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Farthest Distance Function to Strongly Convex Sets

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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The aim of the present paper is twofold. On one hand, we show that the strong convexity of a set is equivalent to the semiconcavity of its associated farthest distance function. On the other hand, we establish that the farthest distance of a point from a strongly convex set is the minimum of farthest distances of the given point from suitable closed balls separating the set and the point. Various other results on strongly convex sets are also provided.

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1. Introduction

Distance functions play a fundamental role in mathematical analysis, including convex analysis ([10, 36, 40]), variational analysis ([15, 25, 31, 36, 40, 44]), differential inclusions ([5, 16, 34, 43]), optimal control ([13, 32]), approximation theory ([20, 42]), shape analysis and PDE ([3, 4, 19]), etc. It is well-known (and easily seen) that the convexity of a given closed subset C in a normed space is equivalent to the convexity of its associated (standard/usual) distance function, namely $d_C(x) := \inf_{c\in C} \|x-c\|$. In the Hilbertian setting M. V. Balashov [6] extended such a characterization for the semiconvexity of the distance function (i.e., its convexity up to a square norm). More precisely, Balashov [6] showed that, in a Hilbert space, the semiconvexity of the distance function d_C (on any convex subset of a suitable enlargement of C) is equivalent to the uniform prox-regularity of the closed set C. Recall that a closed subset C in a Hilbert space X is uniformly prox-regular ([37]) (also known as positively reached, weakly convex, $\varphi-$convex, $\mathcal{O}(2)$-convex, proximally smooth, see, e.g., [12, 17, 21, 41, 45] and the references therein) with constant $r > 0$ provided that the nearest point mapping proj_C is well defined on a suitable enlargement of C (more precisely on the set $U_r(C) := \{x \in X : d_C(x) < r\}$) and continuous therein. Prox-regularity has been recognized as a fundamental concept in variational analysis

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which allows to go beyond convexity in several aspects of mathematical analysis and its applications (see, e.g., [9, 11, 18, 24, 26, 30] and the references therein).

In many situations, it is also of interest to strengthen the convexity property. This can be achieved with R-strongly convex sets ([23, 26, 38, 45, 46]) which are basically intersection of closed balls with common radius $R > 0$. For the use of such sets in diverse applied mathematical problems, we refer, e.g., to [14, 33] for the sweeping process, [22] for optimal control, [29, 39] for numerical optimization, [26, 28] for differential games, etc. In the work [7], it is established that for any closed bounded set in a Hilbert space X, the strong convexity property is equivalent to the semiconcavity of the (convex) farthest distance function $\operatorname{dfar}_C(x) := \sup_{c\in C} \|x - c\|$. The first aim of the present paper is to give an alternative proof of such a characterization of strongly convex sets. Our approach is in the line of [35] based on the fact that the complement of a prox-regular set is nothing but the union of a family of closed balls with common radius.

The distance function from a convex set is known to admit diverse descriptions involving duality relations from convex analysis. Namely, given a closed convex set C in a Hilbert space X, the distance function d_C has been described in the following ways (see, e.g., [20]) :

(i) in terms of the support function $\sigma(\cdot, C)$: for an appropriate $x^\star \in X$

$$d_C(x) = \langle x^\star, x\rangle - \sigma(x^\star, C);$$

(ii) by the duality property:

$$d_C(x) = \max_{\|\bar{x}\|=1} \inf_{y\in C} \langle \bar{x}, x - y\rangle = \inf_{y\in C} \max_{\|\bar{x}\|=1} \langle \bar{x}, x - y\rangle\,;$$

(iii) in terms of supporting hyperplanes: the distance of a point from a convex set is the maximum of distances to appropriate hyperplanes separating the set and the point.

Extensions of (i)–(ii) and (iii) to prox-regular sets have been recently considered in [1]. The second aim of the present work is to provide in details the analysis of the corresponding features (i), (ii) and (iii) when the set C satisfies the strengthened condition of strong convexity.

The paper is organized as follows. In Section 2, we give some preliminaries and notation needed for our analysis. Section 3 is devoted to the semiconcavity property of the farthest distance function associated to a strongly convex set. In the last section of the present work, we develop the analogs of the above properties (i)–(ii) and (iii) for a strongly convex set C.

2. Notation and preliminaries

Throughout this section and the next ones, $\mathbb{R}_+ := [0, +\infty[$ and X stands for a (real) Hilbert space *not reduced to zero* endowed with the inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|$ given by $\|x\|^2 = \langle x, x\rangle$ for all $x \in X$.

The closed (resp. open) ball in X centered at $x \in X$ with radius $r > 0$ is denoted by $B[x, r]$ (resp. $B(x, r)$). The letter $\mathbb{B}$ (resp. $\mathbb{U}$) stands for the closed (resp. open) unit ball in X, that is, $\mathbb{B} := B[0_X, 1]$ (resp. $\mathbb{U} := B(0_X, 1)$). We also set $\mathbb{S} := \{x \in X : \|x\| = 1\}$, that is, the unit sphere of X. The interior (resp. the closure) of a subset A of $(X, \|\cdot\|)$ is denoted by $\operatorname{int} A$ (resp. $\operatorname{cl} A$).

The distance function d_S and the farthest distance function dfar_S from a nonempty subset $S \subset X$ are defined for every $x \in X$ by

$$d_S(x) := d(x, S) := \inf_{y \in S} \|x - y\|$$

and

$$\mathrm{dfar}_S(x) := \mathrm{dfar}(x, S) := \sup_{y \in S} \|x - y\|.$$

To those functions are naturally associated the multimappings $\mathrm{Proj}_S : X \rightrightarrows X$ of nearest points in S and $\mathrm{Far}_S : X \rightrightarrows X$ of farthest points in S defined for every $x \in X$ by

$$\mathrm{Proj}_S(x) := \mathrm{Proj}(S, x) := \{y \in S : d_S(x) = \|x - y\|\}$$

and

$$\mathrm{Far}_S(x) := \mathrm{Far}(S, x) := \{y \in S : \mathrm{dfar}_S(x) = \|x - y\|\}.$$

Whenever $\mathrm{Proj}_S(\overline{x})$ (resp. $\mathrm{Far}_S(\overline{x})$) is reduced to a singleton for some $\overline{x} \in X$, that is, $\mathrm{Proj}_S(\overline{x}) = \{\overline{y}\}$ (resp. $\mathrm{Far}_S(\overline{x}) = \{\overline{y}\}$) the vector $\overline{y} \in S$ will be denoted by $\mathrm{proj}_S(\overline{x})$ (resp. $\mathrm{far}_S(\overline{x})$).

