A History-dependent Sweeping Processes in Contact Mechanics

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Abstract

We consider a special type of sweeping process in real Hilbert spaces, governed by two (possibly history-dependent) operators. We associate to this problem an auxiliary time-dependent inclusion for which we establish an existence and uniqueness result. The proof is based on arguments of convex analysis and fixed point theory. From the unique solvability of the intermediate inclusion, we derive the existence of a unique solution to the considered sweeping processes. Our theoretical results find various applications in contact mechanics. As an example, we consider a frictional contact problem for viscoelastic materials. We list the assumptions on the data and provide a variational formulation of the problem, in a form of a sweeping process for the strain field. Then, we prove the unique solvability of the sweeping process and use it to obtain the existence of a unique weak solution to the viscoelastic contact problem.

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1 Introduction

A convex sweeping process is a constrained differential inclusion governed by the normal cone $N_{K(t)}$ of a closed convex moving set $K(t)$ in a Hilbert space X, namely,

$$\begin{cases} -\dot{u}(t) \in N_{K(t)}(u(t)) \quad \text{a.e. } t \in I, \\ u(t) \in K(t) \quad \text{for all } t \in I, \\ u(0) = u_0 \in K(0). \end{cases}$$

Here $I \subset \mathbb{R}$ is a time interval such that $0 \in I$. Existence and uniqueness results for such evolution problems have been proved by Moreau in his seminal papers *Rafle par un convexe variable* ([23, 24]) published in "Travaux du séminaire d'Analyse Convexe de Montpellier" which, currently, became the Journal of Convex Analysis. The so-called Moreau's sweeping process traces its roots back to 1971 in elastoplasticity theory ([20]) and then has been extended in numerous and various ways: BV inclusions ([13, 18, 22, 24]), stochastic differential inclusions ([8, 9]), inclusions with nonconvex sets ([4, 11, 13]), with a state-dependent moving set ([10, 16, 25]), with external perturbations ([18]), in Banach spaces ([6]), associated to submanifolds ([5]), optimal control theory ([7]) and optimal transport problems ([12]). Sweeping process theory was used in various applications arising in electrical circuits theory ([1]), ressource allocation in economics ([15]), crowd motion models ([17]), contact and solid mechanics ([18, 19, 20, 21]).

Sweeping processes can be handled through different approaches including the famous Moreau's catching-up algorithm introduced in [23] and used in [2, 13, 18, 22, 25], the regularization of the involved normal cone ([23, 28]), the reduction technique ([31]) and fixed point arguments ([3, 10, 25, 26, 27]).

The sweeping process considered in [26] was of the form

$$\begin{cases} -\dot{u}(t) \in N_{K(t)}\big(A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)\big) \qquad \text{for all } \; t \in I, \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $A, B : X \to X$ are Lipschitz operators and $\mathcal{S}$ is an operator on the space of continuous functions defined on I with values in X. There, the crucial assumptions ensuring the well-posedness of (1.1) was a Mosco convergence type property of the moving set and a history-dependent property on the operator $\mathcal{S}$. Moreover, examples of time-dependent convex sets and operators which satisfy these assumptions have been provided. An existence and uniqueness result for a sweeping process of the form (1.1) was previously obtained in [3]. There, it was assumed that $K(t) := C(\mathcal{R}\dot{u}(t), t)$ where $C(\cdot, \cdot)$ had a particular structure and, again, $\mathcal{R}$ was an operator on the space of continuous functions defined on I with values in X. Therefore, it seems that the results in [26] and [3] are not comparable since they have been obtained for sweeping processes with different structures and under different assumptions on the data: for the sweeping process studied in [3] the convex $K(t)$ has a particular form, but could depend on the solution; in contrast, for the sweeping process studied in [26] the convex $K(t)$ has a more general form, but does not depend on the solution.

The aim of this current paper is two folds. The first one is to develop a well-posedness result encompassing the existence and uniqueness results provided in both [3] and [26]. This represents the first novelty of the current work. Our second aim is to illustrate the use of the sweeping processes in the variational analysis of mathematical models which describe the evolution of deformable bodies in contact with an obstacle, the so-called foundation. To this end, we consider a viscoelastic contact problem and prove its unique weak solvability through a new variational formulation, expressed in terms of a sweeping process in which the unknown is the strain field. This represents the second trait of novelty of this paper.

The rest of the manuscript is organized as follows. In Section 2 we present some preliminary material. It includes notation, prerequisites on convex and nonlinear analysis as well as some auxiliary results already obtained in [26], which are needed later in this paper. In Section 3 we introduce the sweeping process we are interestd in, then we state and prove its unique solvability, Theorem 3.1. The proof is based on an intermediate result for time-dependent inclusions, Theorem 3.2, which has some interest in its own. Section 4 is devoted to the proof of Theorem 3.2, based on arguments of convex analysis and fixed point for almost history-dependent operators. Next, in Section 5 we introduce the mathematical model of contact, list the assumptions on the data and derive its variational formulation, in the form of a sweeping process. Then, we state an existence and uniqueness result, Theorem 5.3. Its proof is given in Section 6 and is based on the unique solvability result provided by Theorem 3.1.

2 Preliminaries

In this section we introduce the necessary preliminaries needed in the rest of the paper. The material presented here is quite standard and, for this reason, we present it without proofs. Everywhere below X stands for a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and its associated norm $\|\cdot\|_X := \sqrt{(\cdot,\cdot)_X}$. We use Id_X for the identity mapping on X and the set of parts of X will be denoted by 2^X.

Strongly monotone Lipschitz continuous operators. An operator $B : X \to X$ is said to be strongly monotone if there exists a real $m_B > 0$ such that

$$(Bu - Bv, u - v)_X \geq m_B \|u - v\|_X^2 \quad \text{for all } \ u, v \in X. \tag{2.1}$$

The operator B is said to be Lipschitz continuous if there exists a real $L_B > 0$ such that

$$\|Bu - Bv\|_X \leq L_B \|u - v\|_X \quad \text{for all } \ u, v \in X. \tag{2.2}$$

For any strongly monotone Lipschitz continuous operator $B : X \to X$ we shall denote by m_B and L_B the constants which appear in (2.1) and (2.2). Then, it is easy to see that $L_B \geq m_B$. Nevertheless, without loosing the generality, we shall assume in what follows that $L_B > m_B$, even if we do not mention it explicitly.

Remark 1. *Assume that $B : X \to X$ is a strongly monotone Lipschitz continuous operator. It is not difficult to check that the function $k_B : \left(0, \frac{2m_B}{L_B^2}\right) \to \mathbb{R}$ defined by*

$$k_B(\rho) := \sqrt{\rho^2 L_B^2 - 2\rho m_B + 1} \quad \text{for all } \ \rho \in \left(0, \frac{2m_B}{L_B^2}\right) \tag{2.3}$$

is well defined, takes values in the interval $(0, 1)$ and, moreover,

$$\min_{\rho \in \left(0, \frac{2m_B}{L_B^2}\right)} k_B(\rho) = k_B\left(\frac{m_B}{L_B^2}\right) = \sqrt{1 - \frac{m_B^2}{L_B^2}}. \tag{2.4}$$

Based on Remark 1 we have the following result which ensures that for every $\rho > 0$ small enough the operator $\mathrm{Id}_X - \rho B$ is a contraction on X.

Lemma 2.1. *Let $B : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for any real ρ such that*

$$0 < \rho < \frac{2m_B}{L_B^2}\,, \tag{2.5}$$

one has $k_B(\rho) \in (0, 1)$ and

$$\|(u_1 - \rho B u_1) - (u_2 - \rho B u_2)\|_X \leq k_B(\rho)\|u_1 - u_2\|_X \quad \text{for all } \ u_1,\, u_2 \in X.$$

A proof of Lemma 2.1 can be found in [29, p.22] (see also, [26, Lemma 3.3]). We also recall another lemma which asserts that a strongly monotone Lipschitz operator is invertible.

Lemma 2.2. *Let $A : X \to X$ be a strongly monotone Lipschitz continuous operator with the constants $m_A > 0$ and $L_A > 0$. Then, the operator $A : X \to X$ is invertible and its inverse $A^{-1} : X \to X$ is also strongly monotone and Lipschitz continuous with the constants $m_{A^{-1}} = \frac{m_A}{L_A^2}$ and $L_{A^{-1}} = \frac{1}{m_A}$.*

We refer to [29, p.23] for a proof of Lemma 2.2.

Remark 2. *Using Lemma 2.2 and (2.3), (2.4) it follows that the function $k_{A^{-1}} : \left(0, \frac{2m_A^3}{L_A^2}\right) \to (0, 1)$ defined by*

$$k_{A^{-1}}(\rho) := \sqrt{\rho^2 L_{A^{-1}}^2 - 2\rho m_{A^{-1}} + 1} \quad \text{for all } \ \rho \in \left(0, \frac{2m_A^3}{L_A^2}\right)$$

is well defined and satisfies

$$\min_{\rho \in \left(0, \frac{2m_A^3}{L_A^2}\right)} k_{A^{-1}}(\rho) = k_{A^{-1}}\left(\frac{m_A^3}{L_A^2}\right) = \sqrt{1 - \frac{m_A^4}{L_A^4}}\,. \tag{2.6}$$

Projection and normal cone of a convex set. Let K be a nonempty closed convex subset of X. Recall first that for every $f \in X$, the set

$$\{ u \in K : \|f-u\|_X \leq \|f-v\|_X,\, \forall v \in K \}$$

is a singleton; its unique element is called the projection (or the nearest) point of f on K and is denoted by $P_K(f)$ or $P_K f$. It is not difficult to check that for every $u, f \in X$,

$$u = P_K f \Longleftrightarrow u \in K \quad \text{and} \quad (u, v-u)_X \geq (f, v-u)_X \quad \text{for all } \ v \in K. \tag{2.7}$$

The operator $P_K : X \to K$ defined in this way, called the projection operator on K, is known to be monotone and nonexpansive, that is,

$$(P_K f_1 - P_K f_2, f_1 - f_2)_X \geq 0 \quad \text{for all } \ f_1, f_2 \in X$$

and

$$\|P_K f_1 - P_K f_2\|_X \leq \|f_1 - f_2\|_X \quad \text{for all } \ f_1, f_2 \in X. \tag{2.8}$$

