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FIRST AND SECOND ORDER STATE-DEPENDENT BOUNDED SUBSMOOTH SWEEPING PROCESSES

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Abstract. This paper is devoted to a differential inclusion encompassing in a general Hilbert space first order state-dependent sweeping process and second order one with outward normal at the velocity. Our evolution problem is governed by the normal cone of a bounded subsmooth set which moves in an absolute continuous way. The existence of a trajectory solution is established through an appropriate Moreau's catching-up algorithm.

1. Introduction

The present paper is devoted to the existence of a trajectory solution $\Phi := (\Phi_1, \Phi_2) : I = [T_0, T] \to \mathcal{H}^2$ for the constrained differential inclusion

$$(\mathcal{SP}) \begin{cases} -\dot{\Phi}(t) \in N\big(C(t,\Phi(t)) \times Q; \Phi(t)\big) + G(t,\Phi(t)) \times \{f(t,\Phi_1(t))\} & \lambda\text{-a.e. } t \in I, \\ \Phi(t) \in C(t,\Phi(t)) \times Q \quad \text{for all } t \in I, \\ \Phi(T_0) = (u_0, q_0), \end{cases}$$

which is governed by a generalized normal cone N. Here and below, $C(t,u,v)$ is a (possibly nonconvex) closed set moving in an absolutely continuous way in any Hilbert space $\mathcal{H}$, Q is an autonomous closed convex set of $\mathcal{H}$ while $G : (t,u,v) \mapsto G(t,u,v)$ and $f : (t,u) \mapsto f(t,u)$ are respectively a convex-valued multimapping and a (single-valued) mapping independent of the second state variable.

The latter differential inclusion can be seen as a (perturbed) state-dependent sweeping process, that is,

$$(\mathcal{P}) \begin{cases} -\dot{x}(t) \in N\big(D(t,x(t)); x(t)\big) + P(t,x(t)) & \lambda\text{-a.e. } t \in I, \\ x(t) \in D(t,x(t)) \quad \text{for all } t \in I, \\ x(T_0) = x_0. \end{cases}$$

Such an evolution problem has been introduced and deeply studied by J.J. Moreau in the early's seventies ([28, 29]) in the case of a closed convex moving set $D(t,x) = D(t) \subset \mathcal{H}$ and $P(t,x) \equiv 0$ through two very different ways:
- the so-called *Moreau's catching-up algorithm* (described for the first time in [28], see also [30]) which is the following Euler's explicit scheme

$$x_0^n := x_0 \quad \text{and} \quad x_{i+1}^n := \operatorname{proj}_{D(t_i^n)}(x_i^n) \quad \text{with } t_i^n := T_0 + i\frac{T - T_0}{2^n};$$

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- the (Yosida) regularization of the normal cone (see [28] and the recent survey [31] along with the references therein) through the family of ordinary differential equations

$$u_\nu(T_0) = u_0 \quad \text{and} \quad -\dot{u}_\nu(t) = \frac{1}{2\nu}\nabla d^2_{D(t)}(u_\nu(t)) \quad \text{with } \nu > 0.$$

Due to its numerous applications (mechanical problems ([4]), resource allocation in economics ([22]), crowd motion ([25]), nonregular electrical circuits ([1]), variational inequalities [36], etc.), the original (first order) Moreau's sweeping process has been extended in various directions (see, e.g., [11, 26, 24, 8, 23, 17, 10] and the references therein).

Besides first order works, the second order theory has been also well investigated. Roughly speaking, it is divided into two families of inclusions, depending on the point considered in the outward normal

$$-\ddot{u}(t) \in N\big(C(t); u(t)\big) \quad \text{or} \quad -\ddot{u}(t) \in N\big(C(u(t)); \dot{u}(t)\big).$$

For the first inclusion with outward normal at the state/position, we refer for instance to [34] and the references therein. The second problem with outward normal at the velocity has been introduced by C. Castaing in [12] at the end of eighties and then developed in a great number of works including [26, 13, 8, 2, 3, 5, 32]. Among those studies, we mention the (quite natural in view of $(\mathcal{P})$) problem of second order state-dependent sweeping process, that is,

$$(\mathcal{Q}) \begin{cases} -\ddot{u}(t) \in N\big(C(t, u(t)); \dot{u}(t)\big) + F\big(t, \dot{u}(t), u(t)\big), \\ \dot{u}(t) \in C(t, u(t)), \\ u(0) = u_0, \dot{u}(0) = v_0. \end{cases}$$

The latter inclusion has been first investigated in [13] for a (nonconvex) prox-regular (see the definition below) ball-compact moving set $C(t,x) \subset \mathcal{H}$. For other developments, we refer to [2] for an alternative compactness condition and to [3, 5] for BV versions of $(\mathcal{Q})$.

Recently, a deep link beetwen problems $(\mathcal{P})$ and $(\mathcal{Q})$ has been brough to light by J. Noel ([33]) and M. Yarou ([41]). Loosely speaking, through a clever change of variables, both authors successfully reduce the problem $(\mathcal{Q})$ to a first-order state-dependent sweeping process $(\mathcal{P})$. Unfortunately, such an approach seems to be limited to the finite dimensional setting since it requires a ball-compactness property of the whole space $\mathcal{H}$. To overcome this difficulty, the authors of [32] introduced the following measure differential inclusion (called "First order Mixed partially BV Sweeping Process")

$$(\mathcal{FMSP}) \begin{cases} -d\Phi \in N\big(C(t, \Phi(t)) \times Q; \Phi(t)\big) + G(t, \Phi(t)) \times \{f(t, \Phi_1(t))\}, \\ \Phi(t_0) = (u_0, q_0). \end{cases}$$

The main interest of such an evolution problem lies in the fact that it encompasses both inclusions $(\mathcal{P})$ and $(\mathcal{Q})$, in the sense that if $Q = \{0\}$ (resp. $Q = \mathcal{H}$) any solution Φ of $(\mathcal{FMSP})$ provides a trajectory $x : I \to \mathcal{H}$ satisfying the first-order sweeping process $(\mathcal{P})$ (resp. the second order sweeping process $(\mathcal{Q})$ with outward normal at the velocity). The existence of solutions for $(\mathcal{FMSP})$ is studied in [32] for a prox-regular ball-compact moving set in any Hilbert space. Recall that a (nonempty closed) set S of the Hilbert space $\mathcal{H}$ is *prox-regular with constant* r provided that the nearest point mapping proj_S is well defined on a suitable enlargement of S (namely $U_r(S) := \{d_S < r\}$) and continuous therein. This concept expresses a variational behavior of order two since it is known (see, e.g., [16]) that the r-prox-regularity property is equivalent to

$$\langle v, x' - x \rangle \leq \frac{\|v\|}{2r}\|x' - x\|^2 \quad \text{for all } x, x' \in S, v \in N(S; x).$$

Let us point out that prox-regularity has been well recognized as a powerful tool to go beyond the convexity property in numerous and various contexts: selections and parametrizations, differential inclusions, separation properties, best approximations algorithms (see, e.g., the survey [16] and the references therein). Coming back to the problem $(\mathcal{FMSP})$, we mention that the continuity of $u \mapsto \mathrm{proj}_{C(t,u,y)}(h)$ allows the authors of [32] to apply Schauder's fixed point result in order to construct a solution of $(\mathcal{FMSP})$ through the following appropriate Moreau catching-up algorithm-type:

$$\begin{cases} y_{p+1}^n = \mathrm{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n)ds\big), \\ x_{p+1}^n = \mathrm{proj}_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} \mathrm{proj}_{G(s,x_p^n,y_p^n)}(0)ds\big). \end{cases} \tag{1.1}$$

Our main aim in this work is to provide an existence result for the problem $(\mathcal{SP})$ (that is, $(\mathcal{FMSP})$ in an absolutely continuous setting) driven by a subsmooth moving set $C(t, u, v)$. The class of subsmooth sets ([6]) (see also the recent survey [39]) strictly contains the class of prox-regular ones and describes a variational behavior of order one, that is, for every $\varepsilon > 0$, the following estimate holds

$$\langle v, x' - x \rangle \leq \varepsilon\|v\|\,\|x' - x\|,$$

for appropriate $x, x' \in S$ and every $v \in N(S; x)$. The lack of regularity (continuity) for the nearest point mapping of a subsmooth set leads to replace the second implicit-type equality of (1.1) by the following (explicit-type) inclusion

$$x_{p+1}^n \in \mathrm{Proj}_{C(t_{p+1}^n, x_p^n, y_p^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} \mathrm{proj}_{G(s,x_p^n,y_p^n)}(0)ds\big).$$

As in [32], the existence of solution for our problem will entail the existence of solutions for both subsmooth sweeping processes $(\mathcal{P})$ and $(\mathcal{Q})$. Our present paper complements the very few number of works dealing with sweeping processes governed by a subsmooth set ([21, 23, 5]).

The paper is organized as follows. Section 2 is devoted to the introduction of notation and the necessary preliminaries. In Section 3, we state and prove our main existence result for the problem $(\mathcal{SP})$ governed by a subsmooth set with a Lipschitz variation. In Section 4, we use Moreau's reduction technique to obtain a solution of $(\mathcal{SP})$ in the absolutely continuous framework. Such an existence result is applied in Section 5 to obtain a trajectory solution of the second order state-dependent sweeping process with outward normal at the velocity.

2. Notation and preliminaries

Throughout, $\mathcal{H}$ is a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. The open (resp. closed) ball of $\mathcal{H}$ centered at $x \in \mathcal{H}$ with radius $r > 0$ is denoted by $B(x, r)$ (resp. $B[x, r]$). The letter $\mathbb{B}$ denotes the closed unit ball of $\mathcal{H}$, that is, $\mathbb{B} := B[0, 1]$. The *distance function* from a nonempty subset $S \subset \mathcal{H}$ is defined by

$$d_S(x) =: d(x, S) := \inf_{y \in S} \|x - y\| \quad \text{for all } x \in \mathcal{H}.$$

For any $x \in \mathcal{H}$, the (possibly empty) set of *nearest points* of x in S is defined as

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

If the latter set is reduced to a singleton, we denote $\mathrm{proj}_S(x)$ its unique element. The *Hausdorff-Pompeiu* distance between two nonempty subsets $S_1, S_2 \subset \mathcal{H}$ is defined as the extended real

$$\mathrm{haus}(S_1, S_2) := \max\left\{\sup_{x \in S_1} d(x, S_2), \sup_{x \in S_2} d(x, S_1)\right\}.$$

It is known (and not difficult to prove) that

$$\mathrm{haus}(S_1, S_2) := \sup_{x \in \mathcal{H}} |d(x, S_1) - d(x, S_2)|. \tag{2.1}$$

As usual, $\mathbb{N}$ denotes the set of integers starting from 1 and $\mathbb{R}_+^\star :=]0, +\infty[$ the set of nonnegative reals. In all the paper, λ stands for the Lebesgue measure of an interval $I := [T_0, T] \subset \mathbb{R}$ with $T_0 < T$. The λ-Bochner integrability will play an important role in what follows. Recall that for $p \in [1, +\infty[\cup \{\infty\}$, a (class) of mapping $f : I \to \mathcal{H}$ belongs to the Lebesgue-Bochner space $L^p(I, \mathcal{H}, \lambda)$ whenever it is λ-Bochner (or λ-strongly) measurable on I (see, e.g., [18, Chapter 2]) and $\|f(\cdot)\| \in L^p(I, \mathbb{R}, \lambda)$.