2.1. Normal cones and subdifferentials

Let S be a nonempty set in X. The *proximal normal cone* of S at $x \in S$, denoted by $N^P(S; x)$, is defined as

$$N^P(S; x) := \left\{v \in X : \exists \sigma \geq 0,\, \forall x' \in S, \langle v, x' - x\rangle \leq \sigma \|x' - x\|^2\right\}.$$

If S is convex, it is known that the proximal normal cone $N^P(S; x)$ coincides with the normal cone *in the sense of convex analysis*, that is,

$$N^P(S; x) = \{v \in X : \langle v, x' - x\rangle \leq 0,\, \forall x' \in S\} =: N(S; x).$$

Through the equivalences valid for every $x, y \in X$

$$y \in \mathrm{Proj}_S(x) \Leftrightarrow y \in S \text{ and } \langle x - y, c - y\rangle \leq \frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in S$$

and

$$y \in \mathrm{Far}_S(x) \Leftrightarrow y \in S \text{ and } \langle y - x, c - y\rangle \leq -\frac{1}{2}\|c - y\|^2 \quad \text{for all } c \in S, \tag{1}$$

we see that

$$x - y \in N^P(S; y) \quad \text{for all } (x, y) \in \mathrm{gph}\,\mathrm{Proj}_S \tag{2}$$

and

$$y - x \in N^P(S; y) \quad \text{for all } (x, y) \in \mathrm{gph}\,\mathrm{Far}_S, \tag{3}$$

where

$$\mathrm{gph}\,\mathrm{Proj}_S = \{(x, y) \in X \times X : y \in \mathrm{Proj}_S(x)\}$$

and

$$\mathrm{gph}\,\mathrm{Far}_S = \{(x, y) \in X \times X : y \in \mathrm{Far}_S(x)\}.$$

It is worth pointing out that for any $(x, y) \in \mathrm{gph}\,\mathrm{Far}_S$

$$y = \mathrm{far}_S\big(x + s(x - y)\big) \quad \text{for all } s > 0. \tag{4}$$

Notice also that both graphs $\mathrm{gph}\,\mathrm{Proj}_S$ and $\mathrm{gph}\,\mathrm{Far}_S$ are closed in $X \times X$ whenever the set S is closed.

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in X$ with $|f(x)| < +\infty$, one defines the *proximal subdifferential* of f at x as the set

$$\partial_P f(x) := \big\{\zeta \in X : (\zeta, -1) \in N^P\big(\mathrm{epi}\, f; (x, f(x))\big)\big\},$$

where as usual $\mathrm{epi}\, f := \{(u, r) \in X \times \mathbb{R} : f(u) \le r\}$ stands for the *epigraph* of the function f. Of course, if f is convex then the latter subdifferential is nothing but the *Moreau-Rockafellar subdifferential*, that is,

$$\partial_P f(x) = \{\zeta \in X : \langle \zeta, x' - x\rangle \le f(x') - f(x) \quad \forall x' \in X\} =: \partial f(x).$$

If f is convex near x and continuous at x, then

$$\partial_P f(x) \neq \emptyset \tag{5}$$

and $$f \text{ is Fréchet differentiable at } x \text{ whenever } \partial_P(-f)(x) \neq \emptyset. \tag{6}$$

If f is $C^{1,1}$ near x, it is an exercise to check that for a function $g : X \to \mathbb{R} \cup \{+\infty\}$ finite at x, one has

$$\partial_P (f + g)(x) = \nabla f(x) + \partial_P g(x). \tag{7}$$

Here the $C^{1,1}$-property of f near x means (as usual) that f is differentiable on a neighborhood of x with ∇f Lipschitz continuous there.

If $f : X \to \mathbb{R}$ is convex and Fréchet differentiable a point $\overline{x} \in X$ with $f(\overline{x}) = 0$ and $\nabla f(\overline{x}) \neq 0$, then it is known that for $S := \{x \in X : f(x) \le 0\}$ one has

$$N(S; \overline{x}) = \mathbb{R}_+ \nabla f(\overline{x}). \tag{8}$$

2.2. Strongly convex sets and prox-regular sets

This paragraph is devoted to the needed preliminaries on strong convexity and prox-regularity of sets in Hilbert spaces. We start with the definition of a strongly convex set. For more details on such a class of sets, we refer, e.g., to the book by E. S. Polovinkin and M. V. Balashov [38], to the survey by G. E. Ivanov and V. V. Goncharov ([23]) and to the references therein.

Definition 2.1. Let C be a nonempty subset in X. One says that C is R-*strongly convex* for some real $R > 0$ whenever there is a nonempty set $L \subset X$ such that

$$C = \bigcap_{x \in L} B[x, R]. \qquad \square$$

It is clear that every R-strongly convex set is closed with diameter less or equal than $2R$. Strongly convex sets can be characterized through the farthest distance function. In the statement of next Theorem 2.2 and in the analysis of an R-strongly convex set C of X in all the development of the paper, we will have to use the set

$$\mathcal{E}_R(C) := \{x \in X : \mathrm{dfar}_C(x) > R\}.$$

The statement of Theorem 2.2 collects certain characterizations of strongly convex sets essentially taken from M. V. Balashov and G. E. Ivanov [8], V. V. Goncharov and G. E. Ivanov [23], G. E. Ivanov [27] as explained in the arguments below.

Theorem 2.2. *Let C be a nonempty closed convex bounded subset not reduced to a singleton in X and let $R > 0$ be a real. The following assertions are equivalent:*

(a) *the set C is R-strongly convex;*

(b) *for all $x, x' \in C$ and for all $v \in N(C; x)$, one has*

$$\langle v, x' - x \rangle \le -\frac{\|v\|}{2R} \|x' - x\|^2 ;$$

(c) *the mapping far_C is well defined on $\mathcal{E}_R(C)$, and for every real $s > R$ one has for all $x, x' \in \mathcal{E}_s(C)$*

$$\|\mathrm{far}_C(x) - \mathrm{far}_C(x')\| \le \frac{1}{(s/R) - 1} \|x - x'\|;$$

(d) *the function $\mathrm{dfar}_C^2(\cdot)$ is differentiable on $\mathcal{E}_R(C)$ with a locally Lipschitz derivative and $\nabla \mathrm{dfar}_C^2(x) = 2\big(x - \mathrm{far}_C(x)\big)$ for all $x \in \mathcal{E}_R(C)$;*

(e) *for any pair (u, x) with $u \in \mathcal{E}_R(C)$ and $x \in \mathrm{Far}_C(u)$ one has*

$$x \in \mathrm{Far}_C\big(x + \frac{R}{\mathrm{dfar}_C(u)}(u - x)\big);$$

(f) *for any pair (u, x) with $u \in \mathcal{E}_R(C)$ and $x \in \mathrm{Far}_C(u)$ one has*

$$x = \mathrm{far}_C\big(x + \frac{t}{\mathrm{dfar}_C(u)}(u - x)\big) \quad \text{for all } t \in]R, +\infty[.$$

Proof. The equivalence (a)⇔(c) is shown in [8, Theorem 5] while an equivalent form of (c)⇔(d) can be found in [27, Theorem 1]. For the equivalence (a)⇔(b) we refer to [23, Theorem 2.1]. Regarding the equivalence (a)⇔(e), it is proved in [27, Theorem 2]. To justify (e)⇔(f), assume (e) and take any pair (u, x) with $u \in \mathcal{E}_R(C)$ and $x \in \mathrm{Far}_C(u)$ and take also any real $t > R$. By (e) we have

$$x \in \mathrm{Far}_C\big(x + \frac{R}{\mathrm{dfar}_C(u)}(u - x)\big),$$

hence by (4)

$$x = \mathrm{far}_C\big(x + \frac{t}{\mathrm{dfar}_C(u)}(u - x)\big),$$

so the implication (e)⇒(f) holds. For the converse implication assume that (f) holds. Let (u, x) be a pair with $u \in \mathcal{E}_R(C)$ and $x \in \mathrm{Far}_C(u)$. Take a real $\lambda_0 > 0$ with $u_\lambda := u + \lambda(u - x) \in \mathcal{E}_R(C)$ for every $\lambda \in]0, \lambda_0[$. By (4), for each $\lambda \in]0, \lambda_0[$ we have $x = \mathrm{far}_C(u_\lambda)$, so by (f)

$$x = \mathrm{far}_C\big(u_\lambda + \frac{t}{\mathrm{dfar}_C(u_\lambda)}(u_\lambda - x)\big)$$

for every real $t > R$. Making $t \downarrow R$ yields by the closedness of $\mathrm{gph}\,\mathrm{Far}_C$

$$x \in \mathrm{Far}_C\big(u_\lambda + \frac{R}{\mathrm{dfar}_C(u_\lambda)}(u_\lambda - x)\big),$$

which in turn gives by the closedness of $\mathrm{gph}\,\mathrm{Far}_C$ again

$$x \in \mathrm{Far}_C\big(u + \frac{R}{\mathrm{dfar}_C(u)}(u - x)\big)$$

since $u_\lambda \to u$ as $\lambda \downarrow 0$. This confirms the other implication (f)⇒(e) end finishes the proof. □

We now develop the concept of prox-regularity (see [37]). Historical comments and applications can be found, e.g., in the survey [18] and the book [44].