Next, we recall that the outward normal cone (in the sense of convex analysis) of K is the set-valued mapping $N_K : K \to 2^X$ defined for every $u \in X$ by

$$N_K(u) := \begin{cases} \{ \xi \in X : (\xi, v-u)_X \leq 0,\, \forall v \in K \} & \text{if } u \in K, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.9}$$

It directly follows from (2.9) that for every $u, \xi \in X$

$$\xi \in N_K(u) \iff u \in K \quad \text{and} \quad (\xi, v-u)_X \leq 0 \quad \text{for all } \ v \in K. \tag{2.10}$$

Given a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ its subdifferential (in the sense of convex analysis) is the set-valued mapping $\partial\varphi : X \to 2^X$ defined for all $u \in X$ by

$$\partial\varphi(u) := \begin{cases} \{\xi \in X : (\xi, v-u)_X \leq \varphi(v) - \varphi(u),\, \forall v \in X\} & \text{if } \varphi(u) < +\infty, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.11}$$

The function φ is said to be subdifferentiable at $u \in X$ if $\partial\varphi(u) \neq \emptyset$. According to what precedes, it is clear that N_K is nothing but the subdifferential of the indicator function $\psi_K : X \to \mathbb{R} \cup \{+\infty\}$ defined for all $u \in X$ by

$$\psi_K(u) := \begin{cases} 0 & \text{if } \ u \in K, \\ +\infty & \text{if } \ u \notin K. \end{cases}$$

We now state two results involving the projection operator of a convex set. The first one is elementary and its proof, based on the equivalence (2.7), can be found in [26]. The second one uses Lemma 2.2 and the equivalences (2.7), (2.10).

Lemma 2.3. *Let K be a nonempty closed convex subset of X, $B : X \to X$ an operator and $z, \eta \in X$. Then, the following statements are equivalent:*

(a) $z = P_K\big(z - B(z-\eta)\big)$.

(b) *There exists $\rho > 0$ such that* $z = P_K\big(z - \rho B(z-\eta)\big)$.

(c) $z = P_K\big(z - \rho B(z-\eta)\big)$ *for all* $\rho > 0$.

Lemma 2.4. *Let K be a nonempty closed convex subset of X, $A : X \to X$ a strongly monotone Lipschitz continuous operator, $u, \eta \in X$ and let $z := Au + \eta$. Then, the following equivalence holds:*

$$z = P_K\big(z - A^{-1}(z-\eta)\big) \iff -u \in N_K(Au+\eta).$$

Proof. We use Lemma 2.2 to see that equality $z = Au+\eta$ implies that $A^{-1}(z-\eta) = u$. We now use this equality and equivalences (2.7), (2.10) to get

$$\begin{aligned}
& z = P_K\big(z - A^{-1}(z-\eta)\big) \iff z = P_K(z-u) \\
& \iff z \in K, \quad (z, v-z)_X \geq (z-u, v-z)_X \quad \forall\, v \in K \\
& \iff z \in K, \quad (-u, v-z)_X \leq 0 \quad \forall\, v \in K \\
& \iff -u \in N_K(z) \iff -u \in N_K(Au+\eta),
\end{aligned}$$

which completes the proof. □

Space of continuous functions and history-dependent operators. Let I be an interval of time of the form $I = [0,T]$ with $T > 0$ or the unbounded interval $\mathbb{R}_+ = [0, +\infty)$. For a real normed space $(Y, \|\cdot\|_Y)$, we denote by $C(I;Y)$ the real vector space of continuous functions defined on I with values in Y. The real vector space of continuously differentiable functions on I with values in Y is denoted by $C^1(I;Y)$. Obviously, for any function $v : I \to Y$, the inclusion $v(\cdot) \in C^1(I;Y)$ holds if and only if $v(\cdot) \in C(I;Y)$ and $\dot{v}(\cdot) \in C(I;Y)$. Here and below, $\dot{v}(\cdot)$ stands for the derivative of the function $v(\cdot)$. It is well known that for any function $v(\cdot) \in C^1(I;Y)$, the following equality holds:

$$v(t) = \int_0^t \dot{v}(s)\, ds + v(0) \quad \text{for all } \ t \in I. \tag{2.12}$$

Moreover, we recall that, if $I = [0,T]$ with $T > 0$ and $(Y, \|\cdot\|_Y)$ is a real Banach space, then the real vector spaces $C(I;Y)$ and $C^1(I;Y)$ can be organized in a canonical way as Banach spaces.

Assume now that $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are real normed spaces. Consider two operators $B : Y \to Z$ and $\mathcal{S}\colon C(I;Y) \to C(I;Z)$. For any $t \in I$ and any function $u \in C(I;Y)$, we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u : I \to Z$ at the point $t \in I$, that is, $\mathcal{S}u(t) := (\mathcal{S}u)(t)$. Moreover, $B + \mathcal{S}$

will represent a shorthand notation for the operator which associates to any function $u \in C(I;Y)$ the function $Bu(\cdot) + \mathcal{S}u(\cdot) \in C(I;Z)$.

The next definition introduces two important classes of operators defined on the space of continuous functions.

Definition 2.5. *Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be real normed spaces. An operator $\mathcal{S}\colon C(I;Y) \to C(I;Z)$ is said to be almost history-dependent provided that for every nonempty compact set $\mathcal{J} \subset I$, there exist $l^{\mathcal{S}}_{\mathcal{J}} \in [0,1)$ and $L^{\mathcal{S}}_{\mathcal{J}} > 0$ such that*

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Z \le l^{\mathcal{S}}_{\mathcal{J}} \|u_1(t) - u_2(t)\|_Y + L^{\mathcal{S}}_{\mathcal{J}} \int_0^t \|u_1(s) - u_2(s)\|_Y \, ds,$$

for all $u_1, u_2 \in C(I;Y)$ and $t \in \mathcal{J}$. The operator $\mathcal{S}$ is said to be history-dependent if it is almost history-dependent and, in addition, $l^{\mathcal{S}}_{\mathcal{J}} = 0$, for every nonempty compact set $\mathcal{J} \subset I$.

Almost history-dependent operators enjoy the following fixed point property.

Theorem 2.6. *Let Y be a real Banach space and let $\Lambda\colon C(I;Y) \to C(I;Y)$ be an almost history-dependent operator. Then, Λ has a unique fixed point, i.e., there exists a unique element $\eta^* \in C(I;Y)$ such that $\Lambda\eta^* = \eta^*$.*

The proof of Theorem 2.6 can be found in [30, p. 41–45].

3 An existence and uniqueness result

In this section, we introduce the sweeping process we are interested in, then we state and prove an existence and uniqueness result, Theorem 3.1. Throughout this section and the following one, besides the real Hilbert space X, we consider a real Hilbert space Y endowed with an inner product $(\cdot,\cdot)_Y$ and its associated norm $\|\cdot\|_Y$. We denote by $X \times Y$ the vector product space endowed with the canonical Hilbert product structure given by

$$(\xi_1, \xi_2)_{X\times Y} := (\eta_1, \eta_2)_X + (\theta_1, \theta_2)_Y \quad \text{for all } \xi_1 = (\eta_1, \theta_1),\ \xi_2 = (\eta_2, \theta_2) \in X \times Y.$$

The norm associated to the inner product $(\cdot,\cdot)_{X\times Y}$ will be denoted by $\|\cdot\|_{X\times Y}$. It is an exercise to check that it satisfies the inequalities

$$\|\xi\|_{X\times Y} \le \|\eta\|_X + \|\theta\|_Y \le \sqrt{2}\, \|\xi\|_{X\times Y} \quad \text{for all } \xi = (\eta, \theta) \in X \times Y. \tag{3.1}$$

Consider now a set-valued mapping $K : Y \times I \to 2^X$ and four operators $A : X \to X$, $B : X \to X$, $\mathcal{S} : C(I;X) \to C(I;X)$, $\mathcal{R} : C(I;X) \to C(I;Y)$. With the data above we introduce, for any $u_0 \in X$, the following sweeping process-type inclusion.

Problem 1. *Find a function $u : I \to X$ such that*

$$\begin{cases} -\dot{u}(t) \in N_{K(\mathcal{R}\dot{u}(t),t)}\big(A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)\big) & \text{for all } \ t \in I, \\ u(0) = u_0. \end{cases} \tag{3.2}$$

In the study of Problem 1, we consider the following assumptions.

($\mathcal{K}$) The set-valued mapping $K : Y \times I \to 2^X$ has nonempty closed convex values and, moreover:

(a) The mapping $P_{K(\cdot,\cdot)}u : Y \times I \to X$ is continuous, for any $u \in X$.

(b) There exists $c_0 > 0$ such that

$$\|P_{K(\theta_1,t)}u - P_{K(\theta_2,t)}u\|_X \le c_0\|\theta_1 - \theta_2\|_Y, \tag{3.3}$$

for all θ_1, $\theta_2 \in Y$, $t \in I$ and $u \in X$.

($\mathcal{A}$) The operator $A : X \to X$ is a strongly monotone Lipschitz continuous operator with constants $0 < m_A < L_A$.

($\mathcal{B}$) The operator $B : X \to X$ is Lipschitz continuous with constant $L_B > 0$.

($\mathcal{R}$) For any nonempty compact set $\mathcal{J} \subset I$, there exist two reals $l_{\mathcal{J}}^{\mathcal{R}} > 0$ and $L_{\mathcal{J}}^{\mathcal{R}} > 0$ such that

$$\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_Y \le l_{\mathcal{J}}^{\mathcal{R}}\,\|u_1(t) - u_2(t)\|_X + L_{\mathcal{J}}^{\mathcal{R}} \int_0^t \|u_1(s) - u_2(s)\|_X\, ds, \tag{3.4}$$

for all $u_1, u_2 \in C(I;X)$ and $t \in \mathcal{J}$.

($\mathcal{S}$) For any nonempty compact set $\mathcal{J} \subset I$, there exist $l_{\mathcal{J}}^{\mathcal{S}} > 0$ and $L_{\mathcal{J}}^{\mathcal{S}} > 0$ such that

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \le l_{\mathcal{J}}^{\mathcal{S}}\,\|u_1(t) - u_2(t)\|_X + L_{\mathcal{J}}^{\mathcal{S}} \int_0^t \|u_1(s) - u_2(s)\|_X\, ds, \tag{3.5}$$

for all $u_1, u_2 \in C(I;X)$ and $t \in \mathcal{J}$.

