On prox-regularity and strong convexity with variable radii in Hilbert spaces

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Dedicated to Professor Lionel Thibault

Abstract. This paper provides some new results for the classes of prox-regular sets and strongly convex sets with variable radii in a general Hilbert space.

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1 Introduction

The class of $\rho(\cdot)$-prox-regular sets in Hilbert spaces has been introduced and studied by A. Canino in 1988 ([4]). It has been thoroughly developed by G. Colombo and L. Thibault in the survey [6]. Roughly speaking, a (nonempty closed) subset S of a Hilbert space $\mathcal{H}$ is prox-regular with thickness/radius $\rho(\cdot)$ whenever there is a function $\rho : \Lambda^P(S) := \{x \in S : N^P(S;x) \neq \{0\}\} \to]0,+\infty]$ such that the nearest point mapping proj_S is well defined and continuous on a suitable open neighborhood (depending on $\rho(\cdot)$) of the set S. Here and below, $N^P(S;\cdot)$ stands for the proximal normal cone to the set S (see Section 2 for more details). Whenever the function $\rho(\cdot)$ is constant, say $\rho \equiv r$ for some extended real $r > 0$, the $\rho(\cdot)$-prox-regularity property is nothing but the (usual) uniform prox-regularity[1] with constant r introduced in [17].

Prox-regularity has a long and deep story which goes back to the pioneer work [8] of H. Federer in 1959 (with the class of "positive reached sets" in $\mathbb{R}^n$) and has been involved since in numerous and various topics of mathematical analysis (see the sections "The sweeping process" and "Others applications" in the survey [6], see also the comments at the end of Chapter 15 in the recent book [26] and the references therein).

In this paper, we provide new properties for the class of $\rho(\cdot)$-prox-regular sets, namely:

(π1) Given a (possibly nonregular) thickness $\rho(\cdot)$ and a point $x \in \mathcal{O}_{\rho(\cdot)}(S) := \{\ell < 1\}$ with

$$\ell(x') := \limsup_{\substack{\|x'-y\| \to d_S(x') \\ y \in \Lambda^P(S)}} \frac{\|x'-y\|}{\rho(y)}$$

we estimate the diameter of the following sets of approximate nearest points (with $\eta > 0$)

$$\mathrm{Proj}^2_{S,\eta}(x) := \{c \in S : \|c-x\|^2 \leq d_S^2(x) + \eta\} \tag{1.1}$$

and

$$\mathrm{Proj}_{S,\eta}(x) := \{c \in S : \|c-x\| \leq d_S(x) + \eta\}. \tag{1.2}$$

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[1]This concept is also known as "proximal smoothness", "weak convexity", "$\mathcal{O}(2)$-convexity", φ-convexity, etc.

More precisely, we show that for some constant κ_x (independent of η) we have

$$\operatorname{diam} \operatorname{Proj}^2_{S,\eta}(x) \le \kappa_x \sqrt{\eta}$$

and

$$\operatorname{diam} \operatorname{Proj}_{S,\eta}(x) \le \kappa_x \sqrt{\eta + 2d_S(x)}\sqrt{\eta}.$$

Let us mention that the above constant κ_x satisfies

$$\kappa_x = \frac{2r}{r - d_S(x)} = \frac{2}{1 - \frac{d_S(x)}{r}} \quad \text{whenever } \rho \equiv r,$$

in particular $\kappa_x = 2$ whenever the set S is ∞-prox-regular (i.e., nonempty closed and convex). To the best of our knowledge, the latter estimates are new in the literature (even for a constant radius $\rho \equiv r$). We deduce from this (see Proposition 2.1 below) that the infimum $d_S(x)$ is strongly attained which obviously implies that the set of nearest points $\operatorname{Proj}_S(x)$ is reduced to a singleton.

We also estimate (in a sharp way) the error between the nearest point $\operatorname{proj}_S(x)$ and an approximate nearest point $p \in \operatorname{Proj}^2_{S,\eta}(x)$, namely

$$\sup_{p \in \operatorname{Proj}^2_{S,\eta}(x)} \|p_S(x) - p\| \le \frac{\kappa_x}{2}\sqrt{\eta}.$$

(π_2) We give an upper estimate in terms of closed balls for the $\rho(\cdot)$-open tube $\mathcal{O}_{\rho(\cdot)}(S)$. In the constant case $\rho \equiv r$, we recover the fact that the complement of a uniformly prox-regular set is nothing but the union of closed balls with constant radius ([18]).

(π_3) We give a characterization of $\rho(\cdot)$-prox-regular sets through the proximal subdifferential $\partial_P d_S(x)$ for appropriate ouside points $x \notin S$.

Besides our investigations on the theory of $\rho(\cdot)$-prox-regularity, we also develop (π_1) for the class of $\rho(\cdot)$-strongly convex sets in Hilbert spaces. Such a class has been recently considered by L. Thibault and the author in [19] and can be seen as an extension of the classical R-strong convexity property. Recall that a (nonempty closed) subset of $\mathcal{H}$ is strongly convex of constant $R > 0$ whenever it is the intersection of a family of closed balls in $\mathcal{H}$ with common radius R. It is known (see, e.g., the survey by V.V. Goncharov and G.E. Ivanov [10] and the references therein) that such a property is equivalent to the fact that any $x \in \mathcal{E}_R(C) := \{\operatorname{dfar}_C > R\}$ has one and only one farthest point $\operatorname{far}_C(x)$ from C along with the continuity of the induced mapping $\mathcal{E}_R(C) \ni u \mapsto \operatorname{far}_C(u)$. Here and below, $\operatorname{Far}_C(x)$ is the set of farthest points of x from the set C, that is

$$\operatorname{Far}_C(x) := \{y \in C : \|y - x\| = \operatorname{dfar}_C(x)\},$$

where $\operatorname{dfar}_C(x) := \sup_{c \in C} \|c - x\|$ is the farthest distance of x from C. It is then quite natural to replace the above sets (1.1) and (1.2) in (π_1) by

$$\operatorname{Far}^2_{C,\eta}(x) := \{c \in C : \|c - x\|^2 \ge \operatorname{dfar}^2_C(x) - \eta\} \tag{1.3}$$

and

$$\operatorname{Far}_{C,\eta}(x) := \{c \in C : \|c - x\| \ge \operatorname{dfar}_C(x) - \eta\}. \tag{1.4}$$

We provide (in the line of our development in (π_1) for prox-regularity) estimates for the diameters of the so-called approximate farthest point sets (1.3) and (1.4). Doing so, we complement (through a different approach) the following estimate due to G.E. Ivanov in [11] (stated and proved in the more general context of uniformly convex Banach spaces and for antisun sets)

$$2\delta_{\mathcal{H}}\left(\frac{\|p_1 - p_2\|}{2R}\right) \min\left(R, \operatorname{dfar}_C(x) - R\right) \le \eta,$$

valid for any $x \in \mathrm{Far}_{C,\eta+R}(x)$ and any $p_1, p_2 \in \mathrm{Far}_{C,\eta}(x)$. Here $\delta_{\mathcal{H}}$ stands for the modulus of uniform convexity of $\mathcal{H}$ defined by

$$\delta_{\mathcal{H}}(\varepsilon) := 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \geq \frac{\varepsilon^2}{8} \quad \text{for all } \varepsilon \in [0,2].$$

We also mention that the set of approximate farthest points (1.4) has been also studied by M.V. Balashov ([1]). In [1], the author shows that for two strongly convex sets $C_1, C_2 \subset \mathcal{H}$ of constant R_i (where haus denotes the Hausdorff distance)

$$\mathrm{haus}\big(\mathrm{Far}_{C_1,\eta_1}(x_1), \mathrm{Far}_{C_2,\eta_2}(x_2)\big) \leq 16 \max_{i=1,2} \frac{R_i}{\varepsilon_i}\big(2\|x_1 - x_2\| + 2\mathrm{haus}(C_1, C_2) + |\eta_1 - \eta_2|\big),$$

for all $\eta_i \in]R_i, \mathrm{dfar}_{C_i}(x_i)[$.

The paper is organized as follows:

In Section 2 we recall some important notions which will be used through the paper. Section 3 is devoted to the development of the above properties (π_1), (π_2) and (π_3) for prox-regular sets with variable radii. The property (π_1) is studied in the context of strongly convex sets in the last section of the paper.

2 Notation and preliminaries

In all the paper, $\mathcal{H}$ is a real Hilbert space *not reduced to zero* endowed with its inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. The open (resp. closed) ball centered at $x \in \mathcal{H}$ with radius $r > 0$ is denoted by $B(x,r)$ (resp. $B[x,r]$). We set $\mathbb{B} := B[0,1]$ (the closed unit ball of $\mathcal{H}$) and $\mathbb{S} := \{u \in \mathcal{H} : \|u\| = 1\}$ (the unit sphere of $\mathcal{H}$). The identity mapping of $\mathcal{H}$ is denoted Id.

2.1 Approximate minimizers/maximizers

Let us start by recalling some basic concepts:

Definition 2.1 *Let Y be a nonempty subset of a metric space X and let $f : X \to \mathbb{R}$ be a function and let $\Lambda := \inf_{y\in Y} f(y)$ and $\Gamma := \sup_{y \in Y} f(y)$.*
(i) Any sequence $(y_n)_n$ of Y such that $f(y_n) \to \Lambda$ (resp. $f(y_n) \to \Gamma$) is called a minimizing sequence *(resp. a* maximizing sequence*) for the infimum Λ (resp. the supremum Γ) of the function f over the set Y.*
(ii) The infimum Λ (resp. supremum Γ) is said to be strictly attained *at $\overline{y} \in Y$ if*

$$f(\overline{y}) = \Lambda \quad \textit{and} \quad f(y') > \Lambda \quad \textit{for all}\, y' \in Y \setminus \{\overline{y}\}$$

(resp.

$$f(\overline{y}) = \Gamma \quad \textit{and} \quad f(y') < \Gamma \quad \textit{for all}\, y' \in Y \setminus \{\overline{y}\}).$$

(iii) The infimum Λ (resp. supremum Γ) is said to be strongly attained *at $\overline{y} \in Y$ if any minimizing (resp. maximizing) sequence for the infimum Λ (resp. supremum Γ) converges to the point $\overline{y}$.*
*(iv) For each real $\eta > 0$, an element $\overline{y} \in Y$ is called an η-*approximate minimizer *(resp.* approximate maximizer*) of the function f over Y if*

$$f(\overline{y}) \leq \Lambda + \eta \quad (\text{resp.}\, f(\overline{y}) \geq \Gamma - \eta).$$

The set of η-approximate minimizers (resp. approximate maximizers) is denoted $\underline{M}_{Y,\eta}(f)$ (resp. $\overline{M}_{Y,\eta}(f)$).

Clearly, the infimum Λ (resp. the supremum Γ) is strictly attained at $\overline{y} \in Y$ whenever it is strongly attained at $\overline{y}$.