Definition 2.3. Let S be a nonempty closed subset of X and $r \in]0,+\infty]$. One says that S is *r-prox-regular* (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P(S;x) \cap \mathbb{B}$ and for every real $t \in]0,r]$, one has $x \in \mathrm{Proj}_S(x+tv)$.

A crucial class of uniformly prox-regular sets is given by the complements of open balls. More precisely, given any $x \in X$ and any real $r > 0$, it is known (and not difficult to check) that the set $S := X \setminus B(x,r)$ is r-prox-regular.

The next theorem provides some useful characterizations and properties of uniform prox-regular sets. For its proof, we refer, e.g., to [18, 37, 44]. Given a subset $S \subset X$ and an extended real $r > 0$, we recall the notation in the introduction

$$U_r(S) := \{x \in X : d_S(x) < r\}.$$

Theorem 2.4. *Let S be a nonempty closed subset of X and let $r \in]0,+\infty]$. The following assertions are equivalent:*

(a) *the set S is r-prox-regular;*

(b) *for all $x, x' \in S$, for all $v \in N^P(S;x)$, one has*

$$\langle v, x'-x\rangle \le \frac{1}{2r}\,\|v\|\,\|x-x'\|^2\,;$$

(c) *the mapping $\mathrm{proj}_S(\cdot)$ is well defined on $U_r(S)$, and for every real $s \in]0,r[$ one has for all $x, x' \in U_s(S)$*

$$\|\mathrm{proj}_S(x) - \mathrm{proj}_S(x')\| \le \frac{1}{1-(s/r)}\,\|x-x'\|\,;$$

(d) *the function $d_S^2(\cdot)$ is differentiable on $U_r(S)$ with a locally Lipschitz derivative and*

$$\nabla d_S^2(x) = 2\big(x - \mathrm{proj}_S(x)\big) \quad \textit{for all}\, x \in U_r(S);$$

(e) *for any $u \in U_r(S)\setminus S$ such that $\mathrm{proj}_S(u) =: x$ is well defined, one has*

$$x = \mathrm{proj}_S\big(x + \frac{t}{d_S(u)}(u-x)\big) \quad \textit{for all}\, t \in [0,r[.$$

We end this section with two results on prox-regular sets which will be at the heart of the development below. The first one is concerned with complements of prox-regular sets.

Theorem 2.5. ([35]) *Let S be an r-prox-regular subset of X with $r \in]0,+\infty[$. Then, for all $s \in]0,r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s.*

The second result by G. E. Ivanov [26, Theorem 1.12.3] gives a sufficient condition ensuring the prox-regularity for the Minkowski sum.

Theorem 2.6. ([26]) *Let C be an R-strongly convex subset of X with $R \in]0,+\infty]$ and let S be an r-prox-regular subset of X with $r \in]0,+\infty]$ such that $0 < R < r$. Then, the set $C+S$ is $(r-R)$-prox-regular, in particular closed.*

3. Semiconcavity of the farthest distance function

This section is devoted to the study of the semiconcavity property of the farthest distance function. Given a real $\sigma \geq 0$, a function $f : U \to \mathbb{R} \cup \{+\infty\}$ defined on a (not necessarily open) nonempty convex subset U of X is said to be linearly *σ-semiconvex* (or semiconvex with constant σ) on U provided that for every $t \in]0,1[$ and every $x, y \in U$

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \frac{\sigma}{2} t(1-t) \|x-y\|^2 .$$

It is well known (and not difficult to check) that the function f is σ-semiconvex on U for some $\sigma \geq 0$ if and only if the function $f(\cdot) + \sigma/2 \|\cdot\|^2$ is convex on the set U.

The function f is said to be *locally semiconvex* if f is semiconvex on a neighborhood of each point of U. If $-f$ is σ-semiconvex on U for some real $\sigma \geq 0$, then f is said to be *σ-semiconcave* on U. Similarly, f is said to be *locally semiconcave* if f is semiconcave on a neighborhood of each point in the set U.

Proposition 18 in [18] proved that a closed set S of X is r-prox-regular if and only if, for any positive real $s < r$, the square distance function d_S^2 is $s/(r-s)$-semiconvex on any convex set included in $U_s(S)$. Regarding the distance function d_S itself, it has been shown by M. V. Balashov [6] that the distance d_S associated to an r-prox-regular set S is $(r-s)^{-1}$-semiconvex on any convex set included in the open s-enlargement $U_s(S)$. Another different proof of this property has been recently established by F. Nacry and L. Thibault [35] through Theorem 2.5 above and Lemma 3.1 below. The arguments for Lemma 3.1 are contained in [13, Proposition 2.2.2] and [35, Proposition 7].

Lemma 3.1. ([13]) *Let S be a nonempty subset of X. The following hold.*

(a) *The squared distance function d_S^2 is linearly semiconcave with coefficient* 2.

(b) *For any nonempty convex subset U of X and for any real $\delta > 0$ such that $U \cap \big(S + B(0,\delta)\big) = \emptyset$, the distance function d_S is δ^{-1}-semiconcave on U. So, d_S is locally linearly semiconcave on $X \setminus S$.*

(c) *If S is the union of a collection of closed balls with a common radius $r > 0$, then on each nonempty convex set U included in* $\mathrm{cl}(X \setminus S)$, *the distance function d_S is r^{-1}-semiconcave.*

Our aim in this section is to show that such a lemma can also be successfully used to establish the semiconcavity of the farthest distance function associated to a strongly convex set. As said in the very introduction, our approach and proofs are quite different from those involved in the work [7] by M. V. Balashov and M. O. Golubev; notice that some results in certain Banach spaces are also contained in [7].

We start with the following result which has its own interest and where for a real $r > 0$ and a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, we use notation $\{\varphi \leq r\} := \{x \in X : \varphi(x) \leq r\}$ along with analogous one for $\{\varphi < r\}$, etc.