($\mathcal{U}$) $u_0 \in X$.

Under these assumptions we denote by $\widetilde{c}$ the constant defined by

$$\widetilde{c} = \max\left\{\frac{1 + \kappa_{A^{-1}}}{1 - \kappa_{A^{-1}}}, \frac{c_0}{1 - \kappa_{A^{-1}}}\right\}, \tag{3.6}$$

where

$$\kappa_{A^{-1}} := \sqrt{1 - \frac{m_A^4}{L_A^4}}\,. \tag{3.7}$$

Note that $\widetilde{c}$ depends only on A and K. Moreover, it follows from (2.6) that

$$\kappa_{A^{-1}} = \min_{\rho \in \left(0, \frac{2m_A^3}{L_A^2}\right)} k_{A^{-1}}(\rho) = k_{A^{-1}}\Big(\frac{m_A^3}{L_A^2}\Big).$$

The unique solvability for Problem 1 is given by the following existence and uniqueness result.

Theorem 3.1. *Assume that* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{R})$, $(\mathcal{S})$ *and* $(\mathcal{U})$ *hold. Moreover, assume that for any nonempty compact set* $\mathcal{J} \subset I$ *the following inequality holds:*

$$\sqrt{2}(\widetilde{c}+1)(l_{\mathcal{J}}^{\mathcal{R}} + l_{\mathcal{J}}^{\mathcal{S}}) < m_A. \tag{3.8}$$

Then, Problem 1 *has a unique solution with regularity* $u \in C^1(I;X)$.

Note that the smallness assumption (3.8) does not depend on the constant $L_{\mathcal{J}}^{\mathcal{R}}$ and $L_{\mathcal{J}}^{\mathcal{S}}$.

In order to provide the proof of Theorem 3.1, we introduce the following time-dependent inclusion problem.

Problem 2. *Find a function* $u : I \to X$ *such that*

$$-u(t) \in N_{K(\mathcal{R}u(t),t)}\big(Au(t) + \mathcal{S}u(t)\big) \quad \text{for all } \ t \in I. \tag{3.9}$$

In the study of this auxiliary problem we have the following result which has an interest in its own.

Theorem 3.2. *Assume that* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{R})$ *and* $(\mathcal{S})$ *hold. Moreover, assume that for any nonempty compact set* $\mathcal{J} \subset I$ *the inequality* (3.8) *holds. Then, Problem* 2 *has a unique solution with regularity* $u \in C(I;X)$.

The proof of Theorem 3.2 will be provided in the next section. Here we use this theorem in order to prove the unique solvability of Problem 1.

Proof of Theorem 3.1. We first introduce the operator $\widetilde{\mathcal{S}} : C(I;X) \to C(I;X)$ defined through the following equality

$$\widetilde{\mathcal{S}}v(t) := B\Big(\int_0^t v(s)\,ds + u_0\Big) + \mathcal{S}v(t) \quad \text{for all } v \in C(I;X), \text{ all } t \in I. \tag{3.10}$$

Existence. We use assumptions $(\mathcal{S})$ and $(\mathcal{B})$ to see that for any nonempty compact set $\mathcal{J} \subset I$, any functions v_1, $v_2 \in C(I;X)$ and any $t \in I$, the inequality below holds:

$$\|\widetilde{\mathcal{S}}v_1(t) - \widetilde{\mathcal{S}}v_2(t)\|_X \le l_{\mathcal{J}}^{\mathcal{S}}\|v_1(t) - v_2(t)\|_X + (L_B + L_{\mathcal{J}}^{\mathcal{S}}) \int_0^t \|v_1(s) - v_2(s)\|_X\,ds.$$

This implies that $\widetilde{\mathcal{S}} : C(I;X) \to C(I;X)$ satisfies the assumption $(\mathcal{S})$ with $l_{\mathcal{J}}^{\widetilde{\mathcal{S}}} = l_{\mathcal{J}}^{\mathcal{S}}$. Therefore since the smallness assumption (3.8) holds, we are in a position to apply Theorem 3.2 in order to obtain the existence of a (unique) function $v \in C(I;X)$ which satisfies

$$-v(t) \in N_{K(\mathcal{R}v(t),t)}\big(Av(t) + \widetilde{\mathcal{S}}v(t)\big) \quad \text{for all } \ t \in I.$$

Then, the function $u : I \to X$ defined by

$$u(t) := \int_0^t v(s)\, ds + u_0 \quad \text{for all } \ t \in I$$

is obviously a C^1-solution of Problem 1. This proves the existence part of the theorem.

Uniqueness. To prove the uniqueness part, we consider two solutions u_1, $u_2 \in C^1(I;X)$ of Problem 1. It is readily seen that for each $i \in \{1,2\}$, the function $\dot{u}_i \in C(I;X)$ satisfies

$$-\dot{u}_i(t) \in N_{K(\mathcal{R}\dot{u}_i(t),t)}\big(A\dot{u}_i(t) + \widetilde{\mathcal{S}}\dot{u}_i(t)\big) \quad \text{for all } \ t \in I.$$

Then, by virtue of Theorem 3.2, we know that $\dot{u}_1 = \dot{u}_2$. It remains to invoke (2.12) and the initial conditions $u_1(0) = u_2(0) = u_0$ to deduce that $u_1(t) = u_2(t)$ for all $t \in I$, which concludes the proof. □

We now present the following consequences of Theorem 3.1.

Corollary 3.3. *Assume that $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{U})$ hold. Moreover, assume that $F : X \to Y$ is a Lipschitz continuous operator and $\mathcal{S} : C(I;X) \to C(I;X)$ is a history-dependent operator. Then, there exists a unique function $u \in C^1(I;X)$ such that*

$$\begin{cases} -\dot{u}(t) \in N_{K(Fu(t),t)}\big(A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)\big) & \text{for all } \ t \in I, \\ u(0) = u_0. \end{cases} \tag{3.11}$$

Proof. Consider the operator $\mathcal{R} : C(I;X) \to C(I;Y)$ defined for every $v \in C(I;X)$ by the equality

$$\mathcal{R}v(t) := F\Big(\int_0^t v(s)\, ds + u_0\Big) \quad \text{for all } t \in I. \tag{3.12}$$

Then, for any function $u \in C^1(I;X)$ with $u(0) = u_0$ we have that $\mathcal{R}\dot{u}(t) = Fu(t)$ for all $t \in I$. This implies that a function $u \in C^1(I;X)$ is a solution of (3.11) if and only if u is a solution of (3.2). Moreover, the operator defined through (3.12) is obviously history-dependent as well as $\mathcal{S}$. It follows from here that the operators $\mathcal{R}$ and $\mathcal{S}$ fulfill the conditions $(\mathcal{R})$ and $(\mathcal{S})$ with $l_{\mathcal{J}}^{\mathcal{R}} = l_{\mathcal{J}}^{\mathcal{S}} = 0$. Corollary 3.3 is now a direct consequence of Theorem 3.1. □

Corollary 3.4. *Assume that $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{S})$ and $(\mathcal{U})$ hold. Moreover, assume that the set-valued mapping $K : I \to 2^X$ has nonempty closed convex values and the* *mapping $P_{K(\cdot)}u$ is continuous on I, for every $u \in X$. In addition, assume that for any nonempty compact set $\mathcal{J} \subset I$ the following inequality holds:*

$$\frac{2\sqrt{2}}{1-\kappa_{A^{-1}}}\, l_{\mathcal{J}}^{\mathcal{S}} < m_A. \tag{3.13}$$

Then, there exists a unique function $u \in C^1(I;X)$ such that

$$\begin{cases} -\dot{u}(t) \in N_{K(t)}\big(A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)\big) \quad \text{for all } \ t \in I, \\ u(0) = u_0. \end{cases}$$

Proof. Note that in this case case the set-valued mapping K does not depend on the operator $\mathcal{R}$ and, therefore, the assumption $(\mathcal{K})$ is satisfied with $c_0 = 0$. Then, it follows from (3.6) that $\widetilde{c} = \frac{1+\kappa_{A-1}}{1-\kappa_{A-1}}$ which implies that $\widetilde{c}+1 = \frac{2}{1-\kappa_{A-1}}$ and, since in this case we can take $l_{\mathcal{J}}^{\mathcal{R}} = 0$, we deduce that the smallness assumption (3.13) guarantees that (3.8) holds. Corollary 3.4 (which corresponds to the main result in [26]) is now a direct consequence of Theorem 3.1. □

We end this section with two examples of families of convex sets satisfying assumption $(\mathcal{K})$. A first example is the following.

Example 1. *Let M be a closed linear subspace of X, $A : X \to M$ the projector onto M and $k : Y \times I \to \mathbb{R}_+$ a Lipschitz continuous function. Let $K : I \to 2^X$ be the set-valued mapping defined by*

$$K(\theta, t) := \{\, u \in X : \|Au\|_X \le k(\theta, t) \,\} \quad \text{for all } \ \theta \in Y, \ t \in I.$$

Then, the set-valued mapping K satisfies condition $(\mathcal{K})$.

The proof of this statement is as follows. First, it is routine to check that $K(\theta, t) \subset X$ is nonempty closed and convex, for any $\theta \in Y$ and $t \in I$. Next, assume that θ_1, $\theta_2 \in Y$, t_1, $t_2 \in I$ and $u \in X$. Then, using [26, Proposition 5.1], it follows that

$$\|P_{K(\theta_1,t_1)}u - P_{K(\theta_2,t_2)}u\|_X \le |k(\theta_1, t_1) - k(\theta_2, t_2)|\,. \tag{3.14}$$

On the other hand, since $k(\cdot)$ is a Lipschitz continuous function, we can find some real $L_k > 0$ such that

$$|k(\theta_1, t_1) - k(\theta_2, u_2)| \le L_k\big(\|\theta_1 - \theta_2\|_Y + |t_1 - t_2|\big). \tag{3.15}$$

It remains to combine inequalities (3.14) *and* (3.15) *to see that condition $(\mathcal{K})$ is satisfied.*

Example 2. *Let $K_0 : Y \to 2^X$ be a set-valued mapping with nonempty closed convex values which satisfies the following property: there exists $c_0 > 0$ such that*

$$\|P_{K_0(\theta_1)}u - P_{K_0(\theta_2)}u\|_X \le c_0\, \|\theta_1 - \theta_2\|_Y \tag{3.16}$$

for all θ_1, $\theta_2 \in Y$ and $u \in X$. Moreover, let $f \in C(I;X)$ and let $K : Y \times I \to 2^X$ be the set-valued mapping defined by

$$K(\theta,t) = K_0(\theta) + f(t) \quad \text{for all } \ \theta \in Y,\, t \in I. \tag{3.17}$$

Then, the set-valued mapping K satisfies assumption ($\mathcal{K}$).