2.1. **Normal cones and subdifferentials.**

Let S be a nonempty closed set of the Hilbert space $\mathcal{H}$. A vector v is a *Fréchet normal* vector to the set S at a point $x \in S$ provided that (see, e.g., [37, 27, 40])

$$\limsup_{S \ni x' \to x} \frac{\langle v, x' - x \rangle}{\|x' - x\|} \le 0,$$

that is, for every real $\varepsilon > 0$, there is a real $\delta > 0$ such that

$$\big\langle v, x' - x \big\rangle \le \varepsilon \|x' - x\| \quad \text{for all } x' \in B(x, \delta) \cap S.$$

The set $N^F(S; x)$ of all Fréchet normal vectors at x is a closed convex cone of $\mathcal{H}$ containing 0 called *Fr迣 chet normal cone* of S at x. As usual, we set

$$N^F(S; x) := \emptyset \quad \text{for all } x \in \mathcal{H} \setminus S. \tag{2.2}$$

For each $v \in \mathcal{H}$ with $w \in \mathrm{Proj}_S(v) \neq \emptyset$, we may obviously write $d_S^2(v) = \|w - v\|^2$, or equivalently,

$$\big\langle v - w, x' - w \big\rangle \le \frac{1}{2}\|x' - w\|^2 \quad \text{for all } x' \in S,$$

and this ensures in particular the crucial inclusion

$$v - w \in N^F(S; w). \tag{2.3}$$

Besides the Fr 迦 chet normal cone, we also need to consider the Clarke one. The *Clarke normal cone* of S at $x \in S$ is defined as (see, e.g., [37, 14, 27, 40])

$$N^C(S; x) := \left\{ v \in \mathcal{H} : \langle v, h \rangle \le 0, \forall h \in T^C(S; x) \right\},$$

where $T^C(S; x)$ denotes the so-called *Clarke tangent cone* of S at x

$$T^C(S; x) := \left\{ h \in \mathcal{H} : \forall S \ni x_n \to x, \forall \mathbb{R}_+^\star \ni t_n \downarrow 0, \exists h_n \to h, x_n + t_n h_n \in S, \forall n \in \mathbb{N} \right\}.$$

It is easily checked that $T^C(S; x)$ is a closed convex cone containing 0. As (2.2), one puts $T^C(S; x) := N^C(S; x) := \emptyset$ for every x outside S. It is known that the Fréchet normal cone is always included in the Clarke one, i.e.,

$$N^F(S; x) \subset N^C(S; x) \quad \text{for all } x \in \mathcal{H}. \tag{2.4}$$

If S is (closed) convex, then it is also known that the Fréchet and Clarke normal cones coincide with the one in the sense of convex analysis, that is,

$$N^F(S; x) = N^C(S; x) = \{ v \in \mathcal{H} : \langle v, x' - x \rangle \le 0, \forall x' \in S \} \quad \text{for all } x \in S.$$

Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function defined on a nonempty open subset U of $\mathcal{H}$. Through the above concepts of normal cones, one defines the *Fréchet subdifferential* $\partial_F f(x)$ and the *Clarke subdifferential* $\partial_C f(x)$ of f at $x \in U$ by

$$\partial_F f(x) := \Big\{ v \in \mathcal{H} : \ (v, -1) \in N^F\Big(E_f; \big(x, f(x)\big)\Big) \Big\} \tag{2.5}$$

and

$$\partial_C f(x) := \Big\{ v \in \mathcal{H} : \ (v, -1) \in N^C\Big(E_f; \big(x, f(x)\big)\Big) \Big\}, \tag{2.6}$$

where $\mathcal{H} \times \mathbb{R}$ is endowed with the usual product structure and

$$E_f := \operatorname{epi} f := \{ (u, r) \in U \times \mathbb{R} : f(u) \le r \}.$$

It follows from the very definition of the latter subdifferentials that $\partial_F f(x) = \emptyset$ and $\partial_C f(x) = \emptyset$ whenever f is not finite at $x \in U$. From (2.5), (2.6) and (2.4), it is readily seen that

$$\partial_F f(x) \subset \partial_C f(x) \quad \text{for all } x \in U.$$

The Fréchet subdifferential can also be described in a variational way, namely

$$\partial_F f(x) = \left\{ v \in \mathcal{H} : \liminf_{x' \to x} \frac{f(x') - f(x) - \langle v, x' - x \rangle}{\|x' - x\|} \ge 0 \right\},$$

for any $x \in U$ with $|f(x)| < +\infty$. Of course, when U is convex and the function f is convex on U, the Fr 迦 chet and Clarke subdifferentials coincide with the one in the sense of convex analysis, i.e., for every $x \in U$ with $|f(x)| < +\infty$,

$$\partial_F f(x) = \partial_C f(x) = \left\{ v \in \mathcal{H} : \langle v, x' - x \rangle \le f(x') - f(x), \ \forall x' \in U \right\}. \tag{2.7}$$

If f is the distance function associated to the closed (not necessarily convex) set S (that is, $f = d_S$) then we have the following description of its Fréchet and Clarke subdifferential (see, e.g., [9, 40])

$$\partial_F d_S(x) = N^F(S;x) \cap \mathbb{B} \quad \text{and} \quad \partial_C d_S(x) \subset N^C(S;x) \cap \mathbb{B} \quad \text{for all } x \in S. \tag{2.8}$$

For a function f which is γ-Lipschitz near $x \in U$ for some real $\gamma \geq 0$, it is known that (see [14, 40]) the Clarke subdifferential is nonempty, weakly compact and satisfies

$$\partial_C f(x) = \{v \in \mathcal{H} : \langle v, h\rangle \leq f^o(x;h),\ \forall h \in \mathcal{H}\} \subset \gamma\mathbb{B},$$

where $f^o(x;h)$ is the *Clarke directional derivative* at x in the direction $h \in \mathcal{H}$ defined by

$$f^o(x;h) := \limsup_{t\downarrow 0, x' \to x} t^{-1}\big(f(x'+th) - f(x')\big).$$

Under the latter Lipschitz assumption on the function f, the Clarke derivative $f^o(x;\cdot)$ of f at x is nothing but the support function of the closed convex set $\partial_C f(x)$. Recall that for any subset $A \subset \mathcal{H}$, its *support function* $\sigma(\cdot, A)$ is defined by

$$\sigma(\xi, A) := \sup_{x\in A} \langle \xi, x\rangle \quad \text{for all } \xi \in \mathcal{H}.$$

As a direct consequence of the Hahn-Banach theorem, we get that the support function characterizes the closed convex sets of $\mathcal{H}$, in the sense that for every subsets S_1, S_2 of $\mathcal{H}$,

$$\overline{\text{co}}\, S_1 \subset \overline{\text{co}}\, S_2 \Leftrightarrow \sigma(\cdot, S_1) \leq \sigma(\cdot, S_2). \tag{2.9}$$

Here and below, co (resp. $\overline{\text{co}}$) stands for the convex (resp. closed convex) hull of S. Through the support function, we define the concept of scalar upper semicontinuity as follows: a multimapping $F : \mathcal{T} \rightrightarrows \mathcal{H}$ from a Hausdorff topological space $\mathcal{T}$ to the Hilbert space $\mathcal{H}$ is said to be *scalarly upper semicontinuous* whenever, for any $\xi \in \mathcal{H}$, the extended real-valued function $\sigma(\xi, F(\cdot)) : \mathcal{T} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is upper semicontinuous.

2.2. **Subsmooth sets in Hilbert spaces.** The class of uniformly subsmooth sets will be fundamental in the rest of the paper. It was introduced in the Banach framework by D. Aussel, A. Daniilidis and L. Thibault in [6]. More details can be found in the recent and nice survey [39] (see also the forthcoming monograph [40]). Let us start with the definition of uniformly subsmooth sets.

Definition 2.1. A nonempty closed subset S of $\mathcal{H}$ is said to be uniformly subsmooth whenever for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all $x_1, x_2 \in S$ with $\|x_1 - x_2\| < \delta$, for all $v \in N^C(S;x_1) \cap \mathbb{B}$, one has

$$\langle v, x_2 - x_1\rangle \leq \varepsilon \|x_2 - x_1\|.$$

A nonempty family $(S_j)_{j\in J}$ of nonempty closed subsets of $\mathcal{H}$ is said to be equi-uniformly subsmooth if for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all $j \in J$, for all $x_1, x_2 \in S_j$, for all $v \in N^C(S_j;x_1) \cap \mathbb{B}$, one has

$$\langle v, x_2 - x_1\rangle \leq \varepsilon \|x_2 - x_1\|.$$

An important class of uniformly subsmooth sets is given by the class of uniformly prox-regular sets ([35]). Recall that a nonempty closed subset S of $\mathcal{H}$ is said to be *r-prox-regular* for some extended real $r \in]0, +\infty]$, whenever for all $x_1, x_2 \in S$, for all $v \in N^C(S; x_1) \cap \mathbb{B}$, one has

$$\langle v, x_2 - x_1 \rangle \leq \frac{1}{2r} \|x_1 - x_2\|^2 ,$$

with the convention $\frac{1}{r} := 0$ whenever $r = +\infty$. As mentioned in the very introduction of the paper, such a property expresses a variational behavior of order two while a uniformly subsmooth set is related to the order one. Prox-regularity has been recognized as a key concept in variational analysis and its applications (see, e.g., [16] and the references therein). It is worth pointing out that the subsmoothness property of a set does not imply its prox-regularity (see [21] for a counterexample in $\mathbb{R}^2$).

Proposition 2.2. *The following hold:*

(a) *Any nonempty closed convex subset of $\mathcal{H}$ is uniformly subsmooth.*
(b) *Any nonempty family of nonempty closed convex subsets of $\mathcal{H}$ is equi-uniformly subsmooth.*

More generally, for any extended real $r \in]0, +\infty]$, one has:

(c) *Any r-prox-regular set of $\mathcal{H}$ is uniformly subsmooth.*
(d) *Any nonempty family of r-prox-regular subsets of $\mathcal{H}$ is equi-uniformly subsmooth.*

Subsmooth sets enjoy the following Fr 迦 chet-Clarke regularity properties:

Proposition 2.3. *Let S be a uniformly subsmooth subset of $\mathcal{H}$. The following hold:*

(a) *The set S is Fréchet-Clarke normally regular, i.e.,*

$$N^F(S; x) = N^C(S; x) \quad \textit{for all}\, x \in \mathcal{H}.$$

(b) *The distance function d_S is Fréchet-Clarke regular at every point of S, i.e.,*

$$\partial_F d_S(x) = \partial_C d_S(x) \quad \textit{for all}\, x \in S.$$

According to (a) of Proposition 2.3, we put

$$N(S; x) := N^F(S; x) = N^C(S; x) \quad \text{for all}\, x \in \mathcal{H},$$

whenever the set S is uniformly subsmooth.