We have the following crucial (and classical) result on the diameter of the sets $\underline{M}_{Y,\eta}(f)$ and $\overline{M}_{Y,\eta}(f)$.

Proposition 2.1 *Let Y be a nonempty closed subset of a complete metric space (X, d) and let $f : X \to \mathbb{R}$ be a function. The following hold:*
(a) Assume that f is bounded from below and lower semicontinuous. Then, the infimum $\inf_{y\in Y} f(y)$ is strongly attained at some $\overline{y} \in Y$ if and only if

$$\lim_{\eta\downarrow 0} \underline{M}_{Y,\eta}(f) = 0.$$

(b) Assume that f is bounded from above and upper semicontinuous. Then, the supremum $\sup_{y\in Y} f(y)$ is strongly attained at some $\overline{y} \in Y$ if and only if

$$\lim_{\eta\downarrow 0} \overline{M}_{Y,\eta}(f) = 0.$$

2.2 Distance function

The distance function d_S and the farthest distance function dfar_S from a nonempty subset $S \subset \mathcal{H}$ are defined for every $x \in \mathcal{H}$ by

$$d_S(x) := \inf_{y\in S} \|x - y\| \quad \text{and} \quad \mathrm{dfar}_S(x) := \sup_{y\in S} \|x - y\|.$$

To those functions are naturally associated the multimappings $\mathrm{Proj}_S : \mathcal{H} \rightrightarrows \mathcal{H}$ of nearest points in S and $\mathrm{Far}_S : \mathcal{H} \rightrightarrows \mathcal{H}$ of farthest points in S defined for every $x \in \mathcal{H}$ by

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}$$

and

$$\mathrm{Far}_S(x) := \{y \in S : \mathrm{dfar}_S(x) = \|x - y\|\}.$$

Whenever $\mathrm{Proj}_S(\overline{x})$ (resp. $\mathrm{Far}_S(\overline{x})$) is reduced to a singleton for some $\overline{x} \in \mathcal{H}$, that is, $\mathrm{Proj}_S(\overline{x}) = \{\overline{y}\}$ (resp. $\mathrm{Far}_S(\overline{x}) = \{\overline{y}\}$) the vector $\overline{y} \in S$ will be denoted by $\mathrm{proj}_S(\overline{x})$ (resp. $\mathrm{far}_S(\overline{x})$) or $p_S(\overline{x})$.

It is an exercise to check the following equivalences valid for every $x, y \in \mathcal{H}$

$$y \in \mathrm{Proj}_S(x) \Leftrightarrow y \in S \quad \text{and} \quad \langle x - y, c - y\rangle \le \frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in S \tag{2.1}$$

and

$$y \in \mathrm{Far}_S(x) \Leftrightarrow y \in S \quad \text{and} \quad \langle y - x, c - y\rangle \le -\frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in S.$$

The next result provides fundamental characterizations for the regularity of the (usual) distance function. We refer the reader to [26, Proposition 15.19] for a proof.

Theorem 2.1 *Let C be a nonempty closed subset of $\mathcal{H}$ and let U be a nonempty open subset of $\mathcal{H}$. The following assertions are equivalent:*
(a) The function d_C^2 is C^1 on U;
(b) the function d_C^2 is Fréchet differentiable on U;
(c) the function d_C^2 is Gâteaux differentiable on U and $\|D_G d_C(x)\| = 1$ for every $x \in U$;
(d) the function d_C^2 is Gâteaux differentiable on U and $\mathrm{Proj}_C(x) \neq \emptyset$ for every $x \in U$;
(e) the mapping $\mathrm{proj}_C : U \to C$ is well defined on U and continuous therein;
(f) the mapping $\mathrm{proj}_C : U \to C$ is well defined on U norm-to-weak sequentially continuous therein;
(g) the infimum $d_C(x)$ is strongly attained for all $x \in U$.

Under anyone of the latter assertions, one has

$$\nabla d_C^2(x) = x - \mathrm{proj}_C(x) \quad \textit{for all}\, x \in U.$$

We have a similar result for the farthest distance function (see, e.g., [9]). Note that $\mathrm{dfar}_C(x) > 0$ whenever the set C is nonempty and not reduced to a singleton.

Theorem 2.2 *Let C be a nonempty closed bounded subset of $\mathcal{H}$ not reduced to a singleton and let U be a nonempty open subset of $\mathcal{H}$. The following assertions are equivalent:*
(a) The function dfar_C is C^1 on U;
(b) the function dfar_C is Fréchet differentiable on U;
(c) the function dfar_C is Gâteaux differentiable on U and $\|D_G\mathrm{dfar}_C(x)\| = 1$ for every $x \in U$;
(d) the function dfar_C is Gâteaux differentiable on U and $\mathrm{Far}_C(x) \neq \emptyset$ for every $x \in U$;
(e) the mapping $\mathrm{far}_C : U \to C$ is well defined on U and continuous therein;
(f) the mapping $\mathrm{far}_C : U \to C$ is well defined on U norm-to-weak sequentially continuous therein;
(g) the supremum $\mathrm{dfar}_C(x)$ is strongly attained for all $x \in U$.

Under anyone of the latter assertions, one has

$$\nabla \mathrm{dfar}_C(x) = \frac{x - \mathrm{far}_C(x)}{\mathrm{dfar}_C(x)} \quad \text{for all } x \in U.$$

The theory of $\rho(\cdot)$-prox-regular set makes a great use of the famous Lau theorem [14]. Recall that any Hilbert space is reflexive, sequential Kadec-Klee and strictly convex.

Theorem 2.3 (Lau's theorem of genericity of points with nearest points) *Let X be a reflexive Banach space endowed with a norm $\mathcal{N}$ satisfying the sequential Kadec-Klee property and let C be a nonempty closed subset of X. Then, the set of points of $X \setminus C$ admitting nearest points in C contains a dense G_δ set of $X \setminus C$.*

If in addition the sequential Kadec-Klee norm $\mathcal{N}$ is strictly convex, then there exists a dense G_δ set of points in $X \setminus C$ with unique nearest points in C.

The following result is the farthest points counterpart of Theorem 2.3. It is also due to K.S. Lau [15].

Theorem 2.4 (Lau's theorem of genericity of points with farthest points) *Let C be a nonempty weakly compact set in a Banach space $(X, \mathcal{N})$. Then, the set of points in X with farthest points in C contains a dense G_δ set in X.*

If in addition the norm $\mathcal{N}$ is strictly convex, then there exists a dense G_δ set of points in X with unique farthest points in C.

2.3 Proximal normal cone and its associated subdifferential

We give some preliminaries about normal cones and subdifferentials which will be deeply involved in the rest of the paper. For more details, we refer the reader to the books [5, 16, 24, 25].

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal normal* to a closed set $S \subset \mathcal{H}$ at $x \in S$ whenever there exists a real $r > 0$ such that $x \in \mathrm{Proj}_S(x + r\zeta)$. The set $N^P(S; x)$ (which is a convex cone containing 0 but not necessarily closed) of all proximal normal vectors to S at $x \in S$ is called the *proximal normal cone* of S at x. By convention, if $x \in \mathcal{H} \setminus S$, we put $N^P(S; x) := \emptyset$. Recall that if S is convex, then $N^P(S; x)$ is nothing but the *Moreau-Rockafellar normal* cone, that is,

$$N^P(S; x) = \{v \in \mathcal{H} : \langle v, x' - x\rangle \leq 0 \,\forall x' \in S\}.$$

We point out that we may have $N^P(S; x) = \{0\}$ for some closed convex cone $S \subset \ell^2(\mathbb{N})$ and $x \in \mathrm{bdry}\, S$ (see, e.g., [7, Example 7.7]). Coming back to a general (possibly nonconvex) set S, it should be noted that

$$u - p \in N^P(S; p) \quad \text{for all } (u, p) \in \mathrm{gph}\, \mathrm{Proj}_S \tag{2.2}$$

We can also easily check that

$$q - u \in N^P(S; q) \quad \text{for all } (u, q) \in \mathrm{gph}\, \mathrm{Far}_S. \tag{2.3}$$

Here and below, the graph $\operatorname{gph} M$ (resp. the domain $\operatorname{Dom} M$) of a multimapping $M : \mathcal{H} \rightrightarrows \mathcal{H}$ is defined as

$$\operatorname{gph} M := \{(x, y) \in \mathcal{H}^2 : y \in M(x)\}$$

(resp.

$$\operatorname{Dom} M := \{x \in \mathcal{H} : M(x) \neq \emptyset\}).$$

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal subgradient* of a function $f : U \to \mathbb{R}$ defined on a nonempty open subset U of $\mathcal{H}$ at $\overline{x} \in U$ provided that there are a real $\sigma \geq 0$ and a real $\eta > 0$ such that

$$\langle \zeta, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + \sigma \|y - \overline{x}\|^2 \quad \text{for all } y \in B(\overline{x}, \eta).$$

The set $\partial f(\overline{x})$ of all such proximal subgradients is called the *proximal subdifferential* of f at $\overline{x}$. Given any set $S \subset \mathcal{H}$, we recall that for any $x \in \mathcal{H}$ with $\partial_P d_S(x) \neq \emptyset$, the set $\operatorname{Proj}_S(x)$ is reduced to a singleton (i.e., $\operatorname{proj}_S(x)$ is well defined) and

$$d_S(x)\partial_P d_S(x) = \{x - \operatorname{proj}_S(x)\}. \tag{2.4}$$

At last but not least, we have the crucial equality

$$\partial_P d_S(x) = N^P(S; x) \cap \mathbb{B} \quad \text{for all } x \in S. \tag{2.5}$$

3 Prox-regularity with variable thickness

This section is devoted to the class of $\rho(\cdot)$-prox-regular sets. For more details (proofs, historical comments and further results) we refer the reader to the survey [6] and the book [26]. As usual, we set $1/r := 0$ whenever $r = +\infty$.

3.1 Definition, first properties and remarks

Given a nonempty closed set $S \subset \mathcal{H}$, we set

$$\Lambda^P(S) := \{x \in S : N^P(S; x) \neq \{0\}\} \subset \operatorname{bdry} S.$$

Assume that $S \neq \mathcal{H}$. Let $u \in \mathcal{H} \setminus S =: U$. According to Theorem 2.3 we know that there is a sequence $(u_n)_{n \in \mathbb{N}}$ of the open set U such that $p_n := \operatorname{proj}_S(u_n)$ is well defined (that is, $\operatorname{Proj}_S(u_n)$ is a singleton) for every $n \in \mathbb{N}$. Noticing (see (2.2)) that for each $n \in \mathbb{N}$, $0 \neq u_n - p_n \in N^P(S; p_n)$, we obtain

$$\Lambda^P(S) \neq \emptyset.$$

We are then able to define for a given function $\rho : \Lambda^P(S) \to]0, +\infty]$ the function $\overline{\ell} : U \to [0, +\infty]$ given by

$$\overline{\ell}(x) := \limsup_{\substack{\|x-y\| \to d_S(x) \\ y \in \Lambda^P(S)}} \frac{\|x - y\|}{\rho(y)}. \tag{3.1}$$

It follows from the equality

$$\overline{\ell}(x) = d_S(x) \Big(\liminf_{\substack{\|x - y\| \to d_S(x) \\ y \in \Lambda^P(S)}} \rho(y) \Big)^{-1}$$

that the set

$$\mathcal{O}_{\rho(\cdot)}(S) := \{x \in U : \ell(x) < 1\} \tag{3.2}$$

is open in $\mathcal{H}$. If $\rho \equiv r$, it is evident that $\mathcal{O}_{\rho(\cdot)}(S)$ is nothing but the r-*open tube around* S, that is, $\mathcal{O}_{\rho(\cdot)}(S) = \{0 < d_S < r\}$. It will be convenient to set $\Lambda^P(S) := S$ and $\mathcal{O}_{\rho(\cdot)}(S) := \emptyset$ whenever $S = \mathcal{H}$.