The proof of this statement is as follows. First, it clear that $K(\theta,t) \subset X$ is nonempty closed and convex set, for any $\theta \in Y$ and $t \in I$. Next, assume that θ_1, $\theta_2 \in Y$, t_1, $t_2 \in I$ and $u \in X$. Then, using [26, Proposition 5.3] it follows that

$$P_{K(\theta_1,t)}u = P_{K_0(\theta_1)}\big(u - f(t_1)\big) + f(t_1), \quad P_{K(\theta_2,t)}u = P_{K_0(\theta_2)}\big(u - f(t_2)\big) + f(t_2)$$

and, therefore, using (2.8) *and* (3.16) *we deduce that*

$$\begin{aligned}
&\|P_{K(\theta_1,t_1)}u - P_{K(\theta_2,t_2)}u\|_X \\
&\le \|P_{K_0(\theta_1)}\big(u - f(t_1)\big) - P_{K_0(\theta_2)}\big(u - f(t_1)\big)\|_X \\
&\quad + \|P_{K_0(\theta_2)}(u - f(t_1)) - P_{K_0(\theta_2)}\big(u - f(t_2))\|_X + \|f(t_1) - f(t_2)\|_X \\
&\le c_0\|\theta_1 - \theta_2\|_Y + 2\,\|f(t_1) - f(t_2)\|_X.
\end{aligned}$$

This inequality combined with regularity $f \in C(I;X)$ shows that condition ($\mathcal{K}$) is satisfied.

Note that the sweeping process considered in [3] was of the form (3.2). There, the set-valued mapping $K : Y \times I \to 2^X$ was assumed to be of the form (3.17) with a mapping $K_0(\cdot)$ which has a particular structure and satisfies conditions in Example 2. We conclude from here that Theorem 3.1 extends our results previously obtained in [3].

4 Proof of Theorem 3.2

The proof of Theorem 3.2 is carried out in several steps, based on a number of preliminary results that we present in what follows.

Lemma 4.1. *Let K be a nonempty closed convex subset of X and let $B : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for each $\eta \in X$, there exists a unique element $z_\eta \in X$ such that*

$$z_\eta = P_K\big(z_\eta - B(z_\eta - \eta)\big). \tag{4.1}$$

Proof. Let ρ be a real such that (2.5) hold. Fix any $\eta \in X$ and consider the operator $\Lambda_\rho : X \to X$ defined by

$$\Lambda_\rho z := P_K\big(z - \rho B(z - \eta)\big) \quad \text{for all } z \in X.$$

Let z_1, $z_2 \in X$ and set $u_i := z_i - \eta$, for $i = 1, 2$. We use the definition of Λ_ρ, the nonexpansivity (2.8) of the projection operator P_K and Lemma 2.1 to see that

$$\begin{aligned}\|\Lambda_\rho z_1 - \Lambda_\rho z_2\|_X &\le \|\big(z_1 - \rho B(z_1 - \eta)\big) - \big(z_2 - \rho B(z_2 - \eta)\big)\|_X \\ &= \|(u_1 - u_2) - \rho\big(Bu_1 - Bu_2\big)\|_X \\ &= \|(u_1 - \rho Bu_1) - (u_2 - \rho Bu_2)\|_X \\ &\le k_B(\rho)\|u_1 - u_2\|_X,\end{aligned}$$

where $k_B(\rho) \in (0, 1)$ is given by (2.3). Therefore, taking into account the equality $u_1 - u_2 = z_1 - z_2$ we deduce that Λ_ρ is a contraction on the Hilbert space X. We now use the Banach fixed point theorem to deduce that there exists a unique $z_\eta \in X$ such that $\Lambda_\rho z_\eta = z_\eta$. It remains to use the implication $(b) \Longrightarrow (a)$ in Lemma 2.3 to conclude the proof. □

Lemma 4.2. *Assume that* $(\mathcal{K})$ *holds. Let* $B : X \to X$ *be a strongly monotone Lipschitz continuous operator and let* $\xi = (\eta, \theta) \in C(I; X \times Y)$*. Then, there exists a unique function* $z_\xi : I \to X$ *such that*

$$z_\xi(t) = P_{K(\theta(t),t)}\Big(z_\xi(t) - B\big(z_\xi(t) - \eta(t)\big)\Big) \quad \text{for all } \ t \in I. \tag{4.2}$$

Moreover, the function z_ξ *is continuous, i.e.,* $z_\xi \in C(I; X)$*.*

Proof. The existence and uniqueness of a function $z_\xi(\cdot)$ satisfying (4.2) is a direct consequence of Lemma 4.1.

We now prove the continuity of the function $z_\xi(\cdot)$ and, to this end we consider $t \in I$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of elements of I which converges to t. Fix any $n \in \mathbb{N}$. Denote $\eta_n := \eta(t_n)$, $\theta_n := \theta(t_n)$, $K_n := K(\theta(t_n), t_n)$ and $z_n := z_\xi(t_n)$. Set also $K := K(\theta, t)$, $z := z_\xi(t)$ and $\eta_\infty := \eta(t)$. With these notation we see that

$$z_n = P_{K_n}\big(z_n - B(z_n - \eta_n)\big) \quad \text{and} \quad z = P_K\big(z - B(z - \eta_\infty)\big).$$

Using Lemma 2.3, it follows that

$$z_n = P_{K_n}\big(z_n - \rho B(z_n - \eta_n)\big) \quad \text{and} \quad z = P_K\big(z - \rho B(z - \eta_\infty)\big), \tag{4.3}$$

for every real $\rho > 0$. Fix any real $\rho > 0$ such that (2.5) holds and let $k_B(\rho) \in (0, 1)$ be defined by (2.3). Moreover, let

$$\omega_n := z_n - \rho B(z_n - \eta_n), \qquad \omega := z - \rho B(z - \eta_\infty). \tag{4.4}$$

Then, (4.3) implies that

$$z_n = P_{K_n}\omega_n \quad \text{and} \quad z = P_K\omega,$$

hence

$$\|z_n - z\|_X \le \|P_{K_n}\omega_n - P_{K_n}\omega\|_X + \|P_{K_n}\omega - P_K\omega\|_X\,. \tag{4.5}$$

We now estimate each of the two terms in the right hand side of (4.5). To this end we set

$$u_n := z_n - \eta_n \quad \text{and} \quad u := z - \eta_\infty. \tag{4.6}$$

Thanks to (2.8), (4.4) and (4.6) we see that

$$\begin{aligned}\|P_{K_n}\omega_n - P_{K_n}\omega\|_X &\le \|\omega_n - \omega\|_X = \|z_n - \rho B(z_n - \eta_n) - z + \rho B(z - \eta_\infty)\|_X \\ &= \|u_n - u + \rho(Bu - Bu_n) + \eta_n - \eta_\infty\|_X \\ &\le \|u_n - u + \rho(Bu - Bu_n)\|_X + \|\eta_n - \eta_\infty\|_X. \end{aligned}\tag{4.7}$$

Next, Lemma 2.1 yields

$$\|u_n - u - \rho(Bu_n - Bu)\|_X \le k_B(\rho)\|u_n - u\|_X. \tag{4.8}$$

We now combine inequalities (4.7) and (4.8) to obtain that

$$\|P_{K_n}\omega_n - P_{K_n}\omega\|_X \le k_B(\rho)\|u_n - u\|_X + \|\eta_n - \eta_\infty\|_X. \tag{4.9}$$

On the other hand, (4.6) implies that

$$\|u_n - u\|_X \le \|z_n - z\|_X + \|\eta_n - \eta_\infty\|_X \tag{4.10}$$

We now use inequalities (4.5), (4.9) and (4.10) to find that

$$\big(1 - k_B(\rho)\big)\|z_n - z\|_X \le \big(1 + k_B(\rho)\big)\|\eta_n - \eta_\infty\|_X + \|P_{K_n}\omega - P_K\omega\|_X. \tag{4.11}$$

Now, recall that continuity of the function $\eta : I \to X$ implies that $\|\eta_n - \eta_\infty\|_X \to 0$ while assumption $(\mathcal{K})$(a) ensures that $\|P_{K_n}\omega - P_K\omega\|_X \to 0$, as $n \to \infty$. Therefore, using (4.11) and the inclusion $k_B(\rho) \in (0,1)$, we get that $z_n = z_\xi(t_n) \to z_\xi(t) = z$ in X, as $n \to \infty$. This shows that the function $z_\xi : I \to X$ is continuous, which completes the proof. □

Lemma 4.3. *Assume that* $(\mathcal{K})$ *and* $(\mathcal{A})$ *hold. Let* $\xi = (\eta, \theta) \in C(I; X \times Y)$. *Then, there exists a unique function* $u_\xi : I \to X$ *such that*

$$-u_\xi(t) \in N_{K(\theta(t),t)}\big(Au_\xi(t) + \eta(t)\big) \quad \text{for all } \ t \in I. \tag{4.12}$$

Moreover, the function u_ξ *is continuous, i.e.,* $u_\xi \in C(I; X)$.

Proof. Denote by $z_\xi \in C(I; X)$ the function obtained in Lemma 4.2 with $B := A^{-1}$, where A^{-1} represents the inverse operator of A. Then, using (4.2) we find that

$$z_\xi(t) = P_{K(\theta(t),t)}\Big(z_\xi(t) - A^{-1}\big(z_\xi(t) - \eta(t)\big)\Big) \quad \text{for all } \ t \in I. \tag{4.13}$$

Consider the function $u_\xi : I \to X$ defined by

$$u_\xi(t) := A^{-1}\big(z_\xi(t) - \eta(t)\big) \quad \text{for all } \ t \in I$$

and note that $u_\xi \in C(I;X)$ and satisfies

$$z_\xi(t) = Au_\xi(t) + \eta(t) \quad \text{for all } \ t \in I \tag{4.14}$$

The existence part of the Lemma 4.3 follows now from equalities (4.13) and (4.14) and Lemma 2.4.