We end this section with the following result which provides an upper semicontinuity property.

Proposition 2.4. [39, Proposition 10.4] *Let E be a metric space and let $(C(q))_{q \in E}$ be an equi-uniformly subsmooth family of the Hilbert space $\mathcal{H}$. Let also $Q \subset E$, $\overline{q} \in \operatorname{cl} Q$ (the closure of Q in E) and $x \in C(\overline{q})$.*

Then, for any net $(q_j)_{j \in J}$ in Q converging to $\overline{q}$ with $d_{C(q_j)}(x) \underset{j \in J}{\to} 0$, for any net $(x_j)_{j \in J}$ converging to $x \in \mathcal{H}$, one has

$$\limsup_{j\in J} \sigma\big(h, \partial_C d_{C(q_j)}(x_j)\big) \le \sigma\big(h, \partial_C d_{C(\overline{q})}(x)\big) \quad \textit{for all}\, h \in \mathcal{H}.$$

3. Mixed partially state-dependent sweeping process in Lipschitz setting

The present section is devoted to the development of the main result of the paper. It provides sufficient conditions ensuring the existence of a Lipschitz trajectory solution for the sweeping process $(\mathcal{SP})$. Its proof will require a Gronwall-lemma type that we recall below for the sake of the reader.

Lemma 3.1 (Gronwall)**.** *Let $\varphi : I \to \mathbb{R}$ be an absolutely continuous function on I, $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ be Lebesgue integrable functions on I. If for λ-almost every $t \in I$,*

$$\dot{\varphi}(t) \le b(t) + a(t)\varphi(t),$$

then for all $t \in I$,

$$\varphi(t) \le \varphi(T_0) \exp\left(\int_{T_0}^{t} a(s)ds\right) + \int_{T_0}^{t} b(\tau) \exp\left(\int_{\tau}^{t} a(s)ds\right) d\tau.$$

Now, we are in a position to state and prove the main result of the paper.

Theorem 3.2. *Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings and $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Let Q be a closed convex subset of $\mathcal{H}$, $(u_0, q_0) \in \mathcal{H} \times Q$ with $u_0 \in C(T_0, u_0, q_0)$. Assume that:*

(i) *The family $\big(C(t,x,y)\big)_{t\in I, x,y\in\mathcal{H}}$ is equi-uniformly subsmooth and there exists a real $\rho > 0$ such that*

$$C(t,x,y) \subset \rho\mathbb{B} \quad \textit{for all}\, t \in I,\ x, y \in \mathcal{H};$$

(ii) *for every $x \in \rho\mathbb{B}$, the mapping $f(\cdot, x)$ is λ-Bochner measurable on I and there exist two reals $\beta, l \ge 0$ such that*

$$\|f(t,x)\| \le \beta(1 + \|x\|) \quad \textit{for all } t \in I, x \in \rho\mathbb{B} \tag{3.1}$$

and

$$\big\|f(t,x) - f(t,x')\big\| \le l\, \big\|x - x'\big\| \quad \textit{for all}\, t \in I, x, x' \in \rho\mathbb{B}; \tag{3.2}$$

(iii) *the set $C\big(I \times \rho\mathbb{B} \times (\kappa\mathbb{B} \cap Q)\big)$ is relatively compact with*

$$\kappa := \|q_0\| + 2\beta(1+\rho)(T - T_0);$$

(iv) *there exist a real $L \in [0, 1[$ and two reals $K, L' \ge 0$ such that*

$$\mathrm{haus}\big(C(t_1, x_1, y_1), C(t_2, x_2, y_2)\big) \le K\,|t_2 - t_1| + L\,\|x_1 - x_2\| + L'\,\|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$, all $x_1, x_2 \in \rho\mathbb{B}$ and all $y_1, y_2 \in Q \cap \kappa\mathbb{B}$;

(v) *the multimapping G is nonempty closed convex valued, $G(t,\cdot,\cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $(x,y) \in \rho\mathbb{B} \times (\kappa\mathbb{B} \cap Q)$ the mapping $\mathrm{proj}_{G(\cdot,x,y)}(0) : I \to \mathcal{H}$ is λ-Bochner measurable on I and there exists a real $\alpha \geq 0$ such that*

$$\left\| \mathrm{proj}_{G(t,x,y)}(0) \right\| \leq \alpha(1 + \|x\| + \|y\|),$$

for all $t \in I$, all $x \in \rho\mathbb{B}$ and all $y \in \kappa\mathbb{B} \cap Q$.

Then, there exists a Lipschitz continuous solution $\Phi = (\Phi_1, \Phi_2) : I \to \mathcal{H}^2$ of the differential inclusion $(\mathcal{SP})$, that is:

(a) *the mappings Φ_1, Φ_2 are Lipschitz continuous on I and satisfy $\Phi_1(T_0) = u_0$, $\Phi_2(T_0) = q_0$ along with*

$$\big(\Phi_1(t), \Phi_2(t)\big) \in C\big(t, \Phi_1(t), \Phi_2(t)\big) \times Q \quad \textit{for all}\, t \in I;$$

(b) *for λ-almost every $t \in I$, one has*

$$\dot{\Phi}_2(t) + f\big(t, \Phi_1(t)\big) \in -N\big(Q; \Phi_2(t)\big);$$

(c) *there exists a λ-Bochner integrable mapping $z : I \to \mathcal{H}$ with*

$$z(t) \in G\big(t, \Phi_1(t), \Phi_2(t)\big) \quad \lambda\text{-a.e.}\, t \in I$$

and such that

$$\dot{\Phi}_1(t) + z(t) \in -N\big(C(t, \Phi(t)); \Phi_1(t)\big) \quad \lambda\text{-a.e.}\, t \in I.$$

Proof. Let us start by setting for every $(t,x,y) \in I \times \mathcal{H}^2$,

$$g(t,x,y) := \mathrm{proj}_{G(t,x,y)}(0),$$

that is, $g(t,x,y)$ is the element of minimal norm of the nonempty closed convex set $G(t,x,y)$. It directly follows from assumption (v) that the mapping $g(\cdot,x,y)$ is λ-Bochner integrable on I for any $x \in \rho\mathbb{B}$ and any $y \in \kappa\mathbb{B} \cap Q$. On the other hand, we observe through assumption (iv) and (2.1) that

$$\begin{aligned} \big| d\big(z_1, C(t_1,x_1,y_1)\big) - d\big(z_2, C(t_2,x_2,y_2)\big) \big| \leq \|z_1 - z_2\| + K(t_2 - t_1) \\ + L\|x_1 - x_2\| + L'\|y_1 - y_2\|, \end{aligned} \tag{3.3}$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $x_1, x_2 \in \rho\mathbb{B}$, $y_1, y_2 \in Q \cap \kappa\mathbb{B}$ and $z_1, z_2 \in \mathcal{H}$.

Step 1. Construction of sequences $(x_p^n)_{0 \leq p \leq 2^n}$ and $(y_p^n)_{0 \leq p \leq 2^n}$ $(n \geq 1)$.

Fix for a moment any integer $n \geq 1$. Let us start by setting

$$x_{-1}^n := x_0^n := u_0 \in C(T_0, u_0, q_0) \subset \rho\mathbb{B} \quad \text{and} \quad y_0^n := q_0 \in Q \cap \kappa\mathbb{B} \tag{3.4}$$

and

$$b := \beta(1+\rho),\ c := \alpha(1+\rho+\kappa) \quad \text{and} \quad d := 2c + K + 2L'b.$$

We also need to consider the partition of I by the points

$$t_i^n := T_0 + i\Delta_n \quad \text{for all}\, i \in \{0, \dots, 2^n\} \quad \text{with}\, \Delta_n := \frac{T - T_0}{2^n}.$$

We are going to construct by (finite) induction $x_1^n, \ldots, x_{2^n}^n$ and $y_1^n, \ldots, y_{2^n}^n$, in $\mathcal{H}$ such that for each $p \in \{0, \ldots, 2^n - 1\}$,

$$
\left\{
\begin{array}{l}
\kappa\mathbb{B} \ni y_{p+1}^n = \mathrm{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) ds\big), \\
\|y_{p+1}^n - y_p^n\| \leq 2b\Delta_n, \\
x_{p+1}^n \in \mathrm{Proj}_{C(t_{p+1}^n, x_p^n, y_p^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s, x_p^n, y_p^n) ds\big), \\
\|x_{p+1}^n - x_p^n\| \leq d\Delta_n + L\|x_p^n - x_{p-1}^n\|.
\end{array}
\right. \tag{3.5}
$$

Set $y_1^n := \mathrm{proj}_Q\big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) ds\big)$. Using the definition of y_1^n, the inclusion $y_0^n \in Q$, the inequality $\|x_0^n\| \leq \rho$ (see (3.4)) and the inequality (3.1) in assumption (ii), we obtain

$$
\begin{aligned}
\|y_1^n - y_0^n\| &\leq \left\| y_1^n - \big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) ds\big) \right\| + \int_{t_0^n}^{t_1^n} \|f(s, x_0^n)\| \, ds \\
&= d_Q\big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) ds\big) + \int_{t_0^n}^{t_1^n} \|f(s, x_0^n)\| \, ds \\
&\leq 2 \int_{t_0^n}^{t_1^n} \|f(s, x_0^n)\| \, ds \leq 2\beta(1 + \|x_0^n\|)(t_1^n - t_0^n) \leq 2b\Delta_n,
\end{aligned} \tag{3.6}
$$

from which we easily derive

$$
\|y_1^n\| \leq \|y_0^n\| + 2b\Delta_n \leq \kappa.
$$

Thanks to the (strong) compactness property of $C(t_1^n, x_0^n, y_0^n) \subset C\big(I \times \rho\mathbb{B} \times (\kappa\mathbb{B} \cap Q)\big)$ (see assumption (iii)), we can choose

$$
x_1^n \in \mathrm{Proj}_{C(t_1^n, x_0^n, y_0^n)}\big(x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) ds\big) \neq \emptyset. \tag{3.7}
$$

By (3.7), assumption (v), (3.3) and the inclusions provided by (3.4), we get

$$
\begin{aligned}
\|x_1^n - x_0^n\| &\leq \left\| x_1^n - \big(x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) ds\big) \right\| + \int_{t_0^n}^{t_1^n} \|g(s, x_0^n, y_0^n)\| \, ds \\
&\leq d_{C(t_1^n, x_0^n, y_0^n)}\big(x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) ds\big) + \alpha(1 + \|x_0^n\| + \|y_0^n\|)\Delta_n \\
&\leq d_{C(t_0^n, x_0^n, y_0^n)}\big(x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) ds\big) + K\Delta_n + \alpha(1 + \rho + \kappa)\Delta_n \\
&\leq \int_{t_0^n}^{t_1^n} \|g(s, x_0^n, y_0^n)\| \, ds + K\Delta_n + c\Delta_n \leq (2c + K)\Delta_n \leq d\Delta_n.
\end{aligned} \tag{3.8}
$$