Definition 3.1 *Let S be a nonempty closed subset of $\mathcal{H}$ and let $\rho : \Lambda^P(S) \to]0, +\infty]$ be a function. The set S is* $\rho(\cdot)$-prox-regular *provided that, for any $x \in \Lambda^P(S)$ any $v \in N^P(S;x) \cap \mathbb{S}$ and any $t \in \mathbb{R} \cap]0, \rho(x)]$*

$$x \in \operatorname{Proj}_S\big(x + tv\big). \tag{3.3}$$

In such a case, one says that the function $\rho(\cdot)$ is a prox-regularity radius function *of the set S.*

Note that the radius of prox-regularity $\rho(\cdot)$ in the latter definition is required to be defined on $\Lambda^P(S) \subset \operatorname{bdry} S$. This slightly differs from others developments on that topic ([4, 6, 26]) where $\rho(\cdot)$ is defined on the whole considered set S. Of course if the function $\rho(\cdot)$ is real-valued, the inclusion (3.3) is equivalent to

$$x \in \operatorname{Proj}_S\big(x + \rho(x)v\big)$$

which is also (obviously) equivalent to

$$C \cap B\big(x + \rho(x)v, \rho(x)\big) = \emptyset. \tag{3.4}$$

Whenever the latter equality (3.4) is satisfied one says that the unit proximal normal vector v at x is *realized by a t-ball* with $t = \rho(x)$.

Examples of $\rho(\cdot)$-prox-regular sets (with lower semicontinuous/continuous radius) can be found in [26, Chapter 15].

The following easy result (see, e.g., [26, Theorem 15.6]) provides a fundamental characterization of $\rho(\cdot)$-prox-regular sets.

Proposition 3.1 *Let S be a nonempty closed subset of a Hilbert space $\mathcal{H}$ and let $\rho(\cdot) : \Lambda^P(S) \to]0, +\infty]$ be a function. Then, the set S is $\rho(\cdot)$-prox-regular if and only if*

$$\langle v, x' - x\rangle \le \frac{1}{2\rho(x)}\|x' - x\|^2, \tag{3.5}$$

for any $x \in \Lambda^P(S)$, any $v \in N^P(S;x) \cap \mathbb{S}$ and any $x' \in S$.

Remark 3.1 Keep the notation of the latter proposition. Let $\Lambda^P(S) \subset S' \subset S$ and let $\widehat{\rho} : S' \to]0, +\infty]$ be a function such that $\widehat{\rho}_{|\Lambda^P(S)} = \rho$. The inequality (3.5) is obviously equivalent to

$$\langle v, x' - x\rangle \le \frac{1}{2\widehat{\rho}(x)}\|x' - x\|^2,$$

for all $x, x' \in S$ and all $v \in N^P(S;x) \cap \mathbb{B}$. It is then natural (and convenient) to say that S is $\widehat{\rho}(\cdot)$-prox-regular. ■

Let S be a $\rho(\cdot)$-prox-regular set of $\mathcal{H}$ for some function $\rho(\cdot) : \Lambda^P(S) \to]0, +\infty]$. Through Propositon 3.1 we easily see that the multimapping $N^P(S;\cdot)$ is closed-valued. If the function $\rho(\cdot)$ can be extended to a lower semicontinuous function $\widehat{\rho}(\cdot) : \operatorname{bdry} S \to]0, +\infty]$, we also show in a straightforward way (via Propositon 3.1 below, see also [26, Proposition 15.13]) that the set $\operatorname{gph} N^P(S;\cdot)$ is strongly $\times$ weakly sequentially closed. In such a case, we are able to derive that the set S enjoys a normal/tangential regularity property

$$N^P(S;x) = N^C(S;x) \quad \text{and} \quad T^B(S;x) = T^C(S;x) \quad \text{for all } x \in S.$$

Here T^C (resp. T^B) stands for the Clarke (resp. Boulidang) tangent cone and N^C denotes the Clarke normal cone. The example below shows that such facts may fail for a $\rho(\cdot)$-prox-regular set S with a continuous radius relative to $\Lambda^P(S)$.

Example 1 It is easy to check that the set $S := \{(x,r) \in \mathbb{R}^2 : r \leq |x|\}$ is $\rho(\cdot)$-prox-regular with $\rho : \Lambda^P(S) = \operatorname{bdry} S \setminus \{(0,0)\} \to]0,+\infty[$ with

$$\rho(x,|x|) := \sqrt{2}|x| \quad \text{for all } x \neq 0.$$

Observe that the sets $\Lambda^P(S)$ and $\operatorname{gph} N^P(S;\cdot)$ fail to be closed and

$$T^B(S;(0,0)) \neq T^C(S;(0,0)) \quad \text{and} \quad N^P(S;(0,0)) \neq N^C(S;(0,0)).$$

Note also that the set S is not prox-regular at the point $(0,0)$ in the sense of [26, Definition 15.65]. ■

The above set S satisfies the equality $N^F(S;(0,0)) = N^P(S;(0,0))$ where N^F is the Fréchet normal cone. We point out[2] that the epigraph $S' := \operatorname{epi} f$ of the function $f : \mathbb{R} \to \mathbb{R}$ defined for every $x \in \mathbb{R}$ by

$$f(x) = -|x|^{\frac{3}{2}} \quad \text{if } x \geq 0 \quad \text{and} \quad f(x) = |x|^{\frac{3}{2}} \quad \text{otherwise}$$

is $\rho(\cdot)$-prox-regular for some radius $\rho(\cdot) : \Lambda^P(S) \to]0,+\infty[$ while

$$N^P(S';(0,0)) \neq N^F(S';(0,0)).$$

3.2 Strong attainment of the distance associated to a prox-regular set

Our main aim in that subsection is to show that the infimum $d_S(x)$ is strongly attained for any $x \in \mathcal{O}_{\rho(\cdot)}(S)$ whenever the set S is $\rho(\cdot)$-prox-regular for some radius $\rho : \Lambda^P(S) \to]0,+\infty]$. Recall that the existence and uniqueness of a nearest point $\operatorname{proj}_S(x)$ with $x \in \mathcal{O}_{\rho(\cdot)}(S)$ (which obviously holds whenever $d_S(x)$ is strongly attained) has been previously obtained in [6, 26] under the continuity (on the whole set S) of the radius. Having in mind Proposition 2.1, we are going to estimate (without any regularity assumption on the function $\rho(\cdot)$) the diameter of the sets

$$\operatorname{Proj}^2_{S,\eta}(x) := \{c \in S : \|c-x\|^2 \leq d_S^2(x) + \eta\} = S \cap B[x, \sqrt{d_S^2(x)+\eta}]$$

and

$$\operatorname{Proj}_{S,\eta}(x) := \{c \in S : \|c-x\| \leq d_S(x) + \eta\} = S \cap B[x, d_S(x)+\eta].$$

We have the following nice result:

Proposition 3.2 *Let S be an r-prox-regular set of $\mathcal{H}$ for some real $r > 0$. Then, for any $x \in \mathcal{H}$, the set $\operatorname{Proj}^2_{S,\eta}(x)$ (resp. $\operatorname{Proj}_{S,\eta}(x)$) is $\min(r, \sqrt{d_S^2(x)+\eta})$-prox-regular (resp. $\min(r, d_S(x)+\eta)$-prox-regular).*

Proof. It is a direct consequence of the fact that the intersection of an s-prox-regular set with a closed ball of radius s is s-prox-regular (see, e.g., [26, Corollary 16.17]). ■

We start our study with the following easy lemma:

Lemma 3.1 *Let S be a $\rho(\cdot)$-prox-regular set for some function $\rho : \Lambda^P(S) \to]0,+\infty]$. Then, for all $(x,p) \in \operatorname{gph} \operatorname{Proj}_S$ with $x \notin S$ and $\frac{d_S(x)}{\rho(p)} < 1$ and for all $x' \in S$, one has*

$$\|p-x'\|^2 \leq \frac{1}{1-\frac{d_S(x)}{\rho(p)}}\left(\|x'-x\|^2 - d_S^2(x)\right),$$

in particular

$$\|p-x'\| \leq \frac{1}{\sqrt{1-\frac{d_S(x)}{\rho(p)}}}\|x'-x\|.$$

[2]We thank an anonymous referee and Lionel Thibault for this remark.

Proof. Fix any $(x,p) \in \operatorname{gph} \operatorname{Proj}_S$ with $x \notin S$, $\frac{d_S(x)}{\rho(p)} < 1$ and any $x' \in S$. We may assume that $x \notin S$ (so $p \in \Lambda^P(S)$). Through elementary computations, we see that

$$\|x'-p\|^2 - \|x'-x\|^2 = 2\langle x'-p, x-p\rangle - \|x-p\|^2. \tag{3.6}$$

Combining the $\rho(\cdot)$-prox-regularity of the set S (see Proposition 3.1) and the inclusion $x-p \in N^P(S;p)$ (see (2.2)) yields

$$2\langle x-p, x'-p\rangle \le \frac{\|x-p\|}{\rho(p)}\|x'-p\|^2 = \frac{d_S(x)}{\rho(p)}\|x'-p\|^2. \tag{3.7}$$

It then suffices to put together (3.6) and (3.7) to complete the proof. ■

The following lemma is crucial for our approach.

Lemma 3.2 *Let S be a $\rho(\cdot)$-prox-regular set of $\mathcal{H}$ for some function $\rho : \Lambda^P(S) \to]0,+\infty]$. Let $x \in \mathcal{O}_{\rho(\cdot)}(S)$ and let $c_1,\dots,c_m \in S$. Then, for each real $\varepsilon > 0$, there exists $p \in \operatorname{Proj}_{S,\varepsilon}(x) \cap \operatorname{Proj}^2_{S,\varepsilon}(x)$ such that for all $i \in \{1,\dots,m\}$, one has*

$$\|c_i-p\|^2 \le \big(\frac{1}{1-\overline{\ell}(x)}+\varepsilon\big)\big(\|c_i-x\|^2 - d_S^2(x)+\varepsilon\big)$$

with $\overline{\ell}(x)$ as in (3.1).