To prove the uniqueness part we consider two functions u_1, $u_2 : I \to X$ such that

$$-u_1(t) \in N_{K(\theta(t),t)}\big(Au_1(t) + \eta(t)\big) \quad \text{and} \quad -u_2(t) \in N_{K(\theta(t),t)}\big(Au_2(t) + \eta(t)\big),$$

for every $t \in I$. Fix any $t \in I$. Then, for $i = 1, 2$, we have

$$Au_i(t) + \eta(t) \in K\big(\theta(t), t\big), \quad \big(u_i(t), Au_i(t) + \eta(t) - v\big)_X \leq 0 \quad \text{for all } \ v \in K\big(\theta(t), t\big).$$

This implies that

$$\Big(u_1(t), Au_1(t) + \eta(t) - \big(Au_2(t) + \eta(t)\big)\Big)_X \leq 0,$$

$$\Big(u_2(t), Au_2(t) + \eta(t) - \big(Au_1(t) + \eta(t)\big)\Big)_X \leq 0$$

and, adding these inequalities, we deduce that

$$\big(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\big)_X \leq 0.$$

We now use the strong monotonicity of the operator A to get that $u_1(t) = u_2(t)$ which completes the proof. □

Under the assumptions of Lemma 4.3, we consider the operator $\Lambda : C(I; X \times Y) \to C(I; X \times Y)$ defined by

$$\Lambda\xi := (\mathcal{S}u_\xi, \mathcal{R}u_\xi) \quad \text{for all } \ \xi = (\eta, \theta) \in C(I; X \times Y). \tag{4.15}$$

We have the following result.

Lemma 4.4. *Assume that* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{S})$ *and* $(\mathcal{R})$ *hold. Then, the operator* Λ *has a unique fixed point* $\xi^* \in C(I; X \times Y)$.

Proof. Let $\xi_1 = (\eta_1, \theta_1)$, $\xi_2 = (\eta_2, \theta_2) \in C(I; X \times Y)$ and denote by u_1, u_2 the functions obtained in Lemma 4.3 for $\xi = \xi_1$ and $\xi = \xi_2$, respectively. We have $u_1 \in C(I;X)$, $u_2 \in C(I;X)$ along with

$$-u_1(t) \in N_{K(\theta_1(t),t)}\big(Au_1(t) + \eta_1(t)\big), \quad -u_2(t) \in N_{K(\theta_2(t),t))}\big(Au_2(t) + \eta_2(t)\big), \tag{4.16}$$

for every $t \in I$. Let $z_1, z_2 : I \to X$ be the functions defined by

$$z_1(t) := Au_1(t) + \eta_1(t), \quad z_2(t) := Au_2(t) + \eta_2(t) \quad \text{for all}\, t \in I. \tag{4.17}$$

We fix $t \in I$ and, for simplicity, we use the notation

$$K_1(t) := K\big(\theta_1(t), t\big), \quad K_2(t) := K\big(\theta_2(t), t\big).$$

From (4.16) and Lemma 2.4 we see that, for each $i \in \{1, 2\}$,

$$z_i(t) = P_{K_i}\Big(z_i(t) - A^{-1}\big(z_i(t) - \eta_i(t)\big)\Big).$$

Putting $B := A^{-1}$ and applying Lemma 2.3 we have, for each $i \in \{1, 2\}$,

$$z_i(t) = P_{K_i}\Big(z_i(t) - \rho B\big(z_i(t) - \eta_i(t)\big)\Big) \quad \text{for all} \;\; \rho > 0. \tag{4.18}$$

Next, with $\rho := \frac{m_A^3}{L_A^2}$, we derive from Lemma 2.1 and Remark 2 that $\mathrm{Id}_X - \rho B$ is a contraction on X, namely it is a $\kappa_{A^{-1}}$-Lipschitz continuous mapping with $\kappa_{A^{-1}}$ given by (3.7). Set

$$\omega_i(t) := z_i(t) - \rho B(z_i(t) - \eta_i(t)) \qquad \text{for each} \;\; i \in \{1, 2\} \tag{4.19}$$

and note that (4.18) implies that

$$z_i(t) = P_{K_i}(\omega_i(t)) \qquad \text{for each} \;\; i \in \{1, 2\}. \tag{4.20}$$

Using (2.8) and (4.20), it is not difficult to check that

$$\|z_1(t) - z_2(t)\|_X \le \|\omega_1(t) - \omega_2(t)\|_X + \|P_{K_1}(\omega_2(t)) - P_{K_2}(\omega_2(t))\|_X.$$

The above $\kappa_{A^{-1}}$-Lipschitz property and (4.19) yield

$$\begin{aligned}
\|\omega_1(t) - \omega_2(t)\|_X &\le \|(\mathrm{Id}_X - \rho B)(z_1(t) - \eta_1(t)) - (\mathrm{Id}_X - \rho B)(z_2(t) - \eta_2(t))\|_X \\
&\quad + \|\eta_1(t) - \eta_2(t)\|_X \\
&\le \kappa_{A^{-1}}\|z_1(t) - z_2(t)\|_X + (1 + \kappa_{A^{-1}})\|\eta_1(t) - \eta_2(t)\|_X
\end{aligned}$$

Combining these inequalities we arrive to

$$\begin{aligned}
(1 - \kappa_{A^{-1}})\|z_1(t) - z_2(t)\|_X &\le (1 + \kappa_{A^{-1}})\|\eta_1(t) - \eta_2(t)\|_X \\
&\quad + \|P_{K_1}(\omega_2(t)) - P_{K_2}(\omega_2(t))\|_X.
\end{aligned} \tag{4.21}$$

Taking into account (4.21) and assumption $(\mathcal{K})$(b), we then see

$$\|z_1(t) - z_2(t)\|_X \le \frac{1 + \kappa_{A^{-1}}}{1 - \kappa_{A^{-1}}}\|\eta_1(t) - \eta_2(t)\|_X + \frac{c_0}{1 - \kappa_{A^{-1}}}\|\theta_1(t) - \theta_2(t)\|_Y \tag{4.22}$$

and, using (3.6) we find that

$$\|z_1(t) - z_2(t)\|_X \le \widetilde{c}\,\big(\|\eta_1(t) - \eta_2(t)\|_X + \|\theta_1(t) - \theta_2(t)\|_Y\big). \tag{4.23}$$

On the other hand, (4.17) and Lemma 2.2 show that

$$\|u_1(t) - u_2(t)\|_X \le \frac{1}{m_A}\big(\|z_1(t) - z_2(t)\|_X + \|\eta_1(t) - \eta_2(t)\|_X\big). \tag{4.24}$$

Therefore, combining inequalities (4.23) and (4.24) we find that

$$\|u_1(t) - u_2(t)\|_X \le \frac{\widetilde{c}+1}{m_A}\big(\|\eta_1(t) - \eta_2(t)\|_X + \|\theta_1(t) - \theta_2(t)\|_Y\big).$$

Using (3.1) and keeping in mind that t has been arbitrarily choosen, we arrive to

$$\|u_1(t) - u_2(t)\|_X \le \frac{\sqrt{2}\,(\widetilde{c}+1)}{m_A}\,\|\xi_1(t) - \xi_2(t)\|_{X\times Y} \quad \text{for all}\, t \in I. \tag{4.25}$$

Next, let $\mathcal{J}$ be a nonempty compact subset of I and let $t \in \mathcal{J}$. Then, using (4.15), (3.1) and assumptions $(\mathcal{S})$, $(\mathcal{R})$ we get

$$\begin{aligned} \|\Lambda\xi_1(t) - \Lambda\xi_2(t)\|_{X\times Y} &\le \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X + \|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_Y \\ &\le (l_{\mathcal{J}}^{\mathcal{S}} + l_{\mathcal{J}}^{\mathcal{R}})\|u_1(t) - u_2(t)\|_X + (L_{\mathcal{J}}^{\mathcal{S}} + L_{\mathcal{J}}^{\mathcal{R}})\int_0^t \|u_1(s) - u_2(s)\|_X ds. \end{aligned} \tag{4.26}$$

It remains to combine inequalities (4.26) and (4.25) and to invoke the smallness assumption (3.8) to see that the operator Λ enjoys the almost history-dependent property. Lemma 4.4 is now a direct consequence of Theorem 2.6. $\square$

We are now in a position to provide the proof of Theorem 3.2.

Proof. Let $\xi^* = (\eta^*, \theta^*)$ be the unique fixed point of the operator Λ obtained in Lemma 4.4 and let $u^* = u_{\xi^*} \in C(I;X)$ be the solution of the inclusion (4.12) obtained in Lemma 4.3. We have

$$-u^*(t) \in N_{K(\theta^*(t),t)}\big(Au^*(t) + \eta^*(t)\big) \quad \text{for all}\;\; t \in I \tag{4.27}$$

and, since $\xi^* = \Lambda\xi^*$, the definition (4.15) of the operator Λ implies that

$$\eta^*(t) = \mathcal{S}u^*(t), \qquad \theta^*(t) = \mathcal{R}u^*(t) \quad \text{for all}\;\; t \in I. \tag{4.28}$$

We now combine inclusion (4.27) with equalities (4.28) to see that $u^\star$ is a solution to Problem 2. This proves the existence part of Theorem 3.1. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ, guaranteed by Lemma 4.3. It also can be obtained directly, by using a Gronwall-type argument. $\square$

5 A viscoelastic contact problem

Theorem 3.1 and its consequences are useful in the variational analysis of various boundary boundary-value problems. In this section we use Theorem 3.1 in the study of a mathematical model which describes the equilibrium of a viscoelastic body in contact of with a deformable obstacle, the so-called foundation. The frictional contact conditions we use are classical and have been already considered in various papers, including [14, 29]. Nevertheless, here we associate them to a more general constitutive law. The problem under consideration is stated as follows.