Now, let $p \in \{1, \ldots, 2^n - 1\}$. Assume that $x_1^n, \ldots, x_p^n$ and $y_1^n, \ldots, y_p^n$ have been constructed, so that properties in (3.5) hold true. Set $y_{p+1}^n := \mathrm{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) ds\big)$. Proceeding as in (3.6), we have

$$
\|y_{p+1}^n - y_p^n\| \leq \left\| y_{p+1}^n - \big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) ds\big) \right\| + \int_{t_p^n}^{t_{p+1}^n} \|f(s, x_p^n)\| ds
$$

$$= d_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s,x_p^n)ds\big) + \int_{t_p^n}^{t_{p+1}^n} \big\|f(s,x_p^n)\big\|\, ds$$

$$\le 2\int_{t_p^n}^{t_{p+1}^n} \big\|f(s,x_p^n)\big\|\, ds \le 2\beta(1+\rho)\Delta_n = 2b\Delta_n. \tag{3.9}$$

This and the first inequality in (3.5) ensure that

$$\begin{aligned}\big\|y_{p+1}^n\big\| &\le \big\|y_p^n\big\| + 2\beta(1+\rho)\Delta_n\\ &\le \big\|y_{p-1}^n\big\| + 4\beta(1+\rho)\Delta_n\\ &\ \ \vdots\\ &\le \|y_0^n\| + 2(p+1)\beta(1+\rho)\Delta_n \le \|y_0^n\| + 2^{n+1}b\Delta_n \le \|y_0^n\| + 2b(T-T_0) \le \kappa.\end{aligned}$$

Again, the compactness assumption (iii) allows us to choose some

$$x_{p+1}^n \in \mathrm{Proj}_{C(t_{p+1}^n,x_p^n,y_p^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s,x_p^n,y_p^n)ds\big) \neq \emptyset.$$

In the same way as above in (3.8), we obtain

$$\begin{aligned}\big\|x_{p+1}^n - x_p^n\big\| &\le \|x_{p+1}^n - \big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s,x_p^n,y_p^n)ds\big)\| + \int_{t_p^n}^{t_{p+1}^n} \|g(s,x_p^n,y_p^n)\|ds\\ &\le d_{C(t_{p+1}^n,x_p^n,y_p^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s,x_p^n,y_p^n)ds\big) + \alpha(1+\rho+\kappa)\Delta_n\\ &\le d_{C(t_p^n,x_{p-1}^n,y_{p-1}^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s,x_p^n,y_p^n)ds\big) + c\Delta_n\\ &\quad + K\Delta_n + L\|x_p^n - x_{p-1}^n\| + L'\|y_p^n - y_{p-1}^n\|\\ &\le (2c+K)\Delta_n + L\|x_p^n - x_{p-1}^n\| + 2L'b\Delta_n\\ &\le d\Delta_n + L\|x_p^n - x_{p-1}^n\|,\end{aligned}$$

where the fourth inequality is due to (3.9). This completes the induction.

Fix for a moment any $p \in \{0,\dots,2^n-1\}$. As a direct consequence of the second inequality in (3.5) and (3.8), we have

$$\begin{aligned}\big\|x_{p+1}^n - x_p^n\big\| &\le d\Delta_n + L\|x_p^n - x_{p-1}^n\|\\ &\le d\Delta_n + L\big(d\Delta_n + L\|x_{p-1}^n - x_{p-2}^n\|\big)\\ &= d\Delta_n(1+L) + L^2\|x_{p-1}^n - x_{p-2}^n\|\\ &\le d\Delta_n(1+L) + L^2\big(d\Delta_n + L\|x_{p-2}^n - x_{p-3}^n\|\big)\\ &= d\Delta_n(1+L+L^2) + L^3\|x_{p-2}^n - x_{p-3}^n\|\\ &\ \ \vdots\\ &\le d\Delta_n(1+L+\ldots+L^{p-1}) + L^p\|x_1^n - x_0^n\|\\ &\le d\Delta_n(1+L\ldots+L^p) \le \frac{d}{1-L}\Delta_n.\end{aligned} \tag{3.10}$$

With $\sigma_p^n := \int_{t_p^n}^{t_{p+1}^n} g(s, x_p^n, y_p^n)ds$ and $\omega_p^n := \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n)ds$, it directly follows from the assumptions (v) and (ii) that

$$\|\sigma_p^n\| \le \alpha(1 + \|x_p^n\| + \|y_p^n\|)\Delta_n \le \alpha(1 + \rho + \kappa)\Delta_n = c\Delta_n \tag{3.11}$$

and

$$\left\|\omega_p^n\right\| \le \int_{t_p^n}^{t_{p+1}^n} \|f(s, x_p^n)\|ds \le \beta(1 + \|x_p^n\|)\Delta_n \le b\Delta_n. \tag{3.12}$$

Putting together (3.10), (3.11), (3.5) and (3.12), we obtain with $a := (1-L)^{-1}d + c$

$$\left\|x_{p+1}^n - x_p^n + \sigma_p^n\right\| \le \|x_{p+1}^n - x_p^n\| + \|\sigma_p^n\| \le a\Delta_n \tag{3.13}$$

and

$$\left\|y_{p+1}^n - y_p^n + \omega_p^n\right\| = d_Q(y_p^n - \omega_p^n) \le \|\omega_p^n\| \le b\Delta_n. \tag{3.14}$$

Coming back again to the inclusions provided by (3.5) and using (2.3), (3.13), (2.8) and Proposition 2.3(b) yield

$$\begin{aligned} \frac{1}{a\Delta_n}\left(x_{p+1}^n - x_p^n + \sigma_p^n\right) &\in -N^F\left(C(t_{p+1}^n, x_p^n, y_p^n); x_{p+1}^n\right) \cap \mathbb{B} \\ &= -\partial_F d_{C(t_{p+1}^n, x_p^n, y_p^n)}(x_{p+1}^n) = -\partial_C d_{C(t_{p+1}^n, x_p^n, y_p^n)}(x_{p+1}^n). \end{aligned} \tag{3.15}$$

Similarly, we get with the help of (3.14)

$$\frac{1}{b\Delta_n}\left(y_{p+1}^n - y_p^n + \omega_p^n\right) \in -N^F\left(Q; y_{p+1}^n\right) \cap \mathbb{B} = -\partial_F d_Q(x_{p+1}^n) = -\partial_C d_Q(x_{p+1}^n). \tag{3.16}$$

Step 2. Construction of sequences $(u_n(\cdot))_{n\ge 1}$ and $(q_n(\cdot))_{n\ge 1}$.

Fix for a moment any $n \ge 1$. Let us define the mapping $u_n(\cdot) : I \to \mathcal{H}$ by setting $u_n(T) := x_{2^n}^n$ and

$$u_n(t) := x_p^n + \frac{t - t_p^n}{\Delta_n}\left(x_{p+1}^n - x_p^n + \sigma_p^n\right) - \int_{t_p^n}^t g(s, x_p^n, y_p^n)ds,$$

for all $t \in [t_p^n, t_{p+1}^n[$ with $p \in \{0, \dots, 2^n - 1\}$. Note that

$$u_n(t_p^n) = x_p^n \quad \text{for all}\, p \in \{0, \dots, 2^n\}. \tag{3.17}$$

Besides $u_n(\cdot)$, we also need to consider the mapping $z_n : I \to \mathcal{H}$ defined by $z_n(T) := g(T, x_{2^n}^n, y_{2^n}^n)$ and

$$z_n(t) := g(t, x_p^n, y_p^n) \quad \text{for all}\, t \in [t_p^n, t_{p+1}^n[\, \text{with}\, p \in \{0, \dots, 2^n - 1\}.$$

Fix for a moment any integer $p \in \{0, \dots, 2^n - 1\}$. We easily see that $u_n(\cdot)$ is derivable λ-almost everywhere on I with

$$\dot{u}_n(t) = \frac{1}{\Delta_n}\left(x_{p+1}^n - x_p^n + \sigma_p^n\right) - z_n(t) \quad \lambda\text{-a.e. } t \in]t_p^n, t_{p+1}^n[, \tag{3.18}$$

in particular (see (3.13) and assumption (v))

$$\|\dot{u}_n(t)\| \le a + \alpha(1 + \rho + \kappa) = a + c \quad \lambda\text{-a.e. } t \in I. \tag{3.19}$$

Hence, the mapping $u_n(\cdot)$ is $(a+c)$-Lipschitz continuous on I. On the other hand, from (3.18) and (3.15), we derive that

$$\dot{u}_n(t) + z_n(t) \in -a\partial_C d_{C(t_{p+1}^n, x_p^n, y_p^n)}(x_{p+1}^n) \quad \lambda\text{-a.e. } t \in]t_p^n, t_{p+1}^n[. \tag{3.20}$$

In a similar way as $u_n(\cdot)$, let us define the mapping $q_n : I \to \mathcal{H}$ by setting $q_n(T) := y_{2^n}^n$ and

$$q_n(t) := y_p^n + \frac{t - t_p^n}{\Delta_n}\big(y_{p+1}^n - y_p^n + \omega_p^n\big) - \int_{t_p^n}^t f(s, x_p^n)ds,$$

for all $t \in [t_p^n, t_{p+1}^n[$ with $p \in \{0, \ldots, 2^n - 1\}$. It is clear that

$$q_n(t_p^n) = y_p^n \quad \text{for all } p \in \{0, \ldots, 2^n\}. \tag{3.21}$$

Again, fix any $p \in \{0, \ldots, 2^n - 1\}$. We easily observe that $q_n(\cdot)$ is derivable λ-almost everywhere on I with

$$\dot{q}_n(t) = \frac{1}{\Delta_n}\big(y_{p+1}^n - y_p^n + \omega_p^n\big) - f(t, x_p^n) \quad \lambda\text{-a.e. } t \in]t_p^n, t_{p+1}^n[. \tag{3.22}$$

It follows from the latter equality and from the estimates (3.14) and (3.1) in assumption (ii) that

$$\|\dot{q}_n(t)\| \le b + \beta(1+\rho) \le 2b \quad \lambda\text{-a.e. } t \in I, \tag{3.23}$$

and this guarantees the $2b$-Lipschitz property of the mapping $q_n(\cdot)$ on I. Further, it is clear from (3.22) and (3.16) that

$$\dot{q}_n(t) + f(t, x_p^n) \in -b\partial_C d_Q(y_{p+1}^n) \quad \lambda\text{-a.e. } t \in]t_p^n, t_{p+1}^n[. \tag{3.24}$$

Now, let us consider the mappings $\theta_n, \delta_n : I \to I$ defined by

$$\delta_n(t) := \begin{cases} t_p^n & \text{if } t \in [t_p^n, t_{p+1}^n[\text{ for some } p \in \{0, \ldots, 2^n - 1\}, \\ t_{2^n - 1} & \text{if } t = T \end{cases}$$

and

$$\theta_n(t) := \begin{cases} t_{p+1}^n & \text{if } t \in [t_p^n, t_{p+1}^n[\text{ for some } p \in \{0, \ldots, 2^n - 1\}, \\ T & \text{if } t = T. \end{cases}$$