Proof. We have $S \ne \mathcal{H}$ (since $\mathcal{O}_{\rho(\cdot)}(S) \ne \emptyset$). Let $\varepsilon > 0$ be a real. According to Lau theorem relative to nearest points (see Theorem 2.3) we can find a sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{H}\setminus S$ such that $u_n \to x$ and $p_n := \operatorname{proj}_S(u_n)$ is well defined. Since $\frac{\|x-p_n\|}{\|u_n-p_n\|} \to \frac{d_S(x)}{d_S(x)} = 1$ (keeping in mind that $x \notin S$) and $p_n \in \Lambda^P(S)$ for all $n \in \mathbb{N}$ we have

$$\limsup_{n\to\infty}\frac{\|u_n-p_n\|}{\rho(p_n)} = \limsup_{n\to\infty}\frac{\|x-p_n\|}{\rho(p_n)} \le \overline{\ell}(x) < 1.$$

Pick any $\varepsilon' > 0$ such that

$$\overline{\ell}(x)+\varepsilon' < 1 \quad \text{and} \quad \frac{1}{1-\overline{\ell}(x)-\varepsilon'} \le \frac{1}{1-\overline{\ell}(x)}+\varepsilon.$$

Choose some $N \in \mathbb{N}$ satisfying for all $n \ge N$

$$\frac{d_S(u_n)}{\rho(p_n)} = \frac{\|u_n-p_n\|}{\rho(p_n)} < \overline{\ell}(x)+\varepsilon' < 1$$

along with

$$4\|u_n-x\|\big(d_S(x)+\|u_n-x\|\big) \le \varepsilon \quad \text{and} \quad \|u_n-x\| \le \frac{\varepsilon}{2}. \tag{3.8}$$

For each $i \in \{1,\dots,m\}$, there is $N_i \in \mathbb{N}$ such that

$$\|c_i-u_n\|^2 - d_S^2(u_n) \le \|c_i-x\|^2 - d_S^2(x)+\varepsilon \quad \text{for all}\, n \ge N_i.$$

Set $n_0 := \max\big(N, \max_{i=1,\dots,N} N_i\big)$, $p := p_{n_0}$ and $u := u_{n_0}$. Putting together the 1-Lipschitz property of $d_S(\cdot)$ and second inequality (3.8), we have

$$\|p-x\| \le d_S(u)+\|u-x\| \le d_S(x)+2\|u-x\| \le d_S(x)+\varepsilon.$$

This and the first inequality in (3.8) give

$$\|p-x\|^2 \le d_S^2(x)+4\|u-x\|\big(d_S(x)+\|u-x\|\big) \le d_S^2(x)+\varepsilon.$$

Thus, we have the inclusion $p \in \operatorname{Proj}_{S,\varepsilon}(x) \cap \operatorname{Proj}^2_{S,\varepsilon}(x)$. On the other hand, Lemma 3.1 gives for each $i \in \{1,\dots,m\}$

$$\|c_i-p\|^2 \le \frac{1}{1-\frac{d_S(u)}{\rho(p)}}\big(\|c_i-u\|^2 - d_S^2(u)\big)$$

from which we derive

$$\begin{aligned}\|c_i - p\|^2 &\le \frac{1}{1-\overline{\ell}(x)-\varepsilon'}\big(\|c_i - x\|^2 - d_S^2(x) + \varepsilon\big)\\ &\le \big(\frac{1}{1-\overline{\ell}(x)} + \varepsilon\big)\big(\|c_i - x\|^2 - d_S^2(x) + \varepsilon\big).\end{aligned}$$

This finishes the proof. ■

We are now in a position to establish the strong attainment of the infimum $d_S(x)$ through Proposition 2.1.

Theorem 3.1 *Let S be a $\rho(\cdot)$-prox-regular set of $\mathcal{H}$ for some function $\rho : \Lambda^P(S) \to]0, +\infty]$ and let $x \in \mathcal{O}_{\rho(\cdot)}(S)$. Let $\kappa_x := 2\sqrt{\frac{1}{1-\overline{\ell}(x)}}$ with $\overline{\ell}(x)$ as in (3.1). Then, for each real $\eta > 0$ one has*

$$\operatorname{diam} \operatorname{Proj}^2_{S,\eta}(x) \le \kappa_x \sqrt{\eta} \tag{3.9}$$

and

$$\operatorname{diam} \operatorname{Proj}_{S,\eta}(x) \le \kappa_x \sqrt{\eta\big(\eta + 2d_S(x)\big)}. \tag{3.10}$$

In particular, the infimum $d_S(x)$ is strongly attained.

Proof. Fix any real $\eta > 0$. Let $p_1, p_2 \in \operatorname{Proj}^2_{S,\eta}(x)$ (resp. $p_1, p_2 \in \operatorname{Proj}_{S,\eta}(x)$). Let also $\varepsilon > 0$. According to Lemma 3.2, we can find some $p \in \operatorname{Proj}_{S,\varepsilon}(x) \cap \operatorname{Proj}^2_{S,\varepsilon}(x)$ such that for each $i \in \{1, 2\}$

$$\|p_i - p\|^2 \le \big(\frac{1}{1-\overline{\ell}(x)} + \varepsilon\big)\big(\|p_i - x\|^2 - d_S^2(x) + \varepsilon\big) \le \big(\frac{1}{1-\overline{\ell}(x)} + \varepsilon\big)(\eta + \varepsilon)$$

(resp.

$$\|p_i - p\|^2 \le \big(\frac{1}{1-\overline{\ell}(x)} + \varepsilon\big)\big(\|p_i - x\|^2 - d_S^2(x) + \varepsilon\big) \le \big(\frac{1}{1-\overline{\ell}(x)} + \varepsilon\big)\big(2\eta d_S(x) + \eta^2 + \varepsilon\big)\big).$$

Hence, for each $i \in \{1, 2\}$

$$\|p_i - p\| \le \sqrt{\eta + \varepsilon}\sqrt{\frac{1}{1-\overline{\ell}(x)} + \varepsilon}$$

(resp.

$$\|p_i - p\| \le \sqrt{\eta(2d_S(x) + \eta) + \varepsilon}\sqrt{\frac{1}{1-\overline{\ell}(x)} + \varepsilon}).$$

It remains to apply the triangle inequality and to let $\varepsilon \downarrow 0$ to get the inequality (3.9) (resp. (3.10)). The proof is complete. ■

Remark 3.2 The $\rho(\cdot)$-prox-regularity of the set S then entails every assertion of Theorem 2.1 with the open set $U = \mathcal{O}_{\rho(\cdot)}(S)$. For the converse implications (under the continuity of $\rho(\cdot)$) we refer the reader to [6, 26]. ■

The crucial case of a constant radius $\rho(\cdot)$ deserves to be stated. To the best of our knowledge the estimates below are new.

Corollary 3.1 *Let S be an r-prox-regular set for some $r \in]0, +\infty[$, $x \in \{d_S < r\}$. Let $\kappa_x := 2\sqrt{\frac{r}{r-d_S(x)}}$. Then, for each real $\eta > 0$ one has*

$$\operatorname{diam} \operatorname{Proj}^2_{S,\eta}(x) \le \kappa_x \sqrt{\eta}$$

and

$$\operatorname{diam} \operatorname{Proj}_{S,\eta}(x) \le \kappa_x \sqrt{\eta\big(\eta + 2d_S(x)\big)}.$$

In particular, if S is ∞-prox-regular (i.e., nonempty closed and convex) one has for each $x \in \mathcal{H}$

$$\operatorname{diam} \operatorname{Proj}^2_{S,\eta}(x) \leq 2\sqrt{\eta}$$

and

$$\operatorname{diam} \operatorname{Proj}_{S,\eta}(x) \leq 2\sqrt{\eta\big(\eta + 2d_S(x)\big)}.$$

We now estimate the error $\|p_S(x) - p\|$ between the exact nearest point $p_S(x)$ and an approximate nearest point $p \in \operatorname{Proj}^2_{S,\eta}(x)$.

Corollary 3.2 *Let S be a $\rho(\cdot)$-prox-regular set of $\mathcal{H}$ for some function $\rho : \Lambda^P(S) \to]0, +\infty]$, $x \in \mathcal{O}_{\rho(\cdot)}(S)$ and let $\bar{\ell}(x)$ as in (3.1). Then, $p_S(x)$ is well defined along with*

$$\|p_S(x) - c\| \leq \sqrt{\frac{1}{1-\bar{\ell}(x)}\big(\|x-c\|^2 - d_S^2(x)\big)} \quad \textit{for all}\, c \in S.$$

In particular, for each real $\eta > 0$, one has

$$\sup_{p \in \operatorname{Proj}^2_{S,\eta}(x)} \|p_S(x) - p\| \leq \sqrt{\frac{\eta}{1-\bar{\ell}(x)}}. \tag{3.11}$$

Proof. Let $c \in S$. Pick any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals with $\varepsilon_n \downarrow 0$. By virtue of Lemma 3.2, we can find $p_n \in \operatorname{Proj}^2_{S,\varepsilon_n}(x)$ such that

$$\|p_n - c\|^2 \leq \big(\frac{1}{1-\bar{\ell}(x)} + \varepsilon_n\big)\big(\|x-c\|^2 - d_S^2(x) + \varepsilon_n\big) \quad \text{for all}\, n \in \mathbb{N}. \tag{3.12}$$

Noticing that $(p_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the infimum $d_S^2(x)$, we know by Theorem 3.1 that $p_S(x)$ is well defined and $p_n \to p$. Letting $n \to \infty$, in (3.12) then yields

$$\|p_S(x) - c\|^2 \leq \frac{1}{1-\bar{\ell}(x)}\big(\|x-c\|^2 - d_S^2(x)\big).$$

The proof is complete. ■

Remark 3.3 Of course if $S \subset \mathcal{H}$ is r-prox-regular for some positive real $r > 0$, the inequality (3.11) can be rewritten as

$$\sup_{p \in \operatorname{Proj}^2_{S,\eta}(x)} \|p_S(x) - p\| \leq \sqrt{\frac{\eta r}{r - d_S(x)}}.$$

We are going to show that the latter estimate is sharp. Let $r, \eta > 0$ be two positive reals. Consider the set $S := \mathbb{R}^2 \setminus r\mathbb{U}_{\mathbb{R}^2}$ where $\mathbb{U}_{\mathbb{R}^2}$ denotes the open unit ball of $\mathbb{R}^2$ endowed with its Euclidean norm $\|\cdot\|_{\mathbb{R}^2}$. Fix any $\varepsilon \in]0, r[$ and set $x := (\varepsilon, 0)$. Note that $\operatorname{proj}_S(x) = (r, 0)$ and $d_S(x) = r - \varepsilon$. Recalling that

$$\operatorname{Proj}^2_{S,\eta}(x) = S \cap B[x, \sqrt{(r-\varepsilon)^2 + \eta}]$$

we easily see that

$$\operatorname{Proj}^2_{S,\eta}(x) \cap \{\|\cdot\|_{\mathbb{R}^2} = r\} = \{(u, \pm v)\}$$

with $u := r - \frac{\eta}{2\varepsilon}$ and $v = \sqrt{r^2 - (r - \frac{\eta}{2\varepsilon})^2}$. It remains to see that

$$\|(u,v) - \operatorname{proj}_S(x)\|^2_{\mathbb{R}^2} = \frac{r\eta}{\varepsilon} = \frac{r\eta}{r - d_S(x)}$$

to get with $p := (u, v) \in \operatorname{Proj}^2_{S,\eta}(x)$ that

$$\|p - \operatorname{proj}_S(x)\|_{\mathbb{R}^2} = \sqrt{\frac{\eta r}{r - d_S(x)}}.$$

■

We pass now to the study of the Lipschitz behavior of $x \mapsto \operatorname{proj}_S(x)$ for a prox-regular set S.