Problem 3. *Find a displacement field* $\boldsymbol{u}\colon \Omega\times I \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega\times I \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{F}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big) \qquad \text{in } \Omega, \tag{5.1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{5.4}$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - g) \qquad \text{on } \Gamma_3, \tag{5.5}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \le p_\tau(u_\nu(t) - g),\\ &-\boldsymbol{\sigma}_\tau(t) = p_\nu(u_\nu(t) - g)\, \frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau(t) \neq 0 \end{aligned}\right\} \qquad \text{on } \Gamma_3 \tag{5.6}$$

for all $t \in I$ *and, moreover,*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{5.7}$$

A brief description of the notation used in this problem and below in this section is the following. First $d \in \{2, 3\}$ and $\mathbb{S}^d$ stands for the space of second order symmetric tensors on $\mathbb{R}^d$. Moreover "$\cdot$" and $\|\cdot\|$ represent the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, and I denotes a given interval of time of the form $I = [0, T]$ with $T > 0$ or $I = [0, +\infty)$. In addition, $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary divided intro three mutually disjoint measurable sets Γ_1, Γ_2 and Γ_3, such that the $d-1$ dimensional Lebesgue measure of Γ_1 is positive. The domain Ω represents the reference configuration of a viscoelastic body acted upon by body forces and surface tractions. Equation (5.1) is the viscoelastic constitutive law in which $\mathcal{F}$ is a given function, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain field and, as usual, the dot above represents the derivative with respect to the time variable. Note that here as well as in various places below we skip the dependence of various function with respect to the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equation (5.2) is the equation of equilibrium in which Div denotes the divergence operator and $\boldsymbol{f}_0$ represents the density of body forces. Condition (5.3) is the displacement condition which shows that the body is fixed on Γ_1 and condition (5.4) is the traction boundary condition, in which $\boldsymbol{f}_2$ represents the density of surface tractions acting on Γ_2. Here and below $\boldsymbol{\nu}$ denotes the outward unit normal at Γ. Condition (5.5) and (5.6) represent the frictional contact conditions with normal compliance in which g is the initial gap and p_ν, p_τ are given positive functions which will be described below. Moreover, the indices ν and τ indicate the normal and tangential components of vectors and tensors, i.e., for instance, $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$. Finally, condition (5.7) is the initial condition in which $\boldsymbol{u}_0$ denotes the initial displacement.

In the study of Problem 3 we use the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\}$$

for the displacement field and the space

$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,\}$$

for the stress and the strain fields. The space Q is a real Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx \tag{5.8}$$

and the associated norm $\|\cdot\|_Q$. The space V is a real Hilbert space endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \tag{5.9}$$

and the associated norm $\|\cdot\|_V$ where, here and below, $\boldsymbol{\varepsilon}(\boldsymbol{v})$ denotes the symmetric part of the gradient of $\boldsymbol{v}$. For an element $\boldsymbol{v} \in V$ we use notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential components on the boundary, i.e., $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, respectively. Moreover, we recall that the Sobolev trace theorem yields

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_{tr}\,\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V, \tag{5.10}$$

c_{tr} being a positive constant which depends on Ω, Γ_1 and Γ_3.

We now list the assumptions on the problem data. First, we assume that the constitutive function $\mathcal{F}$ satisfies the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{F} : \Omega\times\mathbb{S}^d\times\mathbb{S}^d \to \mathbb{S}^d.\\[2mm]
\text{(b) The mapping } \boldsymbol{x}\mapsto\mathcal{F}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta}) \text{ belongs to } Q\\
\qquad \text{for all } \boldsymbol{\xi}\in Q,\ \boldsymbol{\eta}\in Q.\\[2mm]
\text{(c) The mapping } t\mapsto\mathcal{F}(\boldsymbol{\xi}(t),\boldsymbol{\eta}(t)) \text{ belongs to } C(I;Q)\\
\qquad \text{for all } \boldsymbol{\xi}\in C(I;Q),\ \boldsymbol{\eta}\in C(I;Q).\\[2mm]
\text{(d) There exists three operators } A,\ B,\ \mathcal{S} \text{ which satisfy conditions}\\
\qquad (\mathcal{A}),\ (\mathcal{B}) \text{ and } (\mathcal{S}) \text{ on the space } Q, \text{ respectively, such that}\\
\qquad\quad (\mathcal{F}(\boldsymbol{\xi}(t),\boldsymbol{\eta}(t)),\boldsymbol{\tau})_Q = (A\boldsymbol{\xi}(t) + B\boldsymbol{\eta}(t) + \mathcal{S}\boldsymbol{\xi}(t),\boldsymbol{\tau})_Q\\
\qquad\qquad \text{for all } \boldsymbol{\xi}\in C(I;Q),\ \boldsymbol{\eta}\in C(I;Q),\ t\in I,\ \boldsymbol{\tau}\in Q.
\end{array}\right. \tag{5.11}$$

A typical example of constitutive law of the form (5.1) is given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))\,ds, \tag{5.12}$$

where $\mathcal{A} : \Omega \times I \to \mathbb{S}^d$ is the viscosity operator, $\mathcal{B} : \Omega \times I \to \mathbb{S}^d$ is the elasticity operator and $\mathcal{C} : \Omega \times I \times \mathbb{S}^d \to \mathbb{S}^d$ is the relaxation tensor. Note that, when $\mathcal{C}$ vanishes, equation (5.12) reduces to the well-known Kelvin-Voigt constitutive law. Following the arguments in [30], it is easy to see that under appropriate assumptions on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the corresponding function $\mathcal{F}$ satisfisfies condition (5.11) with operators A, B and $\mathcal{S}$ defined as follows:

$$\begin{aligned}
&(A\boldsymbol{\xi}, \boldsymbol{\tau})_Q = \int_\Omega \mathcal{A}\boldsymbol{\xi} \cdot \boldsymbol{\tau}\,dx \quad \text{for all } \ \boldsymbol{\xi},\, \boldsymbol{\tau} \in Q,\\
&(B\boldsymbol{\eta}, \boldsymbol{\tau}) = (\mathcal{B}\boldsymbol{\eta}, \boldsymbol{\tau})_Q \quad \text{for all } \ \boldsymbol{\eta}, \boldsymbol{\tau} \in Q,\\
&(\mathcal{S}\boldsymbol{\xi}(t), \boldsymbol{\tau})_Q = (\int_0^t \mathcal{C}(t-s)\boldsymbol{\xi}(s))\,ds, \boldsymbol{\tau})_Q \qquad \text{for all } \ \boldsymbol{\xi} \in C(I;Q),\, \boldsymbol{\tau} \in Q.
\end{aligned}$$

Additional examples of constitutive laws (5.1) in which condition (5.11) is satisfied can be construced by using rheological arguments.

The normal compliance functions p_e ($e = \nu$, τ) are such that

$$\left\{\begin{array}{l}
p_e\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}.\\
\text{(a) There exists } L_e > 0 \text{ such that}\\
\qquad |p_e(\boldsymbol{x}, r_1) - p_e(\boldsymbol{x}, r_2)| \le L_e|r_1 - r_2|\\
\qquad\quad \text{for all } \ r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R},\\
\text{(c) } p_e(\boldsymbol{x}, r) = 0 \ \text{ for all } r \le 0, \ p_e(\boldsymbol{x}, r) \ge 0\\
\qquad \text{for all } r \ge 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{5.13}$$

Finally, the rest of the data satisfy the following conditions.

$$\boldsymbol{f}_0 \in C(I; L^2(\Omega)^d). \tag{5.14}$$

$$\boldsymbol{f}_2 \in C(I; L^2(\Gamma_2)^d). \tag{5.15}$$

$$g \in L^2(\Gamma_3) \quad \text{and} \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

$$\boldsymbol{u}_0 \in V. \tag{5.16}$$

Under these assumptions we consider the functions $j\colon L^2(\Gamma_3) \times V \to \mathbb{R}$, $\boldsymbol{f}\colon I \to V$, the set-valued mapping $\Sigma : L^2(\Gamma_3) \times I \to Q$ and the element $\boldsymbol{\omega}_0$ defined by

$$j(\theta, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(\theta - g) v_\nu \, da + \int_{\Gamma_3} p_\tau(\theta - g) \, \|\boldsymbol{v}_\tau\| \, da \quad \text{for all } \ \boldsymbol{v} \in V, \tag{5.17}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \qquad \text{for all } \boldsymbol{v} \in V, \ t \in I, \tag{5.18}$$

$$\Sigma(\theta, t) = \{ \, \boldsymbol{\tau} \in Q \ : \ (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta, \boldsymbol{v}) \geq (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \, \boldsymbol{v} \in V \, \} \tag{5.19}$$
$$\text{for all } \ \theta \in L^2(\Gamma_3), \, t \in I.$$

$$\boldsymbol{\omega}_0 = \boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{5.20}$$

Related to these notation, we have the following results that we state here and prove in the next section.

Lemma 5.1. *There exists an history-dependent operator $\mathcal{R} : C(I; Q) \to C(I; L^2(\Gamma_3))$ such that for any $\boldsymbol{\omega} \in C^1(I; Q)$ and $\boldsymbol{u} \in C^1(I; V)$ the following implication hold:*

$$\boldsymbol{\omega}(t) = \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \forall \, t \in I \implies u_\nu(t) = \mathcal{R}\dot{\boldsymbol{\omega}}(t) \quad \forall \, t \in I.$$

Lemma 5.2. *The set-valued mapping $\Sigma : L^2(\Gamma_3) \times I \to 2^Q$ satisfies assumption $(\mathcal{K})$ on the spaces $X = Q$ and $Y = L^2(\Gamma_3)$.*

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ represents a regular solution of Problem 3 and let $\boldsymbol{v} \in V$, $t \in I$ be arbitrary fixed. Then, using integration by parts and standard arguments we find that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))) \, dx \\
&\quad + \int_{\Gamma_3} p_\nu(u_\nu(t) - g)(v_\nu - \dot{u}_\nu(t)) \, da + \int_{\Gamma_3} p_\tau(u_\nu(t) - g)(\|\boldsymbol{v}_\tau(s)\| - \|\dot{\boldsymbol{u}}_\tau(s)\|) \, da \\
&\quad \geq \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) \, da.
\end{aligned}$$

Therefore, using notation (5.17) and (5.18) we see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(u_\nu(t), \boldsymbol{v}) - j(u_\nu(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \tag{5.21}$$

and, taking succesively $\boldsymbol{v} = 2\dot{\boldsymbol{u}}(t)$ and $\boldsymbol{v} = \boldsymbol{0}_V$ in this inequality, we obtain that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(u_\nu(t), \dot{\boldsymbol{u}}(t)) = (\boldsymbol{f}(t), \dot{\boldsymbol{u}}(t))_V. \tag{5.22}$$

Then, using (5.21), (5.22) and (5.19) yields

$$\boldsymbol{\sigma}(t) \in \Sigma(u_\nu(t), t), \qquad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0. \tag{5.23}$$

We now introduce the notation $\boldsymbol{\omega} := \boldsymbol{\varepsilon}(\boldsymbol{u})$ and use inequality (5.23), Lemmas 5.1, 5.2 and equivalence (2.10) to see that

$$-\dot{\boldsymbol{\omega}}(t)) \in N_{\Sigma(\mathcal{R}\dot{\boldsymbol{\omega}}(t),t)}\boldsymbol{\sigma}(t). \tag{5.24}$$

On the other hand, the constitutive law (5.1) and assumption (5.11)(d) yields

$$\boldsymbol{\sigma}(t) = A\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + B\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) \tag{5.25}$$

and, finally, the initial condition (5.7) together with notation (5.20) imply that

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{5.26}$$

We now gather relations (5.24)–(5.26) to obtain the folowing variational formulation of Problem 3.