It is readily seen that for every $t \in I$,

$$|\delta_n(t) - t| \le \Delta_n \quad \text{and} \quad |\theta_n(t) - t| \le \Delta_n,$$

in particular

$$\lim_{m\to\infty} \delta_m(t) = t \quad \text{and} \quad \lim_{m\to\infty} \theta_m(t) = t.$$

Fix for a moment any $p \in \{0, \ldots, 2^n - 1\}$. The definitions of $\delta_n(\cdot), \theta_n(\cdot)$ along with (3.17) give in a straightforward way

$$u_n(\delta_n(T)) = x_{2^n-1}^n \quad \text{and} \quad u_n(\delta_n(t)) = u_n(t_p^n) = x_p^n \quad \text{for all } t \in [t_p^n, t_{p+1}^n[,$$

$$u_n(\theta_n(T)) = x_{2^n}^n \quad \text{and} \quad u_n(\theta_n(t)) = u_n(t_{p+1}^n) = x_{p+1}^n \quad \text{for all } t \in [t_p^n, t_{p+1}^n[.$$

In the same spirit with (3.21), we see that

$$q_n(\delta_n(T)) = y_{2^n-1}^n \quad \text{and} \quad q_n(\delta_n(t)) = q_n(t_p^n) = y_p^n \quad \text{for all } t \in [t_p^n, t_{p+1}^n[$$

and

$$q_n(\theta_n(T)) = y^n_{2^n} \quad \text{and} \quad q_n(\theta_n(t)) = q_n(t^n_{p+1}) = y^n_{p+1} \quad \text{for all } t \in [t^n_p, t^n_{p+1}[.$$

It is evident in view of (3.5) and (3.20) that we have for every $t \in I$

$$\begin{aligned} u_n(\theta_n(t)) &\in C\big(\theta_n(t), u_n(\delta_n(t)), q_n(\delta_n(t))\big) =: D_n(t) \\ &\subset C\big(I \times \rho\mathbb{B} \times (\kappa\mathbb{B} \cap Q)\big), \end{aligned} \tag{3.25}$$

along with the inclusion

$$\dot{u}_n(t) + z_n(t) \in -a\partial_C d_{D_n(t)}\big(u_n(\theta_n(t))\big) \quad \lambda\text{-a.e. } t \in I. \tag{3.26}$$

It is also clear from (3.24) that

$$\dot{q_n}(t) + f\big(t, u_n(\delta_n(t))\big) \in -b\partial_C d_Q\big(q_n\big(\theta_n(t)\big)\big) \quad \lambda\text{-a.e. } t \in I. \tag{3.27}$$

Step 3. Convergence of $(u_n(\cdot))_{n\geq 1}$ up to a subsequence.

From the inequality (3.19), we obviously have

$$\{\dot{u}_n(\cdot) : n \geq 1\} \subset L^\infty(I, \mathcal{H}, \lambda) \subset L^2(I, \mathcal{H}, \lambda) \subset L^1(I, \mathcal{H}, \lambda).$$

Hence, we may and do suppose that $\big(\dot{u}_n(\cdot)\big)_{n\geq 1}$ weakly converges in $L^2(I, \mathcal{H}, \lambda)$ (and then, it also weakly converges in $L^1(I, \mathcal{H}, \lambda)$) to a (class of) mapping $v(\cdot) \in L^2(I, \mathcal{H}, \lambda) \subset L^1(I, \mathcal{H}, \lambda)$. Then, for every $t \in I$ and every $\xi \in \mathcal{H}$,

$$\lim_{n\to\infty} \int_{T_0}^T \big\langle \mathbf{1}_{[T_0,t]}(s)\xi, \dot{u}_n(s)\big\rangle\, ds = \int_{T_0}^T \big\langle \mathbf{1}_{[T_0,t]}(s)\xi, v(s)\big\rangle\, ds,$$

where $\mathbf{1}_{[T_0,t]}$ denotes the usual indicator function in the sense of measure theory of the interval $[T_0, t]$. It follows that for every $t \in I$ and every $\xi \in \mathcal{H}$,

$$\lim_{n\to\infty} \left\langle \xi, u_0 + \int_{T_0}^t \dot{u}_n(s) ds \right\rangle = \left\langle \xi, u_0 + \int_{T_0}^t v(s) ds \right\rangle.$$

Thus, we obtain

$$u_n(t) = u_0 + \int_{T_0}^t \dot{u}_n(s) ds \xrightarrow{w} u_0 + \int_{T_0}^t v(s) ds =: u(t) \quad \text{for all } t \in I, \tag{3.28}$$

and this says that the mapping $u(\cdot)$ is absolutely continuous on I with $v(\cdot)$ as a derivative, i.e.,

$$\dot{u}(\cdot) = v(\cdot) \quad \lambda\text{-a.e. } t \in I. \tag{3.29}$$

By virtue of the relative compactness property of $C\big(I \times \rho\mathbb{B} \times (\kappa\mathbb{B} \cap Q)\big)$ (see assumption (iii)) and the inclusions in (3.25), we may assume that for each $t \in I$, the sequence $\big(u_n(\theta_n(t))\big)_{n\geq 1}$ strongly converges to some $\widehat{u}(t) \in \mathcal{H}$. Due to the Lipschitz property in (3.19), we have for every $t \in I$ and every integer $n \geq 1$,

$$\begin{aligned} \|u_n(t) - \widehat{u}(t)\| &\leq \|u_n(t) - u_n(\theta_n(t))\| + \|u_n(\theta_n(t)) - \widehat{u}(t)\| \\ &\leq (a + c)(\theta_n(t) - t) + \|u_n(\theta_n(t)) - \widehat{u}(t)\| \\ &\leq (a + c)\Delta_n + \|u_n(\theta_n(t)) - \widehat{u}(t)\|, \end{aligned}$$

and this obviously entails that $u_n(t) \to \widehat{u}(t)$. The latter strong convergence along with the weak convergence in (3.28) ensure that

$$u_n(t) \to u(t) \quad \text{for all } t \in I.$$

Step 4. Cauchy property of $(q_n(\cdot))_{n\geq 1}$.

Let us start by setting for any integer $k \geq 1$ and any $t \in I$ such that $\dot{q}_k(t)$ exists

$$p_k(t) := \dot{q}_k(t) + f\big(t, u_k(\delta_k(t))\big) \in b\mathbb{B}, \tag{3.30}$$

where the latter inclusion is due to (3.27) and (2.8). Fix any integers $m, n \geq 1$. Putting together (3.27), (2.7) and the inclusion $q_n(\theta_n(t)) \in Q$ valid for every $t \in I$, we have

$$\big\langle b^{-1} p_n(t), q_n(\theta_n(t)) - q_m(t)\big\rangle \leq d_Q(q_m(t)) \quad \lambda\text{-a.e. } t \in I.$$

We derive from the latter inequality, the inclusion (3.30), the $2b$-Lipschitz property of $q_n(\cdot)$ and $q_m(\cdot)$ (coming from the inequality (3.23)) and from $q_m(\theta_m(t)) \in Q$ valid for any $t \in I$

$$\begin{aligned}
\langle p_n(t), q_n(t) - q_m(t)\rangle &= \langle p_n(t), q_n(t) - q_n(\theta_n(t))\rangle + \langle p_n(t), q_n(\theta_n(t)) - q_m(t)\rangle \\
&\leq b\big(\, \|q_n(t) - q_n(\theta_n(t))\| + d_Q(q_m(t))\big) \\
&\leq b\big(\, \|q_n(t) - q_n(\theta_n(t))\| + \|q_m(\theta_m(t)) - q_m(t)\|\,\big) \\
&\leq 2b^2(\theta_n(t) - t) + 2b^2(\theta_m(t) - t) \\
&= 2b^2\big(\theta_m(t) + \theta_n(t) - 2t\big),
\end{aligned} \tag{3.31}$$

for λ-almost every $t \in I$. Since m, n have been arbitrarily choosen, we can also write

$$\langle p_m(t), q_m(t) - q_n(t)\rangle \leq 2b^2\big(\theta_m(t) + \theta_n(t) - 2t\big) \quad \lambda\text{-a.e. } t \in I. \tag{3.32}$$

Adding the inequalities (3.31) and (3.32) yields

$$\begin{aligned}
&\langle \dot{q}_n(t) - \dot{q}_m(t), q_n(t) - q_m(t)\rangle \\
&\leq -\big\langle f\big(t, u_n(\delta_n(t))\big), q_n(t) - q_m(t)\big\rangle - \big\langle f\big(t, u_m(\delta_m(t))\big), q_m(t) - q_n(t)\big\rangle \\
&\quad + 4b^2\big(\theta_m(t) + \theta_n(t) - 2t\big) \\
&\leq \|q_n(t) - q_m(t)\|\, \big\|f\big(t, u_n(\delta_n(t))\big) - f\big(t, u_m(\delta_m(t))\big)\big\| \\
&\quad + 4b^2\big(\theta_m(t) + \theta_n(t) - 2t\big),
\end{aligned} \tag{3.33}$$

for λ-almost every $t \in I$. From the Lipschitz property in (3.2) of assumption (ii) and from the inclusion $\{u_k(\delta_k(t)) : t \in I, k \geq 1\} \subset \rho\mathbb{B}$, we see that for all $t \in I$, all integers $k, k' \geq 1$,

$$\big\|f\big(t, u_k(\delta_k(t))\big) - f\big(t, u_{k'}(\delta_{k'}(t))\big)\big\| \leq l\, \|u_k(\delta_k(t)) - u_{k'}(\delta_{k'}(t))\|\,. \tag{3.34}$$

Through (3.33), (3.34) and the elementary estimate $ab \leq 2^{-1}(a^2 + b^2)$ valid for every $(a, b) \in \mathbb{R}^2$, we see that

$$\begin{aligned}
\langle \dot{q}_n(t) - \dot{q}_m(t), q_n(t) - q_m(t)\rangle \leq\, & 2^{-1}\big(\, \|q_m(t) - q_n(t)\|^2 + l^2\, \|u_n(\delta_n(t)) - u_m(\delta_m(t))\|^2\,\big) \\
&+ 4b^2(\theta_m(t) + \theta_n(t) - 2t).
\end{aligned} \tag{3.35}$$

Now, define $\psi_{m,n}, A_{m,n} : I \to \mathbb{R}$ by

$$\psi_{m,n}(t) := 2^{-1} \|q_m(t) - q_n(t)\|^2 \quad \text{for all } t \in I$$

and

$$A_{m,n}(t) := 2^{-1} l^2 \|u_n(\delta_n(t)) - u_m(\delta_m(t))\|^2 + 4b^2\big(\theta_m(t) + \theta_n(t) - 2t\big) \quad \text{for all } t \in I.$$

With the above definitions and the estimate (3.35) at hands, we get

$$\dot{\psi}_{m,n}(t) \le \psi_{m,n}(t) + A_{m,n}(t) \quad \lambda\text{-a.e. } t \in I.$$

A direct application of Gronwall lemma (see Lemma 3.1) gives

$$\psi_{m,n}(t) \le e^{T-T_0} \int_{T_0}^{T} A_{m,n}(s)ds.$$

On the other hand, it is an exercise to show that $A_{m,n}(t) \to 0$ as $m, n \to \infty$ as well as

$$|A_{m,n}(t)| \le \frac{l^2}{2}(2\rho)^2 + 8b^2(T - T_0) \quad \text{for all } m, n \ge 1.$$

We are then in position to apply the Lebesgue dominated convergence theorem to obtain for every $t \in I$,

$$\psi_{m,n}(t) \to 0 \quad \text{as } m, n \to \infty.$$

Therefore, for every $t \in I$, the sequence $(q_k(t))_{k \ge 1}$ has the Cauchy property in the Hilbert space $\mathcal{H}$ and this guarantees the existence of $q(t) \in \mathcal{H}$ such that

$$q_k(t) \to q(t).$$

Further, from (3.23), we see that $q(\cdot)$ is a $2b$-Lipschitz mapping and this allows us to assume (proceeding as in the beginning of Step 3) that $(\dot{q}_n(\cdot))_{n \ge 1}$ weakly converges in $L^1(I, \mathcal{H}, \lambda)$ to $\dot{q}(\cdot)$.