Proposition 3.3 *Let S be a $\rho(\cdot)$-prox-regular set for some function $\rho : \Lambda^P(S) \to]0, +\infty]$. The following hold:*
(a) The mapping proj_S is well defined on $\mathcal{O}_{\rho(\cdot)}(S)$ and for all $x_1, x_2 \in \mathcal{O}_{\rho(\cdot)}(S)$, one has

$$\|\operatorname{proj}_S(x_1) - \operatorname{proj}_S(x_2)\| \le \big(1 - \frac{d_S(x_1)}{2\rho(\operatorname{proj}_S(x_1))} - \frac{d_S(x_2)}{2\rho(\operatorname{proj}_S(x_2))}\big)^{-1}\|x_1 - x_2\|.$$

(b) For any $\gamma \in]0, 1[$ the mapping proj_S is well defined on the set $\mathcal{O}_{\gamma\rho(\cdot)}(S)$ and $(1-\gamma)^{-1}$-Lipschitz continuous therein.
(c) For any $\gamma \in]0, 1[$ and any real $\eta > 0$ one has

$$\|p_1 - p_2\| \le 2\sqrt{\frac{\eta}{1-\gamma}} + (1-\gamma)^{-1}\|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathcal{O}_{\gamma\rho(\cdot)}(S)$, all $p_1 \in \operatorname{Proj}^2_{S,\eta}(x_1)$ and all $p_2 \in \operatorname{Proj}^2_{S,\eta}(x_2)$.

Proof. (a) According to Theorem 3.1 we know that proj_S is well defined on the (open) set $U := \mathcal{O}_{\rho(\cdot)}(S)$. Let $x_1, x_2 \in U$. For each $i \in \{1, 2\}$, setting $p_i := \operatorname{proj}_S(x_i)$ and $\gamma_i := \frac{d_S(x_i)}{2\rho(p_i)} < \frac{1}{2}$ and using the $\rho(\cdot)$-prox-regularity of the set S we easily get

$$\langle x_i - p_i, c - p_i\rangle \le \gamma_i\|c - p_i\|^2 \quad \text{for all } c \in S.$$

We then have

$$\langle x_1 - p_1, p_2 - p_1\rangle \le \gamma_1\|p_2 - p_1\|^2$$

and

$$\langle x_2 - p_2, p_1 - p_2\rangle \le \gamma_2\|p_2 - p_1\|^2.$$

Adding the two above inequalities yields

$$(1 - \gamma_1 - \gamma_2)\|p_1 - p_2\|^2 \le \langle x_2 - x_1, p_2 - p_1\rangle\,.$$

It remains to invoke Cauchy-Schwarz inequality to conclude that

$$\|p_1 - p_2\| \le (1 - \gamma_1 - \gamma_2)^{-1}\|x_1 - x_2\|.$$

(b) It is a direct consequence of (a) since for each $\gamma \in]0, 1[$ we have

$$\frac{d_S(x)}{2\rho(y)} < \frac{\gamma}{2} \quad \text{for all } (x, y) \in \operatorname{gph}\operatorname{Proj}_S.$$

(c) Fix any $\gamma \in]0, 1[$ and any real $\eta > 0$. For each $i \in \{1, 2\}$, we have $\overline{\ell}(x_i) < \gamma$ so Corollary 3.2 gives

$$\|p_S(x_i) - p_i\| \le \sqrt{\frac{\eta}{1-\gamma}}.$$

It remains to combine the inequality

$$\|p_1 - p_2\| \le \|p_1 - p_S(x_1)\| + \|p_2 - p_S(x_2)\| + \|p_S(x_1) - p_S(x_2)\|$$

with the $(1-\gamma)^{-1}$-Lipschitz property of the mapping $p_S(\cdot)$ on $\mathcal{O}_{\gamma\rho(\cdot)}(S)$ to complete the proof. ■

The following result guarantees that an approximate projection (unit) direction (more precisely, $\frac{x-p}{\|x-p\|}$ with $p \in \operatorname{Proj}^2_{S,\eta}(x)$) can be approximated by an exact one v_x at the same reference point (that is, $v_x := \frac{x - p_S(x)}{\|x - p_S(x)\|}$). In [23], it is established (thanks to the Borwein-Preiss variational principle) that for any closed set $C \subset \mathcal{H}$, the above vector v_x can be estimated by $\frac{\overline{x} - p_S(\overline{x})}{\|\overline{x} - p_S(\overline{x})\|}$.

Proposition 3.4 *Let S be a $\rho(\cdot)$-prox-regular set for some function $\rho : \Lambda^P(S) \to]0,+\infty[$, $x \in \mathcal{O}_{\rho(\cdot)}(S)$ and let $\eta > 0$ be a positive real. Let also $\overline{\ell}(x)$ as in* (3.1)*. Then, for every $p \in \mathrm{Proj}^2_{S,\eta}(x)$ one has*

$$\|\frac{x-p}{\|x-p\|} - \frac{x-p_S(x)}{\|x-p_S(x)\|}\| \leq \frac{2}{d_S(x)}\sqrt{\frac{1}{1-\overline{\ell}(x)}}\sqrt{\eta}.$$

If $\rho(\cdot) \equiv r$ for some constant $r \in]0,+\infty[$, one has

$$\|\frac{x-p}{\|x-p\|} - \frac{x-p_S(x)}{\|x-p_S(x)\|}\| \leq \frac{2}{d_S(x)}\sqrt{\frac{r}{r-d_S(x)}}\sqrt{\eta}.$$

If S is ∞-prox-regular (that is, nonempty closed and convex) one has

$$\|\frac{x-p}{\|x-p\|} - \frac{x-p_S(x)}{\|x-p_S(x)\|}\| \leq \frac{2}{d_S(x)}\sqrt{\eta}.$$

Proof. It suffices to write

$$\|\frac{x-p}{\|x-p\|} - \frac{\|x-p_S(x)}{\|x-p_S(x)\|} \leq \frac{2}{\|x-p\|+d_S(x)}\|(x-p)-(x-p_S(x))\| \leq \frac{\|p_S(x)-p\|}{2d_S(x)}$$

and to invoke Corollary 3.2 to obtain the desired inequality. ■

Remark 3.4 Keep the assumptions of Proposition 3.4. Then, one has $\partial_P d_S(x) = \{\frac{x-p_S(x)}{\|x-p_S(x)\|}\}$, so the latter result says that for each $p \in \mathrm{Proj}^2_{S,\eta}(x)$ there is $\zeta_x \in \partial_P d_S(x)$ such that

$$\|\frac{x-p}{\|x-p\|} - \frac{x-p_S(x)}{\|x-p_S(x)\|}\| \leq \frac{2}{d_S(x)}\sqrt{\frac{r}{r-d_S(x)}}\sqrt{\eta}.$$

■

3.3 Further properties on the open $\rho(\cdot)$-tube $\mathcal{O}_{\rho(\cdot)}(S)$

In this section, we give several important properties of the open $\rho(\cdot)$-tube $\mathcal{O}_{\rho(\cdot)}(S)$ (see (3.2) for the definition) where $S \subset \mathcal{H}$ is a closed set.

Proposition 3.5 *Let S be a $\rho(\cdot)$-prox-regular subset of $\mathcal{H}$ for some function $\rho : \Lambda^P(S) \to]0,+\infty[$. Let also $\overline{\ell}(x)$ as in* (3.1)*. The following hold:*
(a) One always has

$$\frac{d_S(x)}{\rho\big(\mathrm{proj}_S(x)\big)} \leq \overline{\ell}(x) < 1 \quad \textit{for all}\, x \in \mathcal{O}_{\rho(\cdot)}(S),$$

in particular

$$\mathcal{O}_{\rho(\cdot)}(S) \subset \{x \in \mathrm{Dom}\,\mathrm{Proj}_S \setminus S : \exists y \in \mathrm{Proj}_S(x), d_S(x) < \rho(y)\} =: A.$$

(b) One always has with $\mathbb{U} := \{x \in \mathcal{H} : \|x\| < 1\}$

$$A = (\mathcal{H} \setminus S) \cap \mathrm{Dom}\,\big(\mathrm{Id} + \rho(\cdot)\mathbb{U} \cap N^P(S;\cdot)\big)^{-1}.$$

(c) If the function $\rho(\cdot)$ is lower semicontinuous at each point of $\Lambda^P(S)$, then one has $A = \mathcal{O}_{\rho(\cdot)}(S)$.

Proof. (a) This is a direct consequence of the fact that $\mathrm{proj}_S(x)$ is well defined for any $x \in \mathcal{O}_{\rho(\cdot)}(S)$.
(b) Set $M := \big(\mathrm{Id} + \rho(\cdot)\mathbb{U} \cap N^P(S;\cdot)\big)^{-1}$. Fix any $x \in \mathrm{Dom}\,M =: D$ with $x \notin S$. Let $y \in M(x) \neq \emptyset$. We have

$$x - y \in \rho(y)\mathbb{U} \cap N^P(S;y). \tag{3.13}$$

Using the $\rho(\cdot)$-prox-regularity of S we get

$$\langle x-y, c-y\rangle \le \frac{1}{2}\|c-y\|^2 \quad \text{for all } c\in S,$$

and this implies (see (2.1)) that $y\in \operatorname{Proj}_S(x)$. This and the above inclusion (3.13) easily give the inequality

$$d_S(x)=\|x-y\|<\rho(y).$$

Therefore, we have $x\in A$ and this translates the inclusion $D\subset A$. Since the converse inclusion is evident, we get $A=D$ as claimed.