Proposition 3.2. *Let C be a nonempty bounded subset of X and let $R > 0$ be a positive real. The following hold.*

(a) *The equalities*

$$\{\mathrm{dfar}_C \le R\} = \bigcap_{c\in C} B[c,R] \quad \textit{and} \quad \{\mathrm{dfar}_C < R\} = \mathrm{int}\Big(\bigcap_{c\in C} B[c,R]\Big),$$

are satisfied, so in particular the sublevel set $\{\mathrm{dfar}_C \le R\}$ *is* R*-strongly convex.*

(b) *One has* $\{\mathrm{dfar}_C > R\} = (X \setminus R\mathbb{B}) + C.$

(c) *One has* $\mathrm{cl}\,\{\mathrm{dfar}_C > R\} = \{\mathrm{dfar}_C \ge R\},$

or equivalently, $\mathrm{int}\{\mathrm{dfar}_C \le R\} = \{\mathrm{dfar}_C < R\}.$

If in addition the set C *is* R*-strongly convex, then the following assertions also hold true:*

(d) *For every real* $s \ge R$*, one has with* $F_s := \{\mathrm{dfar}_C \le s\}$ *and* $G_s := \{\mathrm{dfar}_C = s\}$

$$\mathrm{dfar}_C(u) = s + d_{F_s}(u) = s + d_{G_s}(u) \quad \textit{for all}\, u \in \mathcal{E}_s(C).$$

(e) *For every real* $s > R$*, one has*

$$\{\mathrm{dfar}_C > s\} = \mathrm{int}\,\{\mathrm{dfar}_C \ge s\} \quad \textit{and} \quad \{\mathrm{dfar}_C \le s\} = \mathrm{cl}\,\{\mathrm{dfar}_C < s\}\,.$$

(f) *For every real* $s > R$*, one has*

$$\{\mathrm{dfar}_C \ge s\} = (X \setminus s\mathbb{U}) + C,$$

in particular the set $\{\mathrm{dfar}_C \ge s\}$ *is* $(s-R)$*-prox-regular.*

Proof. (a) The first equality (on the left) being obvious, we only show the second one. Consider first the inclusion of the left-hand side into the right-hand one. Set $K := \bigcap_{c\in C} B[c,R]$. Let any $x \in X$ with $\mathrm{dfar}_C(x) < R$. Choose some real $\eta > 0$ such that $\mathrm{dfar}_C(x) + \eta < R$. For all $c \in C$ and all $b \in \mathbb{B}$, it is readily seen that

$$\|c - (x + \eta b)\| \le \|c - x\| + \eta\|b\| \le \mathrm{dfar}_C(x) + \eta < R,$$

hence $x + \eta\mathbb{B} \subset K$. This means that $\{\mathrm{dfar}_C < R\} \subset \mathrm{int}\,K$.

Conversely, let $z \in \mathrm{int}\,K$. Choose a real $\delta \in]0,R[$ such that $z + \delta\mathbb{B} \subset B[c,R]$ for every $c \in C$. Then for each $c \in C$ we have

$$z + \delta\mathbb{B} \subset c + (R-\delta)\mathbb{B} + \delta\mathbb{B},$$

hence by the Rådström cancellation lemma (see, e.g., [44, Property (1.22)]) $z \in c + (R-\delta)\mathbb{B}$, that is, $\|z - c\| \le R - \delta$. It ensues that $\mathrm{dfar}_C(z) \le R - \delta < R$, so $\mathrm{int}\,K \subset \{\mathrm{dfar}_C < R\}$. This finishes the proof of (a).

(b) The inclusion $\supset$ is obvious while the converse one $\subset$ follows from the fact that for any $u \in X$ with $\mathrm{dfar}_C(u) > R$, we can find some $c \in C$ with $\|u - c\| > R$, hence $u = (u - c) + c \in (X \setminus R\mathbb{B}) + C$.

(c) The second claimed equality is a consequence of both equalities in (a), whereas the first equality in (c) follows from the second by taking complements. Another proof of the first equality in (c) can also be provided as follows. The inclusion $\subset$ being obvious by continuity of dfar_C, we only show the converse one $\supset$. We may suppose that C is not a singleton. Fix any $x \in X$ with $\mathrm{dfar}_C(x) = R$. Take any real

$\varepsilon > 0$. Using the definition of $\mathrm{dfar}_C(x)$, we can find a sequence $(c_n)_{n\geq 1}$ in C such that

$$R_n := \|c_n - x\| \to \mathrm{dfar}_C(x) = R.$$

Since C is not a singleton, we have $R_n > 0$ for n large enough, and this allows us to choose some integer $N \geq 1$ such that

$$a := \frac{R}{R_N} - 1 < \frac{\varepsilon}{R_N} =: b.$$

Pick any $t \in]a, b[$ and observe that

$$\mathrm{dfar}_C\big(-tc_N + (1+t)x\big) \geq (1+t)\|x - c_N\| = (1+t)R_N > R$$

along with

$$\| -tc_N + (1+t)x - x\| = t\|x - c_N\| = tR_N < \varepsilon.$$

Consequently, we have $B(x, \varepsilon) \cap \{\mathrm{dfar}_C > R\} \neq \emptyset$ and this guarantees the inclusion $x \in \mathrm{cl}\{\mathrm{dfar}_C > R\}$. This finishes the other proof of the first equality in (c).

Assume now that C is R-strongly convex for a given real $R > 0$.

(d) Consider any real $s \geq R$ and put

$$Q_s := \{\mathrm{dfar}_C \leq s\} \text{ (resp. } Q_s := \{\mathrm{dfar}_C = s\}).$$

Fix any $u \in X$ with $\mathrm{dfar}_C(u) > s$. Thanks to (c) in Theorem 2.2, we know that $z := \mathrm{far}_C(u)$ is well defined. Setting $v := z - s\frac{z-u}{\|z-u\|}$, we also have (see (f) in Theorem 2.2) that $\mathrm{far}_C(v)$ is well defined along with $z = \mathrm{far}_C(v)$. Hence, we see that

$$\mathrm{dfar}_C(v) = \|\mathrm{far}_C(v) - v\| = \|z - v\| = s,$$

in particular $v \in Q_s$. This and a direct computation give

$$d_{Q_s}(u) \leq \|u - v\| = \|(u - z)\big(1 - \frac{s}{\|z - u\|}\big)\| = \mathrm{dfar}_C(u) - s. \tag{9}$$

On the other hand, we observe that

$$\mathrm{dfar}_C(u) \leq \mathrm{dfar}_C(x) + \|u - x\| \leq s + \|u - x\| \quad \text{for all } x \in Q_s,$$

and this obviously entails

$$\mathrm{dfar}_C(u) \leq s + d_{Q_s}(u). \tag{10}$$

It suffices then to put together the inequalities (9) and (10) to obtain the desired equalities in (d).

(e) Fix any real $s > R$. Observe that the first claimed equality directly follows from the second one by taking complements. Further, note that $\mathcal{C} := \mathrm{cl}\,\{\mathrm{dfar}_C < s\}$ is obviously included (keeping in mind the continuity of dfar_C) in $\{\mathrm{dfar}_C \leq s\}$. Then, it remains to justify the inclusion $\{\mathrm{dfar}_C \leq s\} \subset \mathcal{C}$. Fix any $u \in X$ with $\mathrm{dfar}_C(u) \leq s$. We may suppose that $\mathrm{dfar}_C(u) = s$ (otherwise there is nothing to prove). Since C is R-strongly convex and $s > R$, we know (see (c) in Theorem 2.2) that $\mathrm{far}_C(u)$ is well defined. Take any sequence $(s_n)_{n\geq 1}$ of $]R,s[$ with $s_n \to s$. The latter convergence and the equality $\mathrm{dfar}_C(u)=s$ obviously give

$$v_n := \mathrm{far}_C(u) - \frac{s_n}{\mathrm{dfar}_C(u)}\big(\mathrm{far}_C(u)-u\big) \to u.$$

On the other hand, (f) in Theorem 2.2 guarantees that for every integer $n \geq 1$, the vector $\mathrm{far}_C(v_n)$ is well defined along with $\mathrm{far}_C(u) = \mathrm{far}_C(v_n)$. Hence, we can write for every integer $n \geq 1$

$$\mathrm{dfar}_C(v_n) = \|\mathrm{far}_C(v_n) - v_n\| = \|\mathrm{far}_C(u) - v_n\| = s_n < s,$$

from which we derive that $u \in \mathrm{cl}\,\{\mathrm{dfar}_C < s\}$, and hence the desired inclusion $\{\mathrm{dfar}_C \leq s\} \subset \mathcal{C}$ holds true.