Step 5. The mapping $\Phi(\cdot) := (u(\cdot), q(\cdot))$ is a trajectory solution of $(\mathcal{SP})$.

Fix for a moment any integer $n \ge 1$ and any real $t \in I$. Using (3.3), (3.25) and the $(a + c)$-Lipschitz property of $u_n(\cdot)$ (see (3.19)), we get

$$\begin{aligned} d\big(u_n(t), C\big(t, u(t), q(t)\big)\big) \le\ & d\big(u_n(\theta_n(t)), D_n(t)\big) \\ & + \|u_n(t) - u_n(\theta_n(t))\| + K(\theta_n(t) - t) \\ & + L\|u(t) - u_n(\delta_n(t))\| + L'\|q(t) - q_n(\delta_n(t))\| \\ \le\ & (K + a + c)(\theta_n(t) - t) + L\|u(t) - u_n(\delta_n(t))\| \\ & + L'\|q(t) - q_n(\delta_n(t))\|. \end{aligned}$$

Since $\theta_n(t) \downarrow t$, $u_n(\delta_n(t)) \to u(t)$ and $q_n(\delta_n(t)) \to q(t)$, the latter estimate guarantees

$$d\big(u_n(t), C(t, u(t), q(t))\big) \to 0,$$

and such a convergence property ensures (thanks to the fact that $C(\cdot, \cdot, \cdot)$ is closed valued)

$$u(t) \in C\big(t, u(t), q(t)\big).$$

On the other hand, from the convergence $Q \ni q_n(\delta_n(t)) \to q(t)$ and the closedness of Q, we obviously have $q(t) \in Q$.

Now, through the inequality in (3.19), we see that the sequence $(z_n(\cdot))_{n\geq 1}$ weakly converges (up to a subsequence) in $L^1(I,\mathcal{H},\lambda)$ to a mapping $z(\cdot)\in L^1(I,\mathcal{H},\lambda)$. Coming back to the beginning of Step 3 and to (3.29), the sequence $\big(\dot u_n(\cdot)+z_n(\cdot)\big)_{n\geq 1}$ weakly converges in $L^1(I,\mathcal{H},\lambda)$ to $\dot u(\cdot)+z(\cdot)$. Then, by the classical Mazur's lemma, we are able to construct a sequence $(\xi_n(\cdot))_{n\geq 1}$ of $L^1(I,\mathcal{H},\lambda)$ which strongly converges to $\dot u_n(\cdot)+z_n(\cdot)$ and such that

$$\xi_n(\cdot)\in \operatorname{co}\{\dot u_k(\cdot)+z_k(\cdot):\ k\geq n\}\quad \text{for all } n\geq 1. \tag{3.36}$$

Extracting a subsequence if necessary, we may and do suppose that $(\xi_n(\cdot))_{n\geq 1}$ also converges λ-almost everywhere to $\dot u(\cdot)+z(\cdot)$. This and the inclusion (3.36) easily give

$$\dot u(t)+z(t)\in \bigcap_{n\geq 1}\overline{\operatorname{co}}\{\dot u_k(\cdot)+z_k(\cdot): k\geq n\}\quad \lambda\text{-a.e. } t\in I,$$

from which we derive for λ-almost every $t\in I$

$$\langle h,\dot u(t)+z(t)\rangle\leq \inf_{n\geq 1}\sup_{k\geq n}\langle h,\dot u_k(t)+z_k(t)\rangle\quad \text{for all } h\in\mathcal{H}.$$

Through (3.26) and the latter inequality, we see that for λ-almost every $t\in I$

$$\langle h,\dot u(t)+z(t)\rangle\leq \limsup_{n\to\infty}\sigma\Big(-ah,\partial_C d_{D_n(t)}\big(u_n(\theta_n(t))\big)\Big)\quad \text{for all } h\in\mathcal{H}.$$

We are then in a position to apply Proposition 2.4 to get for λ-almost every $t\in I$

$$\langle h,\dot u(t)+z(t)\rangle\leq \sigma\Big(-ah,\partial_C d_{C(t,u(t),q(t))}(u(t))\Big)\quad \text{for all } h\in\mathcal{H}.$$

Then, the equivalence in (2.9) coming from Hahn-Banach separation theorem and the inclusion (2.8) guarantee that

$$\dot u(t)+z(t)\in -a\partial_C d_{C(t,u(t),q(t))}\big(u(t)\big)\subset -N\big(C(t,u(t),q(t));u(t)\big), \tag{3.37}$$

for λ-almost every $t\in I$.

Through assumption (ii), it is clear that $\big(f(\cdot,u_n(\delta_n(\cdot)))\big)_{n\geq 1}$ strongly converges in $L^1(I,\mathcal{H},\lambda)$ to $f(\cdot,u(\cdot))$. Taking into account the end of Step 4, we obtain that the sequence $\big(\dot q_n(\cdot)+f(\cdot,u_n(\delta_n(\cdot)))\big)_{n\geq 1}$ weakly converges in $L^1(I,\mathcal{H},\lambda)$ to $\dot q(\cdot)+f(\cdot,u(\cdot))$. Applying as above Mazur's lemma and profiting from (3.27), we obtain

$$\langle h,\dot q(t)+f(t,u(t))\rangle\leq \limsup_{n\to\infty}\sigma\big(-bh,\partial_C d_Q(q_n(\theta_n(t)))\big),$$

for λ-almost every $t\in I$ and all $h\in\mathcal{H}$. Using Proposition 2.4, we get for λ-almost every $t\in I$

$$\langle h,\dot q(t)+f(t,u(t))\rangle\leq \sigma\big(-bh,\partial_C d_Q(q(t))\big)\quad \text{for all } h\in\mathcal{H}.$$

Invoking (2.9) and (2.8), we arrive to

$$\dot q(t)+f(t,u(t))\in -b\partial_C d_Q(q(t))\subset -N\big(Q;q(t)\big)\quad \lambda\text{-a.e. } t\in I. \tag{3.38}$$

Defining $\Phi: I\to\mathcal{H}^2$ by

$$\Phi(t):=(u(t),q(t))\quad \text{for all } t\in I$$

and taking into account all the development above (in particular (3.37) and (3.38)), we obtain that Φ_1, Φ_2 are Lipschitz-continuous mappings on I satisfying

$$\dot{\Phi}_1(t) + z(t) \in -N\big(C(t, \Phi(t)); \Phi_1(t)\big) \quad \lambda\text{-a.e. } t \in I$$

and

$$\dot{\Phi}_2(t) + f\big(t, \Phi_1(t)\big) \in -N\big(Q; \Phi_2(t)\big) \quad \lambda\text{-a.e. } t \in I.$$

It remains to show that $z(t) \in G\big(t, u(t), q(t)\big)$ for λ-almost every $t \in I$. Since $(z_n(\cdot))_{n\geq 1}$ weakly converges to $z(\cdot)$ in $L^1(I, \mathcal{H}, \lambda)$, the Mazur's lemma (up to a subsequence) allows us to write

$$z(t) \in \bigcap_{n\geq 1} \overline{\text{co}}\,\{z_k(t) : k \geq n\} \quad \lambda\text{-a.e. } t \in I.$$

This inclusion along with the following one valid for every $t \in I$ and every $n \geq 1$,

$$z_n(t) \in G\big(t, u_n(\delta_n(t)), q_n(\delta_n(t))\big)$$

allows us to find a Borel subset $\Omega \subset I$ with $\lambda(I \setminus \Omega) = 0$ such that for all $t \in \Omega$ and all $h \in \mathcal{H}$,

$$\langle h, z(t)\rangle \leq \limsup_{n\to\infty} \sigma\Big(h, G\big(t, u_n(\delta_n(t)), q_n(\delta_n(t))\big)\Big).$$

Then, for each $t \in I\backslash\Omega$, using the fact that $G(t, \cdot, \cdot)$ is scalarly upper-semicontinuous, we arrive to the inequality

$$\langle h, z(t)\rangle \leq \sigma\big(h, G(t, u(t), q(t))\big) \quad \text{for all } h \in \mathcal{H},$$

which entails $z(t) \in G\big(t, u(t), q(t)\big)$ by the closedness and convexity of $G(t, u(t), q(t))$ and by (2.9). This finishes the proof. □

Remark 3.3. As mentioned in [32], for any $x, y \in \mathcal{H}$, the mapping $\text{proj}_{G(\cdot,x,y)}(0)$ is λ-Bochner measurable whenever the Hilbert space $\mathcal{H}$ is separable and the multimapping $G(\cdot, x, y)$ is Lebesgue measurable, i.e., its graph belongs to $\mathcal{L}(I) \otimes \mathcal{B}(\mathcal{H})$, where $\mathcal{L}(I)$ and $\mathcal{B}(\mathcal{H})$ denote respectively the Lebesgue σ-field of I and the Borel σ-field of $\mathcal{H}$.

4. Absolutely continuous subsmooth sweeping process

In the present section, we only assume that the moving set in Theorem 3.2 varies in an absolutely continuous way, i.e., there is some absolutely continuous function $v(\cdot) : I \to \mathbb{R}$ such that

$$\text{haus}\big(C(t_1, x_1, y_1), C(t_2, x_2, y_2)\big) \leq |v(t_2) - v(t_1)| + L\|x_1 - x_2\| + L'\|y_1 - y_2\|.$$

The existence of solutions for the problem $(\mathcal{SP})$ under such an assumption is obtained through the reduction method due to J.J. Moreau ([28], see also [31, 21]).