(c) Assume that the function $\rho(\cdot)$ is lower semicontinuous at each point of $\Lambda^P(S)$. Fix any $x\in A$. Let $y\in \operatorname{Proj}_S(x)$ such that $L:=\frac{d_S(x)}{\rho(y)}<1$. Choose some sequence $(y_n)_{n\in\mathbb{N}}$ of $\Lambda^P(S)$ with $\|x-y_n\|\to d_S(x)$ such that

$$\bar{\ell}(x)=\lim_{n\to\infty}\frac{\|x-y_n\|}{\rho(y_n)}$$

By Lemma 3.1, we have

$$\|y-y_n\|^2\le (1-L)^{-1}\big(\|y_n-x\|^2-d_S^2(x)\big) \quad \text{for all } n\in\mathbb{N}.$$

Therefore, we have $y_n\to y$ and this ensures $\bar{\ell}(x)\le L<1$, i.e., $x\in\mathcal{O}_\rho(S)$. ■

Remark 3.5 The above assertions (a) and (b) are known (see, e.g., [26, Proof of Proposition 15.9]). ■

In [18, Theorem 3] L. Thibault and the author establish that for any pair (s,r) of reals with $0<s<r$ the complement of a uniformly prox-regular set with constant r is nothing but the union of closed balls with common radius s. An adaptation of the proof leads to the next theorem. Before stating it, we mention that [18, Theorem 3] has been extended by C. Nour and J. Takche [20, Theorem 1.1] for a closed set $S\subset\mathbb{R}^n$ with $S=\operatorname{cl}(\operatorname{int} S)$ satisfying the exterior sphere condition (which is weaker than the uniform prox-regularity). Recall that a closed set $S\subset\mathcal{H}$ the *r-exterior sphere condition* for some real $r>0$ whenever for each $x\in\operatorname{bdry} C$ there is $v\in N^P(S;x)\cap\mathbb{S}$ which is realized by an r-ball (see (3.4)). Nour and Takche also develop [20, Theorem 1.2] the case where S is not regular closed (that is, $S\neq\operatorname{cl}(\operatorname{int} S)$) through an appropriate extended exterior sphere condition.

Theorem 3.2 *Let S be a $\rho(\cdot)$-prox-regular set of $\mathcal{H}$ for some function $\rho:\Lambda^P(S)\to]0,+\infty[$ and let $\gamma\in]0,1[$. Then, there exists a nonempty set $L\subset\mathcal{H}\times\Lambda^P(S)$ such that*

$$\mathcal{O}_{\rho(\cdot)}(S)\subset\bigcup_{(x,p)\in L}B[x,\gamma\rho(p)]\subset\mathcal{H}\setminus S.$$

Proof. Fix any $y\in\mathcal{O}_{\rho(\cdot)}(S)$. According to Theorem 3.1, $p:=p_S(y)$ is well defined. Note that $p\in\Lambda^P(S)$ (since $y\notin S$). Set $v:=\|y-p\|^{-1}(y-p)\in N^P(S;y)$ and $t:=\rho(p)$. By definition of a $\rho(\cdot)$-prox-regular set, we can write

$$p\in\operatorname{Proj}_S(p+tv)$$

and this entails $B(p+tv,t)\cap S=\emptyset$. On the other hand, we note that (keeping in mind that $\|y-p\|=d_S(y)<t$)

$$\|y-(p+tv)\|=t-d_S(y)=:\alpha. \tag{3.14}$$

Let us distinguish two cases:

Case 1: $\gamma t\ge\alpha$. In view of (3.14), we have $y\in B[p+tv,\alpha]\subset B(p+tv,t)\subset\mathcal{H}\setminus S$.

Case 2: $\gamma t<\alpha$. Observe first that $y\neq p+tv$ (otherwise we would have $\|y-p\|=t$). Set

$$z:=y-\gamma t\|y-p-tv\|^{-1}(y-p-tv).$$

It is readily seen that $y \in B[z, \gamma t]$. We easily check that for any $u \in B[z, \gamma t]$

$$\|u - (p + tv)\| \leq \|u - z\| + \|z - (p + tv)\| = \gamma t + t - d_S(y) - \gamma t < t,$$

hence

$$B[z, \gamma t] \subset B(p + tv, t) \subset \mathcal{H} \setminus S.$$

The proof is complete. ■

Remark 3.6 We obviously have

$$\mathcal{H} \setminus S \subset \mathcal{O}_{\rho(\cdot)}(S) \cup \{x \in \mathcal{H} \setminus S : \bar{\ell}(x) \geq 1\}.$$

Assume that $\rho \equiv r$ for some constant $r > 0$ in the latter theorem. Since any point y in the set $\{\bar{\ell} \geq 1\} = \{d_S \geq r\}$ satisfies $y \in B[y, \gamma r] \subset \mathcal{H} \setminus S$ we obtain that $\mathcal{H} \setminus S$ is the union of closed balls with radius γr.

Now, assume that $\rho(\cdot)$ is lower continuous at each point of $\Lambda^P(S)$ and S is approximatively compact (which is the case if $\mathcal{H} = \mathbb{R}^n$). Let $x_0 \in \{\bar{\ell} \geq 1\}$. There are $\gamma' \in \mathbb{R}$ and a sequence $(y_n)_{n \in \mathbb{N}}$ of $\Lambda^P(S)$ with $\|y_n - x_0\| \to d_S(x_0)$ such that

$$\gamma < \gamma' < 1 \leq \bar{\ell}(x_0) = \lim_{n \to \infty} \frac{\|x_0 - y_n\|}{\rho(y_n)}.$$

Since S is approximatively compact, we have $y_n \to p \in \Lambda^P(S)$ hence

$$d_S(x_0) \geq \gamma' \liminf_{n \to \infty} \rho(y_n) \geq \gamma' \rho(p) > \gamma \rho(p).$$

We derive from this $x_0 \in B[x_0, \gamma \rho(p)] \subset \mathcal{H} \setminus S$. We conclude that $\mathcal{H} \setminus S$ is the union of closed balls with radius $\gamma(p)$ with $p \in \Lambda^P(S)$.

We mention here that sets in $\mathbb{R}^n$ which are union of closed balls with (semicontinuous) variable radius have been studied by C. Nour, R.J. Stern and J. Takche [21] (see also the recent work [22]). ■

3.4 Characterizations of $\rho(\cdot)$-prox-regular sets through outside points

The main aim of the present section is to develop several extensions of the characterization of $\rho(\cdot)$-prox-regularity in Proposition 3.1 through possibly outside points, say $x, x' \in \mathcal{O}_{\rho(\cdot)}(S)$. This can be done with the help of the proximal subdifferential $\partial_P d_S(x)$ (which is nothing but $N^P(S; x) \cap \mathbb{B}$ whenever $x \in S$). We follow the approach utilized by L. Thibault and the author in [18] for uniformly prox-regular sets.

Proposition 3.6 *Let S be a nonempty closed subset of $\mathcal{H}$ and let $\rho : \operatorname{bdry} S \to]0, +\infty]$ be a function. The following assertions are equivalent:*
(a) The set S is $\rho(\cdot)$-prox-regular;
(b) for any $(x', p') \in \operatorname{gph} \operatorname{Proj}_S$ with $\frac{d_S(x')}{\rho(p')} < 1$, for any $(x, p) \in \operatorname{gph} \operatorname{Proj}_S$ and any $\xi \in \partial_P d_S(x)$, one has

$$\langle \xi, x' - x \rangle \leq \frac{1}{2\rho(p)} \frac{1}{1 - \frac{d_S(x')}{\rho(p')}} \left(\|x' - p\|^2 - d_S^2(x') \right) + d_S(x') - d_S(x);$$

(c) for any $x \in S$, for any $(x', p') \in \operatorname{gph} \operatorname{Proj}_S$ with $\frac{d_S(x')}{\rho(p')} < 1$ and any $\xi \in \partial_P d_S(x)$, one has

$$\langle \xi, x' - x \rangle \leq \frac{1}{2\rho(p)} \frac{1}{1 - \frac{d_S(x')}{\rho(p')}} \left(\|x' - x\|^2 - d_S^2(x') \right) + d_S(x');$$

(d) for any $x' \in S$, for any $(x, p) \in \operatorname{gph} \operatorname{Proj}_S$ and any $\xi \in \partial_P d_S(x)$, one has

$$\langle \xi, x' - x \rangle \leq \frac{1}{2\rho(p)} \|x' - p\|^2 - d_S(x).$$

Proof. By Proposition 3.1, we see that anyone of the assertions $(b),(c)$ and (d) implies (a). Further, it is readily seen that the assertion (b) entails (c) and (d). Let us establish the implication $(a) \Rightarrow (b)$. Fix any $(x,p),(x',p') \in \operatorname{gph}\operatorname{Proj}_S$ with $\frac{d_S(x')}{\rho(p')} < 1$ and any $\xi \in \partial_P d_S(x)$.
Case 1. $x \in S$. Note that (see (2.5)) $\xi \in N^P(S;x) \cap \mathbb{B}$. Using the $\rho(\cdot)$-prox-regularity of the set S gives

$$\langle \xi, x'-x\rangle = \langle \xi, p'-x\rangle + \langle \xi, x'-p'\rangle \leq \frac{1}{2\rho(x)}\left\|p'-x\right\|^2 + d_S(x'). \tag{3.15}$$

On the other hand Lemma 3.1 guarantees that

$$\frac{1}{2\rho(x)}\left\|y'-x\right\|^2 \leq \frac{1}{2\rho(x)}\frac{1}{1-\frac{d_S(x')}{\rho(p')}}\big(\left\|x'-x\right\|^2 - d_S^2(x')\big). \tag{3.16}$$

Combining (3.15), (3.16) and the equality $d_S(x)=0$ furnishes the inequality

$$\langle \xi, x'-x\rangle \leq \frac{1}{2\rho(x)}\frac{1}{1-\frac{d_S(x')}{\rho(p')}}\big(\left\|x'-x\right\|^2 - d_S^2(x')\big) + d_S(x') - d_S(x).$$

Case 2. $x \notin S$. In such a case we have (see (2.4))

$$\partial d_S(x) = \{\xi\} \quad \text{with} \quad \xi := \frac{x-p}{d_S(x)} \in N^P(S;p) \cap \mathbb{B} = \partial d_S(p).$$

Due to the definition of ξ, we easily get

$$\langle \xi, x'-x\rangle = \langle \xi, x'-p\rangle + \langle \xi, p-x\rangle = \langle \xi, x'-p\rangle - d_S(x). \tag{3.17}$$

Thanks to the $\rho(\cdot)$-prox-regularity of the set S we have

$$\langle \xi, x'-p\rangle = \langle \xi, x'-p'\rangle + \langle \xi, p'-p\rangle \leq d_S(x') + \frac{1}{2\rho(p)}\|p'-p\|^2. \tag{3.18}$$

As above Lemma 3.1 yields

$$\frac{1}{2\rho(p)}\|p'-p\|^2 \leq \frac{1}{2\rho(p)}\frac{1}{1-\frac{d_S(x')}{\rho(p')}}\big(\|x'-p\|^2 - d_S^2(x')\big). \tag{3.19}$$

It remains to combine (3.17), (3.18) and (3.19) to get

$$\langle \xi, x'-x\rangle \leq \frac{1}{2\rho(p)}\frac{1}{1-\frac{d_S(x')}{\rho(p')}}\big(\left\|x'-p\right\|^2 - d_S^2(x')\big) + d_S(x') - d_S(x).$$

The implication $(a) \Rightarrow (b)$ is then established. This finishes the proof. ■

The Lipschitz property on the (single-valued) mapping proj_S on $\mathcal{O}_{\gamma\rho(\cdot)}(S)$ for a prox-regular is deeply involved in the proof of the next theorem.