Problem 4. *Find a strain field* $\boldsymbol{\omega}\colon I \to V$ *such that*

$$-\dot{\boldsymbol{\omega}}(t) \in N_{\Sigma(\mathcal{R}\dot{\boldsymbol{\omega}}(t),t)}\big(A\dot{\boldsymbol{\omega}}(t) + B\boldsymbol{\omega}(t) + \mathcal{S}\dot{\boldsymbol{\omega}}(t)\big) \quad \text{for all } \ t \in I, \tag{5.27}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{5.28}$$

Note that Problem 4 represents a sweeping process in which the unknown is the strain field. To the best of our knowledge, this problem is new and nonstandard since, usually, the variational formulation of contact models of the form (5.1)–(5.7) is given by an evolutionary variational inequality for the displacement field, as shown in [29] and the references therein. The unique solvability of Problem 4 is provided by the following existence and uniqueness result.

Theorem 5.3. *Assume that* (5.11)*–*(5.16) *hold. Then Problem* 4 *has a unique solution* $\boldsymbol{\omega} \in C^1(I;Q)$.

We complete Theorem 5.3 with the following existence and uniqueness result.

Corollary 5.4. *Assume that* (5.11)*–*(5.16) *hold. Then, there exists a unique couple of functions function* $\boldsymbol{u} \in C^1(I;V)$, $\boldsymbol{\sigma} \in C(I;Q)$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{F}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big) \tag{5.29}$$

$$\boldsymbol{\sigma}(t) \in \Sigma(u_\nu(t),t), \qquad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0 \qquad \forall\, t \in I. \tag{5.30}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{5.31}$$

The proof of Theorem 5.3 and Corollary 5.4 will be presented in the next section. Here, we restrict ourselves to mention that we refer to problem (5.29)–(5.31) as a mixed variational formulation of Problem 3. Moreover, a couple of functions $\boldsymbol{u} : I \to V$ and $\boldsymbol{\sigma} : I \to Q$ which solves (5.29)–(5.31) is called a weak solution to the viscoelastic contact problem 3. We conclude from Corollary 5.4 that Problem 4 has a unique weak solution.

6 Prof of Theorem 5.3

We start this section with the proof of Lemma 5.1.

Proof. First we recall that the range of the deformation operator $\boldsymbol{\varepsilon}\colon V \to Q$, denoted by $\boldsymbol{\varepsilon}(V)$, is a closed subspace of Q. A proof of this result can be find on [30, p.212]. Recall also that, by definition,

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \text{for all} \;\; \boldsymbol{v} \in V. \tag{6.1}$$

Denote by $P\colon Q \to \boldsymbol{\varepsilon}(V)$ the orthogonal projection operator on $\boldsymbol{\varepsilon}(V) \subset Q$ and note that equality (6.1) shows that $\boldsymbol{\varepsilon}\colon V \to \boldsymbol{\varepsilon}(V)$ is a linear invertible operator. In what follows, we denote by $\boldsymbol{\varepsilon}^{-1}\colon \boldsymbol{\varepsilon}(V) \to V$ the inverse of $\boldsymbol{\varepsilon}$. The ingredients above allow us to define the operators $G\colon Q \to L^2(\Gamma_3)$ and $\mathcal{R} : C(I;Q) \to C(I;L^2(\Gamma_3))$ by equalities

$$G\boldsymbol{\omega} = (\boldsymbol{\varepsilon}^{-1}P\boldsymbol{\omega})_\nu \quad \text{for all} \;\; \boldsymbol{\omega} \in Q, \tag{6.2}$$

$$\mathcal{R}\boldsymbol{\omega}(t) = \int_0^t G\boldsymbol{\omega}(s)\,ds + u_{0\nu} \quad \text{for all} \;\; \boldsymbol{\omega} \in C(I,Q) \tag{6.3}$$

where, recall, $u_{0\nu} = \boldsymbol{u}_0 \cdot \boldsymbol{\nu}$. It is obvious to see that G is a linear continuous operator and, therefore, $\mathcal{R}$ is a history-dependent operator.

Assume now that $\boldsymbol{\omega} \in C^1(I;Q)$, $\boldsymbol{u} \in C^1(I;V)$ are such that $\boldsymbol{\omega}(t) = \boldsymbol{\varepsilon}(\boldsymbol{u}(t))$ for all $t \in I$ and let $s \in I$. Then $\dot{\boldsymbol{\omega}}(s) = \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))$ which shows that $\dot{\boldsymbol{\omega}}(s) \in \boldsymbol{\varepsilon}(V)$ and, moreover, $\boldsymbol{\varepsilon}^{-1}P\dot{\boldsymbol{\omega}}(s) = \boldsymbol{\varepsilon}^{-1}\dot{\boldsymbol{\omega}}(s) = \dot{\boldsymbol{u}}(s)$. Using now (6.2) we deduce that $G\dot{\boldsymbol{\omega}}(s) = \dot{u}_\nu(s)$ and, therefore, (2.12) implies that

$$u_\nu(t) = \int_0^t G\dot{\boldsymbol{\omega}}(s)\,ds + u_{0\nu} \quad \text{for all} \;\; t \in I. \tag{6.4}$$

We now combine (6.4) and (6.3) to deduce that $u_\nu(t) = \mathcal{R}\dot{\boldsymbol{\omega}}(t)$ for all $t \in I$, which concludes the proof. □

We proceed with the proof of Lemma 5.2 which will be carried out in several steps.

Proof. i) *We study the properties of the set-valued mapping* $\Sigma_0 : L^2(\Gamma_3) \to 2^Q$ *defined by*

$$\Sigma_0(\theta) = \{\, \boldsymbol{\tau} \in Q \;:\; (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta, \boldsymbol{v}) \geq 0 \quad \forall\, \boldsymbol{v} \in V \,\} \quad \text{for all} \;\; \theta \in L^2(\Gamma_3). \tag{6.5}$$

Let $\theta \in L^2(\Gamma_3)$. Since the function $\boldsymbol{v} \mapsto j(\theta, \boldsymbol{v}) : V \to \mathbb{R}$ is convex and lower semicontinuous, it is subdifferentiable at any point of V. Therefore, since it vanishes in $\mathbf{0}_V$, we deduce from (2.11) that there exists an element $\boldsymbol{g} \in V$ such that $j(\theta, \boldsymbol{v} \geq (\boldsymbol{g}, \boldsymbol{v})_V$ for all $\boldsymbol{v} \in V$. Next, since $(\boldsymbol{g}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\varepsilon}(\boldsymbol{v})_Q$, using the notation $\boldsymbol{\xi} = -\boldsymbol{\varepsilon}(\boldsymbol{g})$ we find that

$$(\boldsymbol{\xi}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta, \boldsymbol{v}) \geq 0 \quad \text{for all} \;\; \boldsymbol{v} \in V. \tag{6.6}$$

We now combine (6.5) and (6.6) to see that $\boldsymbol{\xi} \in \Sigma_0(\theta)$ and, therefore, $\Sigma_0(\theta)$ is not empty. On the other hand, it is easy to see that $\Sigma_0(\theta)$ is a closed convex subset of Q. It follows from above that

$$K_0 : L^2(\Gamma_3) \to 2^Q \ \text{ has nonempty closed convex values.} \tag{6.7}$$

ii) *We now prove that for any* $\boldsymbol{\sigma}$, $\boldsymbol{z} \in Q$ *and* $\theta \in L^2(\Gamma_3)$, *equality* $\boldsymbol{\sigma} = P_{\Sigma_0(\theta)}\boldsymbol{z}$ *implies that there exists a unique element* $\boldsymbol{u} \in V$ *such that*

$$\boldsymbol{\sigma} - \boldsymbol{z} = \varepsilon(\boldsymbol{u}) \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) = 0. \tag{6.8}$$

Indeed, assume that $\boldsymbol{\sigma}$, $\boldsymbol{z} \in Q$ and $\theta \in L^2(\Gamma_3)$ are fixed and, moreover, assume that $\boldsymbol{\sigma} = P_{\Sigma(\theta)}\boldsymbol{z}$. Then, using (2.7) we find that

$$\boldsymbol{\sigma} \in \Sigma_0(\theta), \quad (\boldsymbol{\sigma} - \boldsymbol{z}, \boldsymbol{\tau} - \boldsymbol{\sigma})_Q \geq 0 \quad \text{for all } \ \boldsymbol{\tau} \in \Sigma_0(\theta). \tag{6.9}$$

Let $\widetilde{\boldsymbol{z}} \in \boldsymbol{\varepsilon}(V)^\perp$ where, here and below, $M^\perp$ represents the orthogonal of $M \subset Q$ in Q. Then $(\widetilde{\boldsymbol{z}}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = 0$ for all $\boldsymbol{v} \in V$ which entails that $\boldsymbol{\sigma} \pm \widetilde{\boldsymbol{z}} \in \Sigma_0(\theta)$. Therefore, testing with $\boldsymbol{\tau} = \boldsymbol{\sigma} \pm \widetilde{\boldsymbol{z}}$ in (6.9) we deduce that $(\boldsymbol{\sigma} - \boldsymbol{z}, \widetilde{\boldsymbol{z}})_Q = 0$ which shows that $\boldsymbol{\sigma} - \boldsymbol{z} \in \boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V)$. This ensures that there exists a element $\boldsymbol{u} \in V$ such that

$$\boldsymbol{\sigma} - \boldsymbol{z} = \varepsilon(\boldsymbol{u}). \tag{6.10}$$

Moreover, (6.1) guarantees that $\boldsymbol{u}$ is unique.