(f) Fix any real $s > R$. According to (b) above, we have

$$\{\mathrm{dfar}_C > s\} = (X \setminus s\mathbb{B}) + C,$$

hence by (c) $$\{\mathrm{dfar}_C \geq s\} = \mathrm{cl}\,\{\mathrm{dfar}_C > s\} = \mathrm{cl}\big((X \setminus s\mathbb{B}) + C\big). \tag{11}$$

We claim that $$\mathrm{cl}\big((X \setminus s\mathbb{B}) + C\big) = (X \setminus s\mathbb{U}) + C.$$

First, note that the inclusion $\supset$ holds by the obvious fact $(X\setminus s\mathbb{U})+C \subset \{\mathrm{dfar}_C \geq s\}$ and by (11). Let us justify the converse inclusion. The set $S := \big(X \setminus s\mathbb{U}\big) + C$ is the Minkowski's sum of an s-prox-regular set and an R-strongly convex set with constant $R < s$. According to Theorem 2.6, we know that the set S is closed, so

$$\mathrm{cl}\big((X \setminus s\mathbb{B}) + C\big) \subset \mathrm{cl}\, S = S.$$

The proof is complete. □

We are now in a position to provide our proof of the following characterizations of R-strongly convex sets via the farthest distance function.

Theorem 3.3. *Let C be a nonempty closed bounded subset of X and let $R > 0$ be a positive real. The following assertions are equivalent:*

(a) *the set C is R-strongly convex;*

(b) *for any real $s > R$, the function $-\mathrm{dfar}_C + \frac{1}{2(s-R)}\|\cdot\|^2$ is convex on any nonempty convex subset V of $\mathcal{E}_s(C)$, that is, $-\mathrm{dfar}_C$ is linearly semiconvex on V with $(s-R)^{-1}$ as coefficient;*

(c) *there exists a real $s > R$ such that the function $-\mathrm{dfar}_C + \frac{1}{2(s-R)}\|\cdot\|^2$ is convex on any nonempty convex subset V of $\mathcal{E}_s(C)$;*

(d) *the function $-\mathrm{dfar}_C$ is locally linearly semiconvex on $\mathcal{E}_R(C)$, that is, linearly semiconvex near each point in $\mathcal{E}_R(C)$.*

Proof. (a)⇒(b) Assume that C is R-strongly convex and fix any real $s > R$. Let V be any nonempty convex subset of $\mathcal{E}_s(C)$. Let $t \in]R,s[$. According to (f) in Proposition 3.2, the set $D := \{\mathrm{dfar}_C \geq s\}$ is $(s-R)$-prox-regular, hence (see Theorem 2.5) the set $S := X \setminus D = \{\mathrm{dfar}_C < s\}$ is the union of a collection of closed balls of common radius $s-t$. Thanks to (e) in Proposition 3.2, we know that $\operatorname{cl} S = \{\operatorname{dfar}_C \le s\}$, so

$$d(\cdot, S) = d(\cdot, \operatorname{cl} S) = d\big(\cdot, \{\operatorname{dfar}_C \le s\}\big).$$

On the other hand, Lemma 3.1 says that $d(\cdot, S)$ is $(s-t)^{-1}$-linearly semiconcave on every nonempty convex set $U \subset \operatorname{cl}(X \setminus S) = D$, in particular on the set V. Further, by virtue of Proposition 3.2(d), we can write

$$\operatorname{dfar}_C(u) = s + d\big(u, \{\operatorname{dfar}_C \le s\}\big) \quad \text{for all } u \in \mathcal{E}_s(C).$$

It follows that $\operatorname{dfar}_C(\cdot)$ is $(s-t)^{-1}$-linearly semiconcave on V, that is, the function $\operatorname{dfar}_C(\cdot) - \frac{1}{2(s-t)}\|\cdot\|^2$ is concave on V. Since t has been arbitrarily choosen in $]R, s[$, the desired property is then justified.

(b)⇒(c) and (c)⇒(d): Obvious.

(d)⇒(a) Assume that $-\operatorname{dfar}_C$ is locally linearly semiconvex on the set $\mathcal{E}_R(C)$. Fix any $x \in \mathcal{E}_R(C)$. We can then find two reals $\rho, \delta > 0$ such that the function $f := -\operatorname{dfar}_C(\cdot) + \rho\|\cdot\|^2$ is convex on $B(x, \delta) \subset \mathcal{E}_R(C)$. It directly follows from the $C^{1,1}$-property of $\|\cdot\|^2$ (see (7)) that

$$\emptyset \ne \partial_P f(x) = \partial_P(-\operatorname{dfar}_C)(x) + \rho\nabla\|\cdot\|^2(x),$$

where the non-emptiness is due to (5). Therefore, we have $\partial_P(-\operatorname{dfar}_C)(x) \ne \emptyset$, so $-\operatorname{dfar}_C$ is Fréchet differentiable at x (see (6) at the end of Section 2). We conclude that C is R-strongly convex according to (d) in Theorem 2.2. The proof is complete. □

4. Farthest distance and separating balls

Given a nonempty closed convex set C in the Hilbert space X and an exterior point of C, say $x \in X \setminus C$, it is well-known (and not difficult to check) that we have with $x_\star := d_C(x)^{-1}\big(x - \operatorname{proj}_C(x)\big)$ the following separation property for some real α

$$C \subset \{\langle x_\star, \cdot\rangle < \alpha\} \quad \text{and} \quad \langle x_\star, x\rangle > \alpha. \tag{12}$$

Replacing the above half-space $\{\langle x_\star, \cdot\rangle < \alpha\}$ by a set in the more general form $\big\{ \langle x_\star, x\rangle - \frac{\|x\|^2}{2r} < \alpha\big\}$ (which is nothing else but the complement of a closed ball (see Proposition 4.4)) allows to extend such a separation property, with a suitable vector $x_\star$, to the context of r-prox-regular sets.

Theorem 4.1. ([2]) *Let S be an r-prox-regular subset of X with $r \in]0, +\infty]$, $x \in X$ with $\delta := d_S(x) \in]0, r[$. Then, one has with $x_\star := (\frac{1}{r} - \frac{1}{\delta})\operatorname{proj}_S(x) + \frac{1}{\delta}x$ the following separation property for some $\alpha \in \mathbb{R}$*

$$S \subset \left\{ \langle x_\star, \cdot\rangle - \frac{\|\cdot\|^2}{2r} < \alpha\right\} \quad \textit{and} \quad \langle x_\star, x\rangle - \frac{\|x\|^2}{2r} > \alpha. \tag{13}$$

Our first aim in this section is to see how the above separation property can be reinforced in the case of strongly convex sets.