Theorem 4.1. *Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings and $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Let Q be a closed convex subset of $\mathcal{H}$, $(u_0, q_0) \in \mathcal{H} \times Q$ with $u_0 \in C(T_0, u_0, q_0)$. Assume that (i)-(ii) in Theorem 3.2 hold. Assume also that:*

(iii') *the set $C\big(I \times \rho\mathbb{B} \times Q\big)$ is relatively compact;*

(iv') *there exist a real* $L \in [0,1[$ *a real* $L' \geq 0$ *and an absolutely continuous function* $v : I \to \mathbb{R}$ *such that*

$$\operatorname{haus}\big(C(t_1,x_1,y_1), C(t_2,x_2,y_2)\big) \leq |v(t_2) - v(t_1)| + L\,\|x_1 - x_2\| + L'\,\|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$*, all* $x_1, x_2 \in \rho\mathbb{B}$ *and all* $y_1, y_2 \in Q$*.*

(v') *the multimapping* G *is nonempty closed convex valued,* $G(t,\cdot,\cdot)$ *is scalarly upper semicontinuous for each* $t \in I$*, and for each* $(x,y) \in \rho\mathbb{B} \times Q$ *the mapping* $\operatorname{proj}_{G(\cdot,x,y)}(0) : I \to \mathcal{H}$ *is* λ*-Bochner measurable on* I *and there exists a real* $\alpha \geq 0$ *such that*

$$\left\|\operatorname{proj}_{G(t,x,y)}(0)\right\| \leq \alpha(1 + \|x\| + \|y\|),$$

for all $t \in I$*, all* $x \in \rho\mathbb{B}$ *and all* $y \in Q$*.*

Then, there exists a solution $\Phi = (\Phi_1, \Phi_2) : I \to \mathcal{H}^2$ *of the differential inclusion* $(\mathcal{SP})$*, that is,*

(a) *the mappings* Φ_1, Φ_2 *are absolutely continuous on* I *and satisfy* $\Phi_1(T_0) = u_0$*,* $\Phi_2(T_0) = q_0$ *along with*

$$\big(\Phi_1(t), \Phi_2(t)\big) \in C\big(t, \Phi_1(t), \Phi_2(t)\big) \times Q \quad \text{for all}\, t \in I;$$

(b) *for* λ*-almost every* $t \in I$*,*

$$\dot{\Phi}_2(t) + f(t, \Phi_1(t)) \in -N(Q; \Phi_2(t));$$

(c) *there exists a* λ*-Bochner integrable mapping* $z : I \to \mathcal{H}$ *with*

$$z(t) \in G(t, \Phi(t)) \quad \lambda\text{-a.e. } t \in I,$$

and such that

$$\dot{\Phi}_1(t) + z(t) \in -N\big(C(t,\Phi(t)); \Phi_1(t)\big) \quad \lambda\text{-a.e. } t \in I.$$

Proof. Fix any real $\eta > 0$. The function $w : I \to w(I)$ defined by

$$w(t) := T_0 + \int_{T_0}^{t} (|\dot{v}(\tau)| + \eta)d\tau \quad \text{for all}\, t \in I$$

is obviously absolutely continuous on I and satisfies (see (iv'))

$$\operatorname{haus}\big(C(t_1,x_1,y_1), C(t_2,x_2,y_2)\big) \leq |w(t_2) - w(t_1)| + L\|x_1 - x_2\| + L'\|y_1 - y_2\|, \tag{4.1}$$

for every $t_1, t_2 \in I$, every $x_1, x_2 \in \rho\mathbb{B}$ and every $y_1, y_2 \in Q$. Since the function $w(\cdot)$ is increasing on I, it is has an increasing inverse $w^{-1} : J := [T_0, \overline{T}] \to I$, where $\overline{T} := w(T)$. We claim that the function w^{-1} is η^{-1}-Lipschitz continuous on J. Indeed, for every $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$, there are two reals $T_0 \leq t_1 < t_2 \leq T$ such that $\tau_i = w(t_i)$ for $i \in \{1,2\}$, hence

$$w^{-1}(\tau_2) - w^{-1}(\tau_1) = t_2 - t_1 \leq \eta^{-1} \int_{t_1}^{t_2} \dot{w}(t)dt \leq \eta^{-1}\big(w(t_2) - w(t_1)\big) \leq \eta^{-1}(\tau_2 - \tau_1).$$

With the function w^{-1} at hands, we define the multimapping $\overline{C} : J \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ by setting for every $\tau \in J$ and every $x, y \in \mathcal{H}$,

$$\overline{C}(\tau, x, y) := C\big(w^{-1}(\tau), x, y\big).$$

We also need to define a multimapping $\overline{G} : J \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and a mapping $\overline{f} : J \times \mathcal{H} \to \mathcal{H}$ with

$$\overline{G}(\tau, x, y) := \theta\big(w^{-1}(\tau)\big)^{-1} G(w^{-1}(\tau), x, y) \quad \text{for all } \tau \in J, x, y \in \mathcal{H},$$

and

$$\overline{f}(\tau, x) := \theta\big(w^{-1}(\tau)\big)^{-1} f(w^{-1}(\tau), x) \quad \text{for all } \tau \in J, x \in \mathcal{H},$$

where $\theta : I \to [\eta, +\infty[$ is a λ-measurable function which coincide λ-amost everywhere with the derivative $\dot{w}(\cdot)$. Obviously, the family $(\overline{C}(\tau, x, y))_{\tau \in J, x, y \in \mathcal{H}}$ is equi-uniformly subsmooth and

$$\overline{C}(\tau, x, y) \subset \rho\mathbb{B} \quad \text{for all } \tau \in J, x, y \in \mathcal{H}.$$

Thanks to assumption (ii), it is readily seen that $\overline{f}(\cdot, x)$ is λ-Bochner measurable on J along with

$$\|\overline{f}(\tau, x)\| \leq \eta^{-1}\beta(1 + \|x\|) \quad \text{for all } \tau \in J, x \in \rho\mathbb{B}$$

and

$$\|\overline{f}(\tau, x_1) - \overline{f}(\tau, x_2)\| \leq \eta^{-1} l \|x_1 - x_2\| \quad \text{for all } \tau \in J, x_1, x_2 \in \rho\mathbb{B}.$$

From assumption (iii'), we get that $\overline{C}\big(J \times \rho\mathbb{B} \times Q\big)$ is relatively compact. On the other hand, we derive from (4.1) that

$$\begin{aligned} \operatorname{haus}\big(\overline{C}(\tau_1, x_1, y_1), \overline{C}(\tau_2, x_2, y_2)\big) &= \operatorname{haus}\big(C(w^{-1}(\tau_1), x_1, y_1), C(w^{-1}(\tau_2), x_2, y_2)\big) \\ &\leq \big|w(w^{-1}(\tau_1)) - w(w^{-1}(\tau_2))\big| \\ &\quad + L\|x_1 - x_2\| + L'\|y_1 - y_2\| \\ &\leq |\tau_2 - \tau_1| + L\|x_1 - x_2\| + L'\|y_1 - y_2\|, \end{aligned}$$

for every $\tau_1, \tau_2 \in J$, every $x_1, x_2 \in \rho\mathbb{B}$ and every $y_1, y_2 \in Q$. Due to the properties of G coming from (v'), it is evident to observe that $\overline{G}$ takes nonempty closed convex values and that $\overline{G}(\tau, \cdot, \cdot)$ enjoys for any $\tau \in J$ the scalar upper semicontinuity property. Concerning the element of minimal norm, we can check that for all $\tau \in J$, all $x, y \in \mathcal{H}$,

$$\operatorname{proj}_{\overline{G}(\tau, x, y)}(0) = \theta\big(w^{-1}(\tau)\big)^{-1} \operatorname{proj}_{G\left(w^{-1}(\tau), x, y\right)}(0)$$

and this ensures the λ-Bochner measurability of the mapping $\operatorname{proj}_{\overline{G}(\cdot, x, y)}(0)$ for any $x \in \rho\mathbb{B}$ and any $y \in Q$ along with the estimate

$$\left\|\operatorname{proj}_{\overline{G}(\tau, x, y)}(0)\right\| \leq \eta^{-1}\alpha(1 + \|x\| + \|y\|),$$

valid for every $\tau \in J$ and every $x \in \rho\mathbb{B}$ and every $y \in Q$.

Taking into account what precedes, we can apply Theorem 3.2 to get the existence of two mappings $\varphi_1, \varphi_2 : J \to \mathcal{H}^2$ satisfying the following properties

(a') the mappings φ_1, φ_2 are Lipschitz continuous on J and satisfies $\varphi_1(T_0) = u_0$ and $\varphi_2(T_0) = q_0$ along with

$$\varphi(\tau) := \big(\varphi_1(\tau), \varphi_2(\tau)\big) \in \overline{C}\big(\tau, \varphi_1(\tau), \varphi_2(\tau)\big) \times Q \quad \text{for all } \tau \in J;$$

(b') for λ-almost every $\tau \in J$,

$$\dot{\varphi}_2(\tau) + \overline{f}(\tau, \varphi_1(\tau)) \in -N\big(Q; \varphi_2(\tau)\big);$$

(c') there exists a λ-Bochner integrable mapping $z : J \to \mathcal{H}$ with

$$z(\tau) \in \overline{G}\big(\tau, \varphi(\tau)\big) \quad \lambda\text{-a.e. } \tau \in J,$$

and such that

$$\dot{\varphi}_1(\tau) + z(\tau) \in -N\big(\overline{C}(\tau, \varphi_1(\tau), \varphi_2(\tau)); \varphi_1(\tau)\big) \quad \lambda\text{-a.e. } \tau \in J.$$

For each $i \in \{1, 2\}$, define $\Phi : I \to \mathcal{H}$ by

$$\Phi_i(t) := \varphi_i(w(t)) \quad \text{for all}\, t \in I,$$

which is obviously absolutely continuous on I along with

$$\dot{\Phi}_i(t) = \dot{w}(t)\dot{\varphi}_i(w(t)) \quad \lambda\text{-a.e. } t \in I. \tag{4.2}$$

It directly follows from (a') that $\Phi_1(T_0) = \varphi_1(T_0) = u_0$, $\Phi_2(T_0) = \varphi_2(T_0) = q_0$ and

$$\big(\Phi_1(t), \Phi_2(t)\big) \in C\big(t, \Phi_1(t), \Phi_2(t)\big) \times Q \quad \text{for all}\, t \in I,$$

where the inclusion is due to the definition of $\overline{C}$. Putting together (4.2), (b') and the definition of θ and f give a λ-negligible set of I such that

$$\dot{\Phi}_2(t) + f(t, \Phi_1(t)) \in -N\big(Q; \Phi_2(t)\big) \quad \text{for all}\, t \in I \setminus N_1.$$

It remains to observe that the definition of $\overline{C}, \overline{G}$ and (c') ensure the existence of a negligible set N_2 of I such that the λ-Bochner integrable mapping $Z(\cdot) = \theta(\cdot)z(w(\cdot))$ satisfies for every $t \in I \setminus N_2$

$$Z(t) \in G(t, \Phi(t)) \quad \text{and} \quad \dot{\Phi}_1(t) + Z(t) \in -N\big(C(t, \Phi(t)); \Phi_1(t))\big).$$

The proof is complete. □

The case where $f \equiv 0$ and $Q = \{0\}$ in the latter theorem leads to the existence of solutions for $(\mathcal{P})$.