Theorem 3.3 *Let S be a nonempty closed subset of $\mathcal{H}$ and let $\rho : \operatorname{bdry} S \to]0,+\infty]$ be a function. Then, the set S is $\rho(\cdot)$-prox-regular if and only if for each $\gamma \in]0,1[$, for any $(x,p) \in \operatorname{gph}\operatorname{Proj}_S$ with $x \in \mathcal{O}_{\gamma\rho(\cdot)}(S) \cup S$, any $x' \in \mathcal{O}_{\gamma\rho(\cdot)}(S) \cup S$ and any $\xi \in \partial_P d_S(x)$, one has*

$$\langle \xi, x'-x\rangle \leq \frac{1}{2\rho(p)(1-\gamma)^2}\|x'-x\|^2 + d_S(x') - d_S(x).$$

Proof. $\Rightarrow$, Let $\gamma \in]0,1[$. Fix any $(x,p) \in \operatorname{gph}\operatorname{Proj}_S$ with $x \in \mathcal{O}_{\gamma\rho(\cdot)}(S) \cup S$, any $x' \in \mathcal{O}_{\gamma\rho(\cdot)}(S) \cup S$ and any $\xi \in \partial_P d_S(x)$. Let us distinguish two cases.
Case 1: $x \in S$. Then, one has $\xi \in N^P(S;x) \cap \mathbb{B}$, $p = x$ along with

$$\begin{aligned}\langle \xi, x' - x\rangle &= \langle \xi, x' - p_S(x')\rangle + \langle \xi, p_S(x') - p\rangle .\\ &\leq d_S(x') + \frac{1}{2\rho(p)}\|p_S(x') - p\|^2 \\ &\leq d_S(x') + \frac{1}{2\rho(p)(1-\gamma)^2}\|x' - x\|^2,\end{aligned}$$

where the last inequality is due to Proposition 3.3.
Case 2: $x \notin S$. We know that $\xi = d_S(x)^{-1}(x - p)$ (see (2.4)) and this entails

$$\begin{aligned}\langle \xi, x' - x\rangle &= \langle \xi, x' - p_S(x')\rangle + \langle \xi, p_S(x') - p_S(x)\rangle + \langle \xi, p_S(x) - x\rangle \\ &\leq d_S(x') + \langle \xi, p_S(x') - p_S(x)\rangle - d_S(x). \end{aligned} \tag{3.20}$$

Using the inclusion $\xi \in N^P(S; p_S(x)) \cap \mathbb{B}$ and Proposition 3.1 allows us to write

$$\langle \xi, p_S(x') - p_S(x)\rangle \leq \frac{1}{2\rho(p_S(x))(1-\gamma)^2}\|x' - x\|^2 . \tag{3.21}$$

Combining (3.20) and (3.21) gives the desired inequality.
$\Leftarrow$, Let $x, x' \in S$, $\xi \in N^P(S;x) \cap \mathbb{B}$. Fix any sequence $(\gamma_n)_{n\geq 1}$ of $]0,1[$ with $\gamma_n \to 0$. We have $\xi \in \partial_P d_S(x)$ (see (2.5)) hence

$$\langle \xi, x' - x\rangle \leq \frac{1}{2\rho(p_S(x))(1-\gamma_n)^2}\|x' - x\|^2.$$

Letting $n \to \infty$ in the latter inequality guarantees the $\rho(\cdot)$-prox-regularity of the set S. The proof is complete. ■

4 Approximate farthest points for strongly convex sets

In this last section, we estimate (as in Subsection 3.2) the diameter of the approximate farthest set of points

$$F_1 := \operatorname{Far}_{C,\eta}(x) := \{c \in C : \|x - c\| \geq \operatorname{dfar}_C(x) - \eta\}$$

and

$$F_2 := \operatorname{Far}^2_{C,\eta}(x) := \left\{c \in C : \|x - c\|^2 \geq \operatorname{dfar}^2_C(x) - \eta\right\},$$

where $C \subset \mathcal{H}$ is a $\rho(\cdot)$-strongly convex set. Of course, if $r_1 := \operatorname{dfar}_C(x) - \eta > 0$ (resp. $r_2 := \operatorname{dfar}^2_C(x) - \eta > 0$) we have

$$F_1 := C \cap \mathcal{H} \setminus B(x, \sqrt{r_1}) \quad \text{(resp.} \quad F_2 := C \cap \mathcal{H} \setminus B(x, \sqrt{r_2})).$$

Let us first recall the definition of a strongly convex set with a variable radius. Such a concept has been introduced in the recent work [19].

Definition 4.1 *Let C be a nonempty closed subset of the Hilbert space $\mathcal{H}$ with $C \neq \mathcal{H}$ and let $\rho : \Lambda^P(C) \to]0, +\infty[$ be a positive function. One says that C is $\rho(\cdot)$-*strongly convex *whenever for any $x \in \Lambda^P(C)$ and $v \in N^P(C;x) \cap \mathbb{S}$, one has*

$$x \in \operatorname{Far}_C\big(x - \rho(x)v\big).$$

*Whenever $\rho(\cdot) \equiv R$, one says that C is an R-*strongly convex *set (or* uniformly strongly convex *set of constant R).*

It can be checked that such a set is nothing but the intersection of closed balls with variable radii ([19]). In particular, any $\rho(\cdot)$-strongly convex set is (closed bounded) convex.

The constant case $\rho(\cdot) \equiv R$ has a long story for which we refer to the survey by V.V. Goncharov and G.E. Ivanov [10] (see also the references therein). It is known that *any* intersection of closed balls of $\mathcal{H}$ with radius R is uniformly strongly convex of constant R.

We start with the following result which complements Proposition 3.2. Note that it has been first observed in [1, Proof of Theorem 3.2].

Proposition 4.1 *Let C be an R-strongly convex set for some real $R > 0$, $x \in \mathcal{H}$ and $\eta > 0$ with $r_1 := \mathrm{dfar}_C(x) - \eta > 0$ (resp. $r_2 := \mathrm{dfar}_C^2(x) - \eta > 0$). If $R \le \sqrt{r_1}$ (resp. $R \le \sqrt{r_2}$) then the set $\mathrm{Far}_{C,\eta}(x)$ is $\sqrt{r_1}$-prox-regular (resp. $\sqrt{r_2}$-prox-regular).*

Proof. It suffices to apply [26, Corollary 16.17] (having in mind that the complement of an open ball with radius $r > 0$ is r-prox-regular. ■

The following easy characterization of $\rho(\cdot)$-strongly convex sets will be needed (see [19]).

Proposition 4.2 *Let C be a nonempty closed subset of the Hilbert space $\mathcal{H}$ and let $\rho : \Lambda^P(C) \to]0, +\infty[$ be a function. Then, the set C is $\rho(\cdot)$-strongly convex if and only if*

$$\langle v, x' - x\rangle \le -\frac{1}{2\rho(x)} \|x' - x\|^2 ,$$

for any $x \in \Lambda^P(C)$, any $v \in N^P(C; x) \cap \mathbb{S}$ and any $x' \in C$.

Given a nonempty closed bounded set $C \subset \mathcal{H}$ not reduced to a singleton and a function $\rho(\cdot) : \Lambda^P(C) \to]0, +\infty]$, we define (in the same way as [19])

$$\mathcal{D}_{\rho(\cdot)}(C) := \left\{ x \in \mathcal{H} : \liminf_{\substack{\|x-y\| \to \mathrm{dfar}_C(x) \\ y \in \Lambda^P(C)}} \frac{\|x - y\|}{\rho(y)} > 1 \right\}.$$

Observe that the above limit inferior is well defined thanks to Theorem 2.4 and to the inclusion (2.3). If C is reduced to a singleton, say $C = \{c\}$, it will be convenient to set $\Lambda^P(C) = C$ and $\mathcal{D}_{\rho(\cdot)}(C) = \mathcal{H} \setminus B[c, \rho(c)]$. Note that the set $\mathcal{D}_{\rho(\cdot)}(C)$ is always open.

The first lemma of our study is easy.

Lemma 4.1 *Let C be a nonempty closed bounded subset of the Hilbert space $\mathcal{H}$ and let $\rho : \Lambda^P(C) \to]0, +\infty[$ be a given function. Let also $(x, y) \in \mathrm{gph}\, \mathrm{Far}_C$ with $x \in \mathcal{D}_{\rho(\cdot)}(C)$ satisfying the inclusion*

$$y \in \mathrm{Far}_C\big(y - \frac{\rho(y)}{\|x - y\|}(y - x)\big). \tag{4.1}$$

Then, one has

$$\|y - c\|^2 \le \frac{1}{\frac{\mathrm{dfar}_C(x)}{\rho(y)} - 1} \big(\mathrm{dfar}_C^2(x) - \|x - c\|^2\big) \quad \textit{for all}\, c \in C.$$

Proof. Fix any $c \in C$. Note first that $\frac{\mathrm{dfar}_C(x)}{\rho(y)} > 1$. From the inclusion (4.1) we see that

$$2\langle y - x, c - y\rangle \le -\frac{\|x - y\|}{\rho(y)} \|y - c\|^2 = -\frac{\mathrm{dfar}_C(x)}{\rho(y)} \|y - c\|^2. \tag{4.2}$$

Putting together this and the following elementary equality

$$\|c - y\|^2 - \|c - x\|^2 = 2\langle c - y, x - y\rangle - \|x - y\|^2 \tag{4.3}$$

easily gives the desired inequality. ■

Remark 4.1 The inclusion (4.1) obviously holds (see (2.3)) whenever C is $\rho(\cdot)$-strongly convex. Observe that there are nonconvex sets (as $C := \mathbb{S}\cup\{0\}$) which satisfy the inclusion (4.1) below. ■

The following lemma is in the line of Lemma 3.2.