Lemma 4.2. *Let C be an R-strongly convex subset of X for some real $R > 0$. Let also $(x, y) \in \operatorname{gph} \operatorname{Far}_C$ with $\operatorname{dfar}_C(x) > 0$ and let $\rho \in]0, \operatorname{dfar}_C(x)[$. Define $x^\star := \operatorname{dfar}_C(x)^{-1}(x - y)$. Then, one has for every $c \in C$*

$$\langle x^\star, x\rangle - \frac{1}{\rho}\operatorname{dfar}^2_C(x) < \langle x^\star, y\rangle \le \langle x^\star, c\rangle - \frac{1}{2R}\|c - y\|^2,$$

in particular there exists $\alpha \in \mathbb{R}$ such that for every $c \in C$

$$\langle x^\star, x\rangle - \frac{1}{\rho}\operatorname{dfar}^2_C(x) < \alpha < \langle x^\star, c\rangle - \frac{1}{2R}\|c - y\|^2.$$

Proof. Fix any $c \in C$. Using the inclusion $-x^\star \in N(C; y) \cap \mathbb{S}$ (see (3)) and (b) in Theorem 2.2, we obtain

$$\langle -x^\star, c - y\rangle \le -\frac{1}{2R}\|c - y\|^2,$$

or equivalently,
$$\langle x^\star, y\rangle \le \langle x^\star, c\rangle - \frac{1}{2R}\|c - y\|^2. \tag{14}$$

On the other hand, we easily see (thanks to the definition of $x^\star$ and the inclusion $\rho \in]0, \operatorname{dfar}_C(x)[$) that

$$\langle x^\star, x\rangle - \frac{1}{\rho}\operatorname{dfar}^2_C(x) < \langle x^\star, y\rangle\,. \tag{15}$$

It then suffices to combine (14) and (15) to finish the proof. □

In (12) and (13) we have a separation property with half-spaces and complements of balls for convex sets and prox-regular sets, respectively. Here, we are in a position to refine these results for strongly convex sets in establishing for such sets a separation property with balls. Given any $x^\star \in X$, any real $R > 0$ and any R-strongly convex set C in X, it will be convenient for us to set

$$q_{x^\star,R}(x) := \langle x^\star, x\rangle - \frac{\|x\|^2}{2R} \quad \text{for all } x \in X, \text{ and} \tag{16}$$

$$\Upsilon_{C,R}(x) := \Big(\frac{1}{R} - \frac{1}{\operatorname{dfar}_C(x)}\Big)\operatorname{far}_C(x) + \frac{1}{\operatorname{dfar}_C(x)}x \quad \text{for all } x \in \mathcal{E}_R(C). \tag{17}$$

Theorem 4.3. *Let C be an R-strongly convex set in X for some real $R > 0$ and let $x \in X$ with $\delta := \operatorname{dfar}_C(x) > R$. Then one has with*

$$x^\star = \Upsilon_{C,R}(x) := \big(R^{-1} - \delta^{-1}\big)\operatorname{far}_C(x) + \delta^{-1}x$$

$$C \subset \Big\{ \langle x^\star, \cdot\rangle - \frac{\|\cdot\|^2}{2R} \ge \inf_{c\in C} q_{x^\star,R}(c)\Big\} \quad \textit{and} \quad q_{x^\star,R}(x) \le \inf_{c\in C} q_{x^\star,R}(c). \tag{18}$$

If in addition $\delta > 2R$ (so $x \notin C$), then one has the following strict separation property for some $\alpha \in \mathbb{R}$,

$$C \subset \Big\{ \langle x^\star, \cdot\rangle - \frac{\|\cdot\|^2}{2R} > \alpha\Big\} \quad \textit{and} \quad q_{x^\star,R}(x) < \alpha \le \inf_{c\in C} q_{x^\star,R}(c).$$

Proof. Set $y := \operatorname{far}_C(x)$, $\delta := \operatorname{dfar}_C(x)$ and $u^\star := \delta^{-1}(x - y)$ (so $x^\star = u^\star + R^{-1}y$). Now assume first that $\delta > R$. The inclusion in (18) is obvious since we have $\kappa := \inf_{c\in C} q_{x^\star,R}(c) < +\infty$.

Noticing that $u^\star \in -N(C;y)\cap \mathbb{S}$ by (3) and using (b) in Theorem 2.2, we get

$$\langle u^\star, c-y\rangle \geq \frac{1}{2R}\|c-y\|^2 \quad \text{for all } c\in C,$$

or equivalently,

$$\left\langle u^\star + \frac{y}{R}, c\right\rangle - \frac{\|c\|^2}{2R} \geq \left\langle u^\star + \frac{y}{R}, x\right\rangle - \frac{\|x\|^2}{2R} = q_{x^\star,R}(x) \quad \text{for all } c\in C.$$

We then arrive to the inequality $q_{x^\star,R}(x)\leq \kappa$.

Now, assume that $\delta > 2R$. Applying Lemma 4.2 with $\rho := 2R$ gives some real β such that for all $c\in C$

$$\langle u^\star, x\rangle - \frac{\delta^2}{2R} < \beta < \langle u^\star, c\rangle - \frac{1}{2R}\|c-y\|^2. \tag{19}$$

Fix any $c\in C$. Through elementary computations, we observe that

$$\langle u^\star, c\rangle - \frac{1}{2R}\|c-y\|^2 = \langle x^\star, c\rangle - \frac{1}{2R}(\|y\|^2+\|c\|^2) \tag{20}$$

and
$$\langle u^\star, x\rangle - \frac{\delta^2}{2R} = \langle x^\star, x\rangle - \frac{1}{2R}(\|x\|^2+\|y\|^2). \tag{21}$$

Putting together (19), (20) and (21) yields

$$\langle x^\star, x\rangle - \frac{\|x\|^2}{2R} < \beta + \frac{\|y\|^2}{2R} < \langle x^\star, c\rangle - \frac{\|c\|^2}{2R}.$$

It remains to set $\alpha := \beta + \frac{\|y\|^2}{2R}$ to finish the proof. □

Setting
$$\Phi_{C,R}(x^\star) := \inf_{c\in C} q_{x^\star,R}(c) \quad \text{and} \quad L_{x^\star,R,\alpha} := \{q_{x^\star,R}\geq \alpha\} \tag{22}$$

and noticing the elementary equalities

$$\|Rx^\star - x\|^2 = R^2\|x^\star\|^2 - 2R\,\langle x^\star, x\rangle + \|x\|^2$$

along with

$$R^2\|x^\star\|^2 - 2R\Phi_{C,R}(x^\star) = R^2\|x^\star\|^2 + \sup_{c\in C}\big(-2R\,\langle x^\star, c\rangle + \|c\|^2\big) = \sup_{c\in C}\|Rx^\star - c\|^2,$$

it is not difficult to establish the following important description of the upper level set $L_{x^\star,R,\alpha}$.

Proposition 4.4. *Let $x^\star\in X$, $\alpha\in\mathbb{R}$ and $R\in]0,+\infty[$. Let also C be a nonempty subset of X. The following hold with $\rho := R^2\|x^\star\|^2 - 2R\alpha$.*

(a) *One has*
$$L_{x^\star,R,\alpha} = \begin{cases} B[Rx^\star, \sqrt{\rho}] & \text{if } \rho\geq 0,\\ \emptyset & \text{otherwise.}\end{cases}$$

(b) *If $\alpha = \Phi_{C,R}(x^\star)$, then one has $\rho = \mathrm{dfar}_C^2(Rx^\star)$, in particular*

$$L_{x^\star,R,\Phi_{C,R}(x^\star)} = B[Rx^\star, \mathrm{dfar}_C(Rx^\star)].$$

Theorem 4.3 naturally leads to investigate some properties of $\Upsilon_{C,R}(x)$. The following lemma will be needed.