Corollary 4.2. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be two multimappings, $u_0 \in \mathcal{H}$ with $u_0 \in C(T_0, u_0)$. Assume that:*

(i) *the family $\big(C(t, x)\big)_{t \in I, x \in \mathcal{H}}$ is equi-uniformly subsmooth and there exists a real $\rho > 0$ such that*

$$C(t, x) \subset \rho\mathbb{B} \quad \textit{for all}\, t \in I, x \in \mathcal{H};$$

(ii) *the set $C\big(I \times \rho\mathbb{B}\big)$ is relatively compact;*

(iii) *there exist a real $L \in [0, 1[$ and an absolutely continuous function $v : I \to \mathbb{R}$ such that*

$$\operatorname{haus}\big(C(t_1, x_1), C(t_2, x_2)\big) \le |v(t_2) - v(t_1)| + L\, \|x_1 - x_2\|\,,$$

for all $t_1, t_2 \in I$, all $x_1, x_2 \in \rho\mathbb{B}$;

(iv) *the multimapping G is nonempty closed convex valued, $G(t, \cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $x \in \rho\mathbb{B}$ the mapping $\operatorname{proj}_{G(\cdot,x)}(0) : I \to \mathcal{H}$ is λ-Bochner measurable on I and there exists a real $\alpha \ge 0$ such that*

$$\left\|\operatorname{proj}_{G(t,x)}(0)\right\| \le \alpha(1 + \|x\|) \quad \textit{for all}\, t \in I, x \in \rho\mathbb{B}.$$

Then, there exists an absolutely continuous mapping $u(\cdot) : I \to \mathcal{H}$ *such that*

$$\begin{cases} -\dot{u}(t) \in N\big(C(t,u(t)); u(t)\big) + G(t,u(t)) & \lambda\text{-}a.e.\ t \in I, \\ u(t) \in C(t,u(t)) \quad \text{for all}\, t \in I, \\ u(T_0) = u_0. \end{cases}$$

Proof. It suffices to apply Theorem 4.1 with $Q := \{0\}$ and $f \equiv 0$ along with the multimappings $\widehat{C} : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $\widehat{G} : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ defined by

$$\widehat{C}(t,x,y) := C(t,x) \quad \text{and} \quad \widehat{G}(t,x,y) := G(t,x) \quad \text{for all}\, (t,x,y) \in I \times \mathcal{H}^2.$$

□

5. Reduction of state-dependent second order sweeping process to the first order one

In this last section, we derive from our existence results for $(\mathcal{SP})$ (see Theorem 3.2 and Theorem 4.1) the existence of a trajectory solution for the problem $(\mathcal{Q})$, i.e., for the second order state-dependent sweeping process with outward normal at the velocity. As mentioned in the introduction, such a reduction has been first independently observed by J. Noël ([33]) and M. Yarou ([41]) in $\mathbb{R}^d$ with a prox-regular set moving in an absolute continuous way. The extension in any Hilbert space has been developped in [32] through the introduction of the BV differential inclusion $(\mathcal{FMSP})$ driven by a prox-regular moving set.

The following result lies at the heart of the reduction technique provided by [33, 41, 32].

Proposition 5.1. *Let* $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ *and* $F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ *be two multimappings. Let also* $(u_0, v_0) \in \mathcal{H}^2$ *with* $v_0 \in C(T_0, u_0)$.

If $\Phi_1, \Phi_2 : I \to \mathcal{H}$ *are absolutely continuous mappings satisfying with* $\Phi := (\Phi_1, \Phi_2)$ *the following differential inclusion (in the sense of Theorem 4.1)*

$$(\mathcal{R}) \begin{cases} -\dot{\Phi}(t) \in N^C\big(C(t,\Phi_2(t)) \times \mathcal{H}; \Phi(t)\big) + F\big(t,\Phi(t)\big) \times \{-\Phi_1(t)\} & \lambda\text{-}a.e.\ t \in I, \\ \Phi(t) \in C\big(t,\Phi_2(t)\big) \times \mathcal{H} \quad \text{for all}\, t \in I, \\ \Phi(T_0) = (v_0, u_0), \end{cases}$$

then $\dot{\Phi}_2 = \Phi_1$ λ*-almost everywhere on* I *and the mapping* $\Phi_2(\cdot)$ *is a solution of the second order sweeping process*

$$\begin{cases} -\ddot{\Phi}_2(t) \in N^C\big(C(t,\Phi_2(t)), \dot{\Phi}_2(t)\big) + F\big(t, \dot{\Phi}_2(t), \Phi_2(t)\big), \\ \dot{\Phi}_2(t) \in C(t,\Phi_2(t)), \\ \Phi_2(T_0) = u_0, \dot{\Phi}_2(T_0) = v_0, \end{cases}$$

that is,

(a) *the mapping* Φ_2 *is absolutely continuous on* I *and* $\Phi_2(T_0) = u_0$*;*
(b) *there exists an absolutely continuous mapping* $\phi : I \to \mathcal{H}$ *with* $\phi(T_0) = v_0$, $\dot{\Phi}_2 = \phi$ *Lebesgue-almost everywhere on* I *and*

$$\phi(t) \in C\big(t, \Phi_2(t)\big) \quad \text{for all}\, t \in I;$$

(c) *there exists a λ-Bochner integrable mapping $z : I \to \mathcal{H}$ with*

$$z(t) \in F\big(t, \dot{\Phi}_2(t), \Phi_2(t)\big) \quad \lambda\text{-}a.e.\ t \in I,$$

and such that

$$\ddot{\Phi}_2(t) + z(t) \in -N^C\big(C(t, \Phi_2(t)); \dot{\Phi}_2(t)\big) \quad \lambda\text{-}a.e.\ t \in I.$$

Proof. Assume that $\Phi(\cdot) : I \to \mathcal{H}^2$ is a solution of $(\mathcal{R})$. Let us first define the multimappings $S_C, G_F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ by setting for every $(t, x, y) \in I \times \mathcal{H}^2$,

$$S_C(t, x, y) := C(t, y) \times \mathcal{H} \quad \text{and} \quad G_F(t, x, y) := F(t, x, y) \times \{-x\}.$$

Note that

$$\big(\Phi_1(T_0), \Phi_2(T_0)\big) = (v_0, u_0) \in S_C\big(T_0, \Phi(T_0)\big) = C\big(T_0, \Phi_2(T_0)\big) \times \mathcal{H}.$$

Now, we are going to show that the properties (a)-(b)-(c) claimed above hold true. Since it is evident that (a) holds, we only focus on (b) and (c). Thanks to the fact that $\Phi(\cdot)$ is a solution of $(\mathcal{R})$, we have

$$\big(\Phi_1(t), \Phi_2(t)\big) = \Phi(t) \in S_C\big(t, \Phi(t)\big) \quad \text{for all } t \in I, \tag{5.1}$$

and

$$\dot{\Phi}_2(t) - \Phi_1(t) \in -N\big(\mathcal{H}; \Phi_2(t)\big) = \{0\} \quad \lambda\text{-a.e. } t \in I, \tag{5.2}$$

i.e., $\dot{\Phi}_2 = \Phi_1$ λ-almost everywhere on I. Further, we know that there is a λ-Bochner integrable mapping $z(\cdot) : I \to \mathcal{H}$ satisfying

$$z(t) \in G_F\big(t, \Phi(t)\big) = F\big(t, \Phi_1(t), \Phi_2(t)\big) \quad \lambda\text{-a.e. } t \in I$$

along with

$$\dot{\Phi}_1(t) + z(t) \in -N\big(C(t, \Phi_2(t)); \Phi_1(t)\big) \quad \lambda\text{-a.e. } t \in I.$$

This and the equality $\dot{\Phi}_2 = \Phi_1$ valid λ-almost everywhere on I (see (5.2)) guarantee that (b) holds true. Concerning (c), it suffices to set $\phi := \Phi_1$ and to invoke (5.1) to get

$$\phi(t) \in C\big(t, \Phi_2(t)\big) \quad \text{for all } t \in I.$$

All together means that the mapping $\Phi_2(\cdot)$ satisfies properties (a)-(b) and (c) as desired. The proof is complete. □

Remark 5.2. It is worth pointing out that we can replace in the latter proposition the Clarke normal cone N^C by any concept of normals coming from Variational Analysis.

We are now in a position to reduce the second order sweeping process with outward normal at the velocity to the Moreau's one.

Theorem 5.3. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings, $(u_0, v_0) \in \mathcal{H}^2$ with $v_0 \in C(T_0, u_0)$. Assume that:*

(i) *the family $\big(C(t, y)\big)_{t \in I, y \in \mathcal{H}}$ is equi-uniformly subsmooth and there exists a real $\rho > 0$ such that*

$$C(t, y) \subset \rho\mathbb{B} \quad \textit{for all } t \in I, y \in \mathcal{H};$$

(ii) *the set* $C(I \times \mathcal{H})$ *is relatively compact;*

(iii) *there exist a real* $L' \geq 0$ *and an absolutely continuous function* $v : I \to \mathbb{R}$ *such that*

$$\operatorname{haus}\big(C(t_1, y_1), C(t_2, y_2)\big) \leq |v(t_2) - v(t_1)| + L' \|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$ *and* $y_1, y_2 \in \mathcal{H}$*;*

(iv) *the multimapping* F *is nonempty closed convex valued,* $F(t, \cdot, \cdot)$ *is scalarly upper semicontinuous for each* $t \in I$*, and for each* $(x, y) \in \rho\mathbb{B} \times \mathcal{H}$*, the mapping* $\operatorname{proj}_{F(\cdot,x,y)}(0) : I \to \mathcal{H}$ *is* λ*-Bochner measurable on* I *and there exists a real* $\alpha \geq 0$ *such that*

$$\left\|\operatorname{proj}_{F(t,x,y)}(0)\right\| \leq \alpha(1 + \|x\| + \|y\|),$$

for all $t \in I$*,* $x \in \rho\mathbb{B}$ *and all* $y \in \mathcal{H}$*.*

Then, there exists a solution $u(\cdot) : I \to \mathcal{H}$ *of the second order sweeping process (in the sense of (a)-(b)-(c) in Proposition 5.1)*

$$\begin{cases} -\ddot{u}(t) \in N\big(C(t, u(t)); \dot{u}(t)\big) + F(t, \dot{u}(t), u(t)), \\ \dot{u}(t) \in C\big(t, u(t)\big), \\ u(T_0) = u_0, \dot{u}(T_0) = v_0. \end{cases}$$

Proof. All conditions of Theorem 4.1 are satisfied with $S_C, G_F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ defined by

$$S_C(t,x,y) := C(t,y) \times \mathcal{H} \quad \text{and} \quad G_F(t,x,y) := F(t,x,y) \times \{-x\} \quad \text{for all}(t,x,y) \in I \times \mathcal{H}^2.$$

This ensures the existence of a mapping $\Phi(\cdot)$ satisfying

$$\begin{cases} -\dot{\Phi}(t) \in N\big(S_C(t, \Phi(t)); \Phi(t)\big) + G_F\big(t, \Phi(t)\big) & \lambda\text{-a.e. } t \in I, \\ \Phi(t) \in S_C(t, \Phi(t)) \quad \text{for all } t \in I, \\ \Phi(T_0) = (v_0, u_0). \end{cases}$$

It remains to apply Proposition 5.1 to get that $u(\cdot) := \Phi_2(\cdot)$ is a solution of the considered second order sweeping process. This completes the proof. ☐

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