proof of the theorem is complete. □

Remark 3.17. It should be noted that there are nonconvex sets C such that $\mathrm{dfar}_C(\cdot)$ is Fréchet differentiable on $\mathcal{E}_{\rho(\cdot)}(C)$ for $\rho(\cdot) \equiv R > 0$ large enough. Indeed, setting $C := \mathbb{S} \cup \{0\} \subset \mathcal{H}$, we observe that $\mathrm{dfar}_C(\cdot) = 1 + \|\cdot\|$ is Fréchet differentiable on $\mathcal{H} \setminus \{0\} \supset \mathcal{E}_R(C) = \mathcal{H} \setminus (R-1)\mathbb{B}$ for every real $R > 1$. □

We end the paper with the next proposition which provides, in addition to Theorem 3.3(c), another description of $\rho(\cdot)$-strongly convex sets as intersection of closed balls.

Proposition 3.18. *Let C be a $\rho(\cdot)$-strongly convex subset of the Hilbert space $\mathcal{H}$ for some continuous function $\rho(\cdot) : \operatorname{bdry} C \to]0,+\infty[$. Assume that C is not reduced to a singleton. Then, one has $\Omega_{C,\rho(\cdot)} := \{(u,y) \in \mathcal{H}^2 : y \in \operatorname{Far}_C(\mathcal{H}), \operatorname{dfar}_C(u) = \rho(y)\} \neq \emptyset$ and*

$$C = \bigcap_{(u,y)\in L} B[u,\rho(y)],$$

for every set L such that $\Omega_{C,\rho(\cdot)} \subset L \subset \{(u,y) \in \mathcal{H}^2 : \operatorname{dfar}_C(u) = \rho(y)\}$.

Proof. Fix any $x \in \mathcal{E}_{\rho(\cdot)}(C)$. According to Theorem 3.16, we know that $y := \operatorname{far}_C(x)$ is well defined along with

$$y \in \operatorname{Far}_C\left(y - \rho(y)\frac{y-x}{\|y-x\|}\right).$$

Setting $u := y - \rho(y)\frac{y-x}{\|y-x\|}$ we then see that

$$(u,y) \in \operatorname{gph}\operatorname{Far}_C \quad \text{and} \quad \operatorname{dfar}_C(u) = \|y-u\| = \rho(y),$$

that is, $(u,y) \in \Omega_{C,\rho(\cdot)} \neq \emptyset$.

Let L be a set such that $\Omega_{C,\rho(\cdot)} \subset L \subset \{(u,y) \in \mathcal{H}^2 : \operatorname{dfar}_C(u) = \rho(y)\}$. We are going to show that $D = C$ with $D := \bigcap_{(u,y)\in L} B[u,\rho(y)]$. Given any $c \in C$, we see that

$$\|c-u\| \le \operatorname{dfar}_C(u) = \rho(y) \quad \text{for all } (u,y) \in L,$$

or equivalently, $c \in \bigcap_{(u,y)\in L} B[u,\rho(y)]$. This translates the inclusion $C \subset D$. Let us show the converse inclusion. By contradiction, suppose that we can find some $z \in D \setminus C$. Set $p := \operatorname{proj}_C(z)$, $d := d_C(z) > 0$ and $v := \frac{z-p}{d} \in N(C;p)$. Fix any real $\kappa > \rho(p)$ and set $q := p - \kappa v$. We obtain by Theorem 3.9 that for every $c \in C$

$$\langle p-q, c-p\rangle = \kappa\,\langle v, c-p\rangle \le -\frac{\kappa}{2\rho(p)}\|c-p\|^2 \le -\frac{1}{2}\|c-p\|^2.$$

By (5) we have the inclusion $p \in \operatorname{Far}_C(q)$. It follows from this

$$\operatorname{dfar}_C(q) = \|p-q\| = \kappa > \rho(p),$$

hence (see Theorem 3.10) $q \in \mathcal{E}_{\rho(\cdot)}(C)$. This inclusion allows us to apply Theorem 3.16 to get

$$p = \operatorname{far}_C(q) \in \operatorname{Far}_C\left(p - \rho(p)\frac{p-q}{\|p-q\|}\right).$$

Thus, the point $w := p - \rho(p)\frac{p-q}{\|p-q\|}$ satisfies

$$(w,p) \in \operatorname{gph}\operatorname{Far}_C \quad \text{and} \quad \operatorname{dfar}_C(w) = \|p-w\| = \rho(p),$$

in particular, $(w,p) \in \Omega_{C,\rho(\cdot)} \subset L$. We deduce from this and the definition of D that

$$z \in D \subset B[w,\rho(p)].$$

On the other hand, we also have

$$z - w = z - p + \rho(p)\frac{\kappa v}{\|\kappa v\|} = z - p + \rho(p)v = \frac{d+\rho(p)}{d}(z-p).$$

Taking the norm $\|\cdot\|$ of the extreme members in what precedes and keeping in mind that $z \in B[w,\rho(p)]$, we then arrive to

$$d + \rho(p) \le \rho(p).$$

This must imply $d = 0$, that is, $z \in C$ since C is closed in $\mathcal{H}$, which contradicts the fact that $z \in D \setminus C$. The proof of the proposition is complete. □

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