Lemma 4.5. *Let $x^\star \in X$, $\alpha \in \mathbb{R}$ and $R \in]0,+\infty[$. Let $x \in X$ with $x \notin L_{x^\star,R,\alpha}$ and let C be a nonempty subset of X such that $C \subset L_{x^\star,R,\alpha}$. Then, one has $\beta := \Phi_{C,R}(x^\star) \geq \alpha$ and*

$$C \subset L_{x^\star,R,\beta} \subset L_{x^\star,R,\alpha} \quad \textit{and} \quad x \notin L_{x^\star,R,\beta},$$

where $\Phi_{C,R}$ and $L_{x^\star,R,\alpha}$ are as defined in (22). In particular, one has

$$\mathrm{dfar}(x, C) \leq \mathrm{dfar}(x, L_{x^\star,R,\beta}) \leq \mathrm{dfar}(x, L_{x^\star,R,\alpha}).$$

Proof. Due to the definition of β, it is evident that $C \subset L_{x^\star,R,\beta}$. On the other hand, we observe that

$$\langle x^\star, c\rangle - \frac{\|c\|^2}{2R} \geq \alpha \quad \text{for all } c \in C,$$

hence $\beta \geq \alpha$. This obviously implies that $L_{x^\star,R,\beta} \subset L_{x^\star,R,\alpha}$ along with (keeping in mind that $x \notin L_{x^\star,R,\alpha}$) $x \notin L_{x^\star,R,\beta}$. The proof is complete. □

Now, we can state and prove:

Proposition 4.6. *Let C be an R-strongly subset of X with $R \in]0,+\infty[$ and let $x \in X$ with $\delta := \mathrm{dfar}_C(x) > R$. Let $\Upsilon_{C,R}(x)$ be as defined in (17). The following hold with $t := \delta^{-1}R \in]0,1[$.*

(a) *The inclusion $\mathrm{far}_C(x) \in \mathrm{Far}_C\big(R\Upsilon_{C,R}(x)\big)$ is satisfied along with the equalities*

$$R\Upsilon_{C,R}(x) = tx + (1-t)\mathrm{far}_C(x) \quad \textit{and} \quad \|R\Upsilon_{C,R}(x) - x\| = \delta - R. \tag{23}$$

(b) *One has*

$$\{R\Upsilon_{C,R}(u) : u \in \mathcal{E}_R(C)\} = \{\mathrm{dfar}_C = R\} \cap \mathrm{Dom}\,\mathrm{Far}_C =: \Lambda_R(C) \tag{24}$$

along with $$\mathrm{cl}_{\|\cdot\|}\big(\Lambda_R(C)\big) = \{\mathrm{dfar}_C = R\}.$$

(c) *One has with $x^\star := \Upsilon_{C,R}(x)$ and $L_{x^\star,R,\Phi_{C,R}(x^\star)}$ as defined in (22)*

$$R^2\|x^\star\|^2 - 2R\Phi_{C,R}(x^\star) = R^2 \quad \textit{and} \quad L_{x^\star,R,\Phi_{C,R}(x^\star)} = B[Rx^\star, R].$$

(d) *One has $\mathcal{E}_R(C) = \{\tau R\Upsilon_{C,R}(u) + (1-\tau)\mathrm{far}_C(u) : u \in \mathcal{E}_R(C), \tau \in]1,+\infty[\}$.*

(e) *For all $\tau \in]1,+\infty[$, one has*

$$\Upsilon_{C,R}\big(\tau R\Upsilon_{C,R}(x) + (1-\tau)\mathrm{far}_C(x)\big) = \Upsilon_{C,R}(x).$$

(f) *One has* $$C = \bigcap_{y \in \Lambda_R(C)} B[y, R].$$

Proof. (a) The inclusion is a direct consequence of (e) in Theorem 2.2 while the equalities directly follow from the definition of $\Upsilon_{C,R}(x)$.

(b) Fix any $u \in \mathcal{E}_R(C)$. Thanks to (c) in Theorem 2.2 and to (a) above we know that $\mathrm{far}_C(u)$ is well defined along with $\mathrm{far}_C(u) \in \mathrm{Far}_C\big(R\Upsilon_{C,R}(u)\big)$.

The latter inclusion and the definition of $v := R\Upsilon_{C,R}(u)$ easily give

$$\mathrm{dfar}_C(v) = \|\mathrm{far}_C(u) - v\| = R,$$

which justifies the inclusion $\subset$ concerning the first equality in (b). Let us show the converse inclusion. Fix any $z \in \mathrm{Dom}\,\mathrm{Far}_C$ such that $\mathrm{dfar}_C(z) = R$. Pick any $v \in \mathrm{Far}_C(z)$ and any real $\lambda > 1$. It is easy to check that $u := \lambda z + (1-\lambda)v \in \mathcal{E}_R(C)$ along with $\mathrm{far}_C(u) = v$ (see (4)). Set $\sigma := \mathrm{dfar}_C(u)$ and note that (having in mind $\|z - v\| = R$)

$$\sigma := \mathrm{dfar}_C(u) = \|u - v\| = \lambda\|z - v\| = \lambda R,$$

hence $\lambda = \sigma/R$. This and the obvious equality $u - z = (\lambda - 1)(z - v)$ gives

$$\|u - z\| = \frac{\sigma - R}{R}\|z - v\| = \sigma - R$$

and $$\langle v - z, z - u\rangle = (\lambda - 1)\|v - z\|^2 = (\lambda - 1)R^2 = (\sigma - R)R.$$

We then arrive at

$$\begin{aligned}
&\Big\|(1 - \frac{R}{\sigma})v + \frac{R}{\sigma}u - z\Big\|^2 = \Big\|(1 - \frac{R}{\sigma})(v - z) - \frac{R}{\sigma}(z - u)\Big\|^2 \\
&= \big(1 - \frac{R}{\sigma}\big)^2\|v - z\|^2 + \frac{R^2}{\sigma^2}\|z - u\|^2 - 2\frac{R}{\sigma}(1 - \frac{R}{\sigma})\langle v - z, z - u\rangle \\
&= (\sigma - R)^2\frac{R^2}{\sigma^2} + \frac{R^2}{\sigma^2}(\sigma - R)^2 - 2\frac{R^2}{\sigma^2}(\sigma - R)^2 = 0,
\end{aligned}$$

that is, $z = (1 - \frac{R}{\sigma})v + \frac{R}{\sigma}u$. This means that $z = R\Upsilon_{C,R}(u)$ with $u \in \mathcal{E}_R(C)$, which confirms the desired inclusion $\supset$ for the first equality in (b). The first equality in (b) is then proved.

Regarding the second equality in (b), take any $y \in X$ such that $\mathrm{dfar}_C(y) = R$. According to Proposition 3.2, we can choose a sequence $(u_n)_{n\geq 1}$ in $\{\mathrm{dfar}_C > R\}$ with $u_n \to y$. Keeping in mind that C is bounded and $\mathrm{dfar}_C(\cdot)$ is continuous, we easily see that $v_n := R\Upsilon_{C,R}(u_n) \to y$. It remains to use (24) above to get that $\mathrm{dfar}_C(v_n) = R$ for every integer $n \geq 1$. This justifies the second equality in (b).