Next, by the subdifferentibility of the function $j(\theta, \cdot)$ at $\boldsymbol{u}$ we know that there exists an element $\boldsymbol{h} \in V$ such that

$$j(\theta, \boldsymbol{v}) - j(\theta, \boldsymbol{u}) \geq (\boldsymbol{h}, \boldsymbol{v} - \boldsymbol{u})_V = (\boldsymbol{\varepsilon}(\boldsymbol{h}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q$$

and, taking $\boldsymbol{\tau}_0 := -\boldsymbol{\varepsilon}(\boldsymbol{h})$ we deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{v}) - j(\theta, \boldsymbol{u}) \geq 0 \quad \text{for all } \ \boldsymbol{v} \in V. \tag{6.11}$$

We now test with $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \mathbf{0}_V$ in this inequality to obtain that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) = 0. \tag{6.12}$$

Therefore, combining (6.11) and (6.12) we find that $(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta, \boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in V$ and, therefore, $\boldsymbol{\tau}_0 \in \Sigma_0(\theta)$. This regularity, (6.9) and (6.10) imply that $(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q$ and, therefore, (6.12) yields

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) \leq 0. \tag{6.13}$$

On the other hand, since $\boldsymbol{\sigma} \in \Sigma_0(\theta)$ and $\boldsymbol{u} \in V$ the converse inequality holds, i.e.,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) \geq 0. \tag{6.14}$$

We now combine (6.10), (6.13) and (6.14) to see that (6.8) holds, as claimed.

iii) *We now prove that there exists* $c_0 > 0$ *such that*

$$\|P_{\Sigma_0(\theta_1)}\boldsymbol{z} - P_{\Sigma_0(\theta_2)}\boldsymbol{z}\|_Q \le c_0\,\|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} \tag{6.15}$$

for all θ_1, $\theta_2 \in Q$ *and* $\boldsymbol{z} \in Q$. To this end, let θ_1, $\theta_2 \in Q$ and $\boldsymbol{z} \in Q$ be fixed and let $\boldsymbol{\sigma}_1 = P_{\Sigma_0(\theta_1)}\boldsymbol{z}$, $\boldsymbol{\sigma}_2 = P_{\Sigma_0(\theta_2)}\boldsymbol{z}$. Then (6.5) implies that

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta_1, \boldsymbol{v}) \ge 0 \quad \text{for all } \ \boldsymbol{v} \in V, \tag{6.16}$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta_2, \boldsymbol{v}) \ge 0 \quad \text{for all } \boldsymbol{v} \in V. \tag{6.17}$$

Moreover, the step ii) guarantees that there exist $\boldsymbol{u}_1,\ \boldsymbol{u}_2 \in V$ such that

$$\boldsymbol{\sigma}_1 - \boldsymbol{z} = \varepsilon(\boldsymbol{u}_1) \quad \text{and} \quad (\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\boldsymbol{u}_1))_Q + j(\theta_1, \boldsymbol{u}_1) = 0, \tag{6.18}$$

$$\boldsymbol{\sigma}_2 - \boldsymbol{z} = \varepsilon(\boldsymbol{u}_2) \quad \text{and} \quad (\boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_Q + j(\theta_2, \boldsymbol{u}_2) = 0. \tag{6.19}$$

We now use (6.16) and (6.18), to see that

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1))_Q + j(\theta_1, \boldsymbol{v}) - j(\theta_1, \boldsymbol{u}_1) \ge 0 \quad \text{for all } \ \boldsymbol{v} \in V,$$

then we take $\boldsymbol{v} = \boldsymbol{u}_2$ in this inequality and use the identity $\boldsymbol{\varepsilon}(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1) = \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1$ to find that

$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)_Q + j(\theta_1, \boldsymbol{u}_2) - j(\theta_1, \boldsymbol{u}_1) \ge 0. \tag{6.20}$$

Similar arguments, based on (6.17) and (6.19) yield

$$(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)_Q + j(\theta_2, \boldsymbol{u}_1) - j(\theta_2, \boldsymbol{u}_2) \ge 0. \tag{6.21}$$

Therefore, adding the inequalities (6.20) and (6.21) we find that

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q^2 \le j(\theta_1, \boldsymbol{u}_2) - j(\theta_1, \boldsymbol{u}_1) + j(\theta_2, \boldsymbol{u}_1) - j(\theta_2, \boldsymbol{u}_2). \tag{6.22}$$

On the other hand, a standard calculation, based on the definition (5.17), the properties (5.13) of the functions p_e $(e = \nu, \tau)$ and the trace inequality (5.10), shows that there exists a positive constant $c_0 > 0$ such that

$$j(\theta_1, \boldsymbol{u}_2) - j(\theta_1, \boldsymbol{u}_1) + j(\theta_2, \boldsymbol{u}_1) - j(\theta_2, \boldsymbol{u}_2) \le c_0\|\theta_1 - \theta_2\|_{L^2(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V. \tag{6.23}$$

We now combine inequalities (6.22) and (6.23) and use the identity $\boldsymbol{\varepsilon}(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1) = \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1$, again, to deduce that

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q \le c_0\|\theta_1 - \theta_2\|_{L^2(\Gamma_3)}. \tag{6.24}$$

Recall also that $\boldsymbol{\sigma}_1 = P_{\Sigma_0(\theta_1)}\boldsymbol{z}$ and $\boldsymbol{\sigma}_2 = P_{\Sigma_0(\theta_1)}\boldsymbol{z}$. Thus, using these equalities in (6.24) we deduce that (6.15) holds, as claimed.

iv) *End of proof.* We note that assumptions (5.14) and (5.15) imply that the element $\boldsymbol{f}$ given by (5.18) has the regularity $\boldsymbol{f} \in C(I;V)$ and, therefore, $\boldsymbol{\varepsilon}(\boldsymbol{f}) \in C(I;Q)$. On the other hand, it is obviously to see that

$$\Sigma(\theta,t) = \Sigma_0(\theta) + \boldsymbol{\varepsilon}(\boldsymbol{f}(t)) \quad \text{for all } \theta \in L^2(\Gamma_3),\, t \in I. \tag{6.25}$$

We now use (6.7), (6.15), (6.25) and the arguments in Example 2 to see that the set-valued mapping Σ satisfies assumption $(\mathcal{K})$, which concludes the proof. □

We now have all the ingredients to provide the proof of Theorem 5.3.

Proof. We use Theorem 3.1 on the spaces $X = Q$, $Y = L^2(\Gamma_3)$ and, to this end, we check the validity of conditions of this theorem, in the functional frame above. First, we note that Lemma 5.2 guarantees that condition $(\mathcal{K})$ is satisfied. Next, assumption (5.11)(d) implies that conditions $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{S})$ are satisfied. Moreover, Lemma 5.1 shows that the operator $\mathcal{R}$ defined by (6.3) satisfies condition $(\mathcal{R})$, too. On the other hand, assumption (5.16) and (5.20) show that $\boldsymbol{\omega}_0 \in Q$ and, therefore, it satisfies condition $(\mathcal{U})$. It follows from above that we are in a position to apply Theorem 3.1 to conclude the proof. □

We end this section with the proof of Corollary 5.4.

Proof. Let $\boldsymbol{\omega} \in C^1(I;Q)$ be the solution of Problem 4 obtained in Theorem 5.3 and denote by $\boldsymbol{\sigma}$ the function given by by $\boldsymbol{\sigma}(t) = A\dot{\boldsymbol{\omega}}(t) + B\boldsymbol{\omega}(t) + \mathcal{S}\dot{\boldsymbol{\omega}}(t)$ for all $t \in I$. Then $\boldsymbol{\sigma} \in C(I,Q)$ and

$$-\dot{\boldsymbol{\omega}}(t) \in N_{\Sigma(\mathcal{R}\dot{\boldsymbol{\omega}}(t),t)}(\boldsymbol{\sigma}(t)) \quad \text{for all } t \in I, \tag{6.26}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{6.27}$$

Moreover, (5.11)(d) implies that (5.29) holds and (6.26), (2.10) yield

$$\boldsymbol{\sigma}(t) \in \Sigma(\mathcal{R}\dot{\boldsymbol{\omega}}(t),t), \qquad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\boldsymbol{\omega}}(t))_Q \geq 0 \qquad \forall\, t \in I. \tag{6.28}$$

Let $t \in I$ and let $\boldsymbol{z} \in Q$ be such that

$$(\boldsymbol{z}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = 0 \qquad \text{for all } \boldsymbol{v} \in V. \tag{6.29}$$

Then, it is easy to see that $\boldsymbol{\tau} = \boldsymbol{\sigma}(t) \pm \boldsymbol{z} \in \Sigma(\mathcal{R}\dot{\boldsymbol{\omega}}(t),t)$ and, testing with these elements in (6.28), we find that

$$(\boldsymbol{z}, \dot{\boldsymbol{\omega}}(t))_Q = 0. \tag{6.30}$$

Equalities (6.29) and (6.30) show that $\dot{\boldsymbol{\omega}}(t) \in \boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V)$. Therefore, since $\boldsymbol{\varepsilon} : V \to \boldsymbol{\varepsilon}(V)$ is a linear invertible operator, the function $\boldsymbol{v} := \boldsymbol{\varepsilon}^{-1}\dot{\boldsymbol{\omega}}$ has the regularity $\boldsymbol{v} \in C(I;V)$. Consider now the the function $\boldsymbol{u} : I \to V$ given by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{\varepsilon}^{-1}\dot{\boldsymbol{\omega}}(s)\, ds + \boldsymbol{u}_0 \qquad \text{for all } t \in I. \tag{6.31}$$

Then, using notation (5.20) we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \boldsymbol{\omega}(t)$ for all $t \in I$. Using now Lemma 5.1 we deduce that $\mathcal{R}\dot{\boldsymbol{\omega}}(t) = u_\nu(t)$ for all $t \in I$ and, therefore, inequality (6.28) implies that (5.30) holds. In addition, equality (5.31) is a direct consequence of the equalities $\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \boldsymbol{\omega}(t)$ for all $t \in I$ and $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 = \boldsymbol{\varepsilon}(\boldsymbol{u}_0)$, see (5.28) and (5.20). This proves the existence part in Corollary 5.4. The uniqueness part follows from the uniqueness of the solution to Problem 4, guaranteed by Theorem 5.3. $\square$

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