Lemma 4.2 *Let C be a nonempty closed bounded subset not reduced to a singleton of the Hilbert space $\mathcal{H}$ such that there exists a function $\rho : \Lambda^P(C) \to]0, +\infty[$ satisfying*

$$y \in \mathrm{Far}_C\big(y + \frac{\rho(y)}{\|x-y\|}(x-y)\big) \quad \textit{for all}\,(x,y) \in \mathrm{gph}\,\mathrm{Far}_C \;\textit{with}\, x \in \mathcal{D}_{\rho(\cdot)}(C).$$

Let $x \in \mathcal{D}_{\rho(\cdot)}(C)$ and let $c_1, \ldots, c_m \in C$. Then, for each real $\varepsilon > 0$, there exists $q \in \mathrm{Far}_{C,\varepsilon}(x) \cap \mathrm{Far}^2_{C,\varepsilon}(x)$ such that for all $i \in \{1, \ldots, m\}$, one has

$$\|c_i - q\|^2 \le \big(\frac{1}{\underline{\ell}(x)-1} + \varepsilon\big)\big(\mathrm{dfar}^2_C(x) - \|c_i - x\|^2 + \varepsilon\big)$$

with

$$\underline{\ell}(x) := \liminf_{\substack{\|x-y\|\to \mathrm{dfar}_C(y)\\ y\in\Lambda^P(C)}} \frac{\|x-y\|}{\rho(y)}. \tag{4.4}$$

Proof. We may suppose that C is not a singleton (in particular $\mathrm{dfar}_C(x) > 0$). Let $\varepsilon > 0$ be a real. By Lau theorem relative to farthest points (see Theorem 2.4) we can find a sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ such that $u_n \to x$ and $q_n := \mathrm{far}_C(u_n)$ is well defined. Since $\frac{\|x-q_n\|}{\|u_n-q_n\|} \to \frac{\mathrm{dfar}_C(x)}{\mathrm{dfar}_C(x)} = 1$ we have

$$\liminf_{n\to\infty} \frac{\|u_n - q_n\|}{\rho(q_n)} = \liminf_{n\to\infty} \frac{\|x - q_n\|}{\rho(q_n)} \ge \underline{\ell}(x) > 1.$$

Pick any $\varepsilon' > 0$ such that

$$\underline{\ell}(x) - \varepsilon' > 1 \quad \text{and} \quad \frac{1}{\underline{\ell}(x) - 1 - \varepsilon'} \le \frac{1}{\underline{\ell}(x)-1} + \varepsilon.$$

Choose some $N \in \mathbb{N}$ satisfying for all $n \ge N$

$$\frac{\mathrm{dfar}_C(u_n)}{\rho(q_n)} = \frac{\|u_n - q_n\|}{\rho(q_n)} > \underline{\ell}(x) - \varepsilon' > 1$$

along with

$$4\|u_n - x\|\big(\mathrm{dfar}_C(x) - \|u_n - x\|\big) \le \varepsilon \quad \text{and} \quad \|u_n - x\| \le \frac{\varepsilon}{2}. \tag{4.5}$$

For each $i \in \{1, \ldots, m\}$, there is $N_i \in \mathbb{N}$ such that

$$\mathrm{dfar}^2_C(u_n) - \|c_i - u_n\|^2 \le \mathrm{dfar}^2_C(x) - \|c_i - x\|^2 + \varepsilon \quad \text{for all}\, n \ge N_i.$$

Set $n_0 := \max\big(N, \max_{i=1,\ldots,N} N_i\big)$, $p := p_{n_0}$ and $u := u_{n_0}$. Writing

$$\|q - x\| \le \mathrm{dfar}_C(u) - \|u - x\| \ge \mathrm{dfar}_C(x) - 2\|u - x\| \ge \mathrm{dfar}_C(x) - \varepsilon$$

we see that

$$\|q - x\|^2 \le \mathrm{dfar}^2_C(x) - 4\|u - x\|(\mathrm{dfar}_C(x) - \|u - x\|) \ge \mathrm{dfar}^2_C(x) - \varepsilon.$$

Thus, we have the inclusion $q \in \mathrm{Far}_{C,\varepsilon}(x) \cap \mathrm{Far}^2_{C,\varepsilon}(x)$. Applying Lemma 4.1 gives for each $i \in \{1, \ldots, m\}$

$$\|c_i - q\|^2 \le \frac{1}{\frac{\mathrm{dfar}_C(u)}{\rho(q)} - 1}\big(\mathrm{dfar}^2_C(u) - \|c_i - u\|^2\big)$$

from which we derive

$$\begin{aligned}\|c_i-p\|^2 &\le \frac{1}{1-\underline{\ell}(x)-\varepsilon'}\big(\mathrm{dfar}_C^2(x)-\|c_i-x\|^2+\varepsilon\big)\\ &\le \big(\frac{1}{1-\underline{\ell}(x)}+\varepsilon\big)\big(\mathrm{dfar}_C^2(x)-\|c_i-x\|^2+\varepsilon\big)\big).\end{aligned}$$

This finishes the proof. ■

Now we can state and prove the following result:

Theorem 4.1 *Let C be a nonempty closed bounded subset of the Hilbert space $\mathcal{H}$ such that there exists a function $\rho : \Lambda^P(C) \to]0,+\infty[$ satisfying*

$$y \in \mathrm{Far}_C\big(y+\frac{\rho(y)}{\|x-y\|}(x-y)\big) \quad \textit{for all}\,(x,y)\in \mathrm{gph}\,\mathrm{Far}_C \ \textit{with}\, x\in \mathcal{D}_{\rho(\cdot)}(C).$$

Let $x \in \mathcal{D}_{\rho(\cdot)}(C)$. Then, for $\kappa_x := 2\sqrt{\frac{1}{\underline{\ell}(x)-1}}$ and for each real $\eta > 0$ one has

$$\mathrm{diam}\,\mathrm{Far}^2_{C,\eta}(x) \le \kappa_x\sqrt{\eta} \tag{4.6}$$

and

$$\mathrm{diam}\,\mathrm{Far}_{C,\eta}(x) \le \kappa_x\sqrt{\eta(2\mathrm{dfar}_C(x)+\eta)} \quad \textit{if}\,\mathrm{dfar}_C(x)\ge\eta \tag{4.7}$$

with $\underline{\ell}(x)$ as in (4.4). In particular, the supremum $\mathrm{dfar}_C(x)$ is strongly attained.

Proof. Fix any real $\eta > 0$. Let $q_1, q_2 \in \mathrm{Far}^2_{C,\eta}(x)$ (resp. $q_1, q_2 \in \mathrm{Far}_{C,\eta}(x)$). Let also $\varepsilon > 0$. According to Lemma 4.2, we can find some $q \in \mathrm{Far}_{C,\varepsilon}(x) \cap \mathrm{Far}^2_{C,\varepsilon}(x)$ such that for each $i \in \{1,2\}$

$$\|q_i-q\|^2 \le \big(\frac{1}{\underline{\ell}(x)-1}+\varepsilon\big)\big(\mathrm{dfar}_C^2(x)-\|q_i-x\|^2+\varepsilon\big) \le \big(\frac{1}{\underline{\ell}(x)-1}+\varepsilon\big)(\eta+\varepsilon)$$

(resp.

$$\begin{aligned}\|q_i-q\|^2 &\le \big(\frac{1}{\underline{\ell}(x)-1}+\varepsilon\big)\big(\mathrm{dfar}_C^2(x)-\|q_i-x\|^2+\varepsilon\big)\\ &\le \big(\frac{1}{\underline{\ell}(x)-1}+\varepsilon\big)\big(2\eta\mathrm{dfar}_C(x)+\eta^2+\varepsilon\big)\big).\end{aligned}$$

Hence, for each $i \in \{1,2\}$

$$\|q_i-q\| \le \sqrt{\eta+\varepsilon}\sqrt{\frac{1}{\underline{\ell}(x)-1}+\varepsilon}$$

(resp.

$$\|q_i-q\| \le \sqrt{\eta(2\mathrm{dfar}_C(x)+\eta)+\varepsilon}\sqrt{\frac{1}{\underline{\ell}(x)-1}+\varepsilon}).$$

It remains to apply the triangle inequality and to let $\varepsilon \downarrow 0$ to obtain the inequalities in the statement. The proof is complete. ■

For an R-strongly convex set, we have the following estimates:

Corollary 4.1 *Let C be an R-strongly convex set for some $R \in]0,+\infty[$ and let $x \in \mathcal{E}_R(C) := \{\mathrm{dfar}_C > R\}$. Let $\kappa_x := 2\sqrt{\frac{R}{R-\mathrm{dfar}_C(x)}}$. Then, for each real $\eta > 0$ one has*

$$\mathrm{diam}\,\mathrm{Far}^2_{C,\eta}(x) \le 2\kappa_x\sqrt{\eta}$$

and

$$\mathrm{diam}\,\mathrm{Far}_{C,\eta}(x) \le 2\kappa_x\sqrt{\eta(\mathrm{dfar}_C(x)-\eta)} \quad \textit{if}\,\mathrm{dfar}_C(x)\ge\eta.$$

We now estimate the error between $\mathrm{far}_C(x)$ and any approximate farthest point $q \in \mathrm{Far}^2_{C,\eta}(x)$.

Corollary 4.2 *Let C be a $\rho(\cdot)$-strongly convex set of $\mathcal{H}$ for some function $\rho : C \to]0, +\infty[$, $x \in \mathcal{D}_{\rho(\cdot)}(S)$ and let $\underline{\ell}(x)$ as in (4.4). Then, $\mathrm{far}_C(x)$ is well defined along with*

$$\|\mathrm{far}_C(x) - c\| \leq \sqrt{\frac{1}{\underline{\ell}(x) - 1}(\mathrm{dfar}^2_C(x) - \|x - c\|^2)} \quad \textit{for all}\, c \in C.$$

In particular, for each real $\eta > 0$, one has

$$\sup_{q \in \mathrm{Far}^2_{C,\eta}(x)} \|\mathrm{far}_C(x) - q\| \leq \sqrt{\frac{1}{\underline{\ell}(x) - 1}}\sqrt{\eta}. \tag{4.8}$$

We end the paper with the following result which encompasses the Lipschitz behavior of the farthest point mapping relative to a strongly convex set.

Proposition 4.3 *Let C be a $\rho(\cdot)$-strongly convex set of $\mathcal{H}$ for some function $\rho : \Lambda^P(C) \to]0, +\infty[$. The following hold.*
(a) The mapping far_C is well defined on the open set $\mathcal{D}_{\rho(\cdot)}(C)$ and for all $x_1, x_2 \in \mathcal{D}_{\rho(\cdot)}(C)$, one has

$$\|\mathrm{far}_C(x_1) - \mathrm{far}_C(x_2)\| \leq \Big(\frac{\mathrm{dfar}_C(x_1)}{2\rho(\mathrm{far}_C(x_1))} + \frac{\mathrm{dfar}_C(x_2)}{2\rho(\mathrm{far}_C(x_2))} - 1\Big)^{-1}\|x_1 - x_2\|.$$

(b) For every real $\gamma > 1$ the mapping far_C is well defined on the open set $\mathcal{D}_{\gamma\rho(\cdot)}(C)$ and $(\gamma - 1)^{-1}$-Lipschitz continuous therein.
(c) For any real $\gamma > 1$ and any real $\eta > 0$ one has

$$\|q_1 - q_2\| \leq 2\sqrt{\frac{\eta}{\gamma - 1}} + (\gamma - 1)^{-1}\|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathcal{D}_{\gamma\rho(\cdot)}(S)$, all $q_1 \in \mathrm{Far}^2_{C,\eta}(x_1)$ and all $q_2 \in \mathrm{Far}^2_{C,\eta}(x_2)$.

Proof. Proceed as Proposition 3.3. ■

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