(c) According to Proposition 4.4, we have $R^2\|x^\star\|^2 - 2R\Phi_{C,R}(x^\star) = \mathrm{dfar}_C^2(Rx^\star)$ and

$$L_{x^\star,R,\Phi_{C,R}(x^\star)} = B[Rx^\star, \mathrm{dfar}_C(Rx^\star)].$$

It then suffices to use (a) above to get $\mathrm{dfar}_C(Rx^\star) = \|Rx^\star - \mathrm{far}_C(x)\| = R$.

(d) Since any $u \in \mathcal{E}_R(C)$ can be written by the first equality in (23) as

$$u = \theta R\Upsilon_{C,R}(u) + (1 - \theta)\,\mathrm{far}_C(u) \quad \text{with } \theta := \frac{\mathrm{dfar}_C(u)}{R} > 1,$$

the inclusion $\subset$ directly follows.

Conversely, fix $u \in \mathcal{E}_R(C)$ and $\tau > 1$. Set $\omega := \tau R\Upsilon_{C,R}(u) + (1 - \tau)\mathrm{far}_C(u)$. Using the inclusion $\mathrm{far}_C(u) \in C$ and the definition of $\Upsilon_{C,R}$, we easily see that

$$\mathrm{dfar}_C(\omega) \geq \|\omega - \mathrm{far}_C(u)\| = \tau\|R\Upsilon_{C,R}(u) - \mathrm{far}_C(u)\| = \tau R > R.$$

The desired equality is then established.

(e) Fix any real $\tau > 1$. According to (d) above we know that

$$y := \tau R\varUpsilon_{C,R}(x) + (1-\tau)\mathrm{far}_C(x) \in \mathcal{E}_R(C).$$

Further, the inclusion $\mathrm{far}_C(x) \in \mathrm{Far}_C\big(R\varUpsilon_{C,R}(x)\big)$ in (a) above combined with the property (4) gives that

$$\begin{aligned}\mathrm{far}_C(x) &= \mathrm{far}_C\Big(R\varUpsilon_{C,R}(x) + (\tau-1)\big(R\varUpsilon_{C,R}(x) - \mathrm{far}_C(x)\big)\Big)\\ &= \mathrm{far}_C\big(\tau R\varUpsilon_{C,R}(x) + (1-\tau)\mathrm{far}_C(x)\big) = \mathrm{far}_C(y).\end{aligned} \tag{25}$$

Hence, by definition of $\varUpsilon_{C,R}(x)$, elementary computations yield with $\delta' := \mathrm{dfar}_C(y)$ (recalling that $\delta := \mathrm{dfar}_C(x)$)

$$\begin{aligned}w := \varUpsilon_{C,R}(x) - \varUpsilon_{C,R}(y) &= \big(\frac{1}{\delta'} - \frac{1}{\delta}\big)\mathrm{far}_C(x) + \frac{x}{\delta} - \frac{y}{\delta'}\\ &= \big(\frac{1}{\delta'} - \frac{1}{\delta}\big)\big(\mathrm{far}_C(x) - x\big) + \frac{1}{\delta'}(x-y).\end{aligned} \tag{26}$$

On the other hand, by the definitions of y and $\varUpsilon_{C,R}(x)$, we can easily check that

$$y - x = \frac{\delta - \tau R}{\delta}\big(\mathrm{far}_C(x) - x\big) \tag{27}$$

and
$$y - \mathrm{far}_C(x) = y - x + x - \mathrm{far}_C(x) = \frac{\tau R}{\delta}\big(x - \mathrm{far}_C(x)\big). \tag{28}$$

If $\delta - \tau R \geq 0$ (resp. $\delta - \tau R < 0$), we have by (27) and (28) and by the equality $\mathrm{far}_C(x) = \mathrm{far}_C(y)$ in (25) that

$$\|y - x\| = \delta - \tau R = \delta - \|y - \mathrm{far}_C(x)\| = \delta - \delta'$$

(resp.
$$\|y - x\| = \tau R - \delta = \|y - \mathrm{far}_C(x)\| - \delta = \delta' - \delta).$$

In both cases $\tau R = \delta'$, so (26) and (27) easily give

$$w = \big(\frac{1}{\tau R} - \frac{1}{\delta} + \frac{\tau R - \delta}{\delta \tau R}\big)\big(\mathrm{far}_C(x) - x\big) = 0,$$

hence (e) holds true.

(f) Set $C_0 := \bigcap_{x^\star \in \Lambda_R(C)} B[x^\star, R]$. According to Theorem 4.3 and to (c) and (b) above, we can write

$$C \subset \bigcap_{x^\star \in \varUpsilon_{C,R}(\mathcal{E}_R(C))} L_{x^\star, R, \Phi_{C,R}(x^\star)} = \bigcap_{x^\star \in \varUpsilon_{C,R}(\mathcal{E}_R(C))} B[Rx^\star, R] = C_0.$$

We are now going to show that $C_0 \subset C$. By contradiction, suppose that there is $z \in C_0 \setminus C$. Set $p := \mathrm{proj}_C(z)$ and $d := d_C(z) > 0$. Note first (see (2)) that we have $v := \frac{z-p}{d} \in N(C;p)$. Fix any real $R' > R$ and put $q := p - R'v$. Thanks to (b) in Theorem 2.2, we see that for every $c \in C$

$$\langle p - q, c - p\rangle = R'\,\langle v, c - p\rangle \leq -\frac{R'}{2R}\|c - p\|^2 \leq -\frac{1}{2}\|c - p\|^2,$$

and this entails (see (1)) that $p \in \mathrm{Far}_C(q)$. In fact, we have $p = \mathrm{far}_C(q)$ since $\mathrm{dfar}_C(q) = \|p - q\| = R' > R$.

Therefore, we can apply (e) in Theorem 2.2 to get

$$p = \mathrm{far}_C(q) \in \mathrm{Far}_C\big(p - R\frac{p-q}{\mathrm{dfar}_C(q)}\big) = \mathrm{Far}_C\big(p - \frac{R}{R'}(p-q)\big).$$

Set $w := p - \frac{R}{R'}(p-q)$ and note that $\mathrm{dfar}_C(w) = \|p-w\| = \frac{R}{R'}\|p-q\| = R$. Then, it follows from the definition of C_0 and from the definition of $\Lambda_R(C)$ in (24) that $z \in B[w,R]$. On the other hand, we easily observe that

$$z - w = z - p + \frac{R}{R'}(p-q) = z - p + Rv = \big(1 + \frac{R}{d}\big)(z-p),$$

hence
$$d + R = \big(1 + \frac{R}{d}\big)\|z-p\| = \|z-w\| \le R.$$

Consequently, we obtain $d = 0$, and this is the desired contradiction. The equality $C = C_0$ is then established and the proof of the proposition is complete. □

Our second aim in the present section is to provide some analytic formulation for the farthest distance function from a strongly convex set. Doing so, we complement the following well known formula (see, e.g., [20, Theorem 6.23])

$$d_C(x) = \langle x_\star, x\rangle - \sigma(x_\star, C), \tag{29}$$

where x is an exterior point of a convex set C and $\sigma(x_\star, C)$ denotes the support function of the set C at $x_\star := d_C(x)^{-1}(x - \mathrm{proj}_C(x))$. We point out that such an equality has been extended to the context of a prox-regular set in [35]. Keeping notation of Theorem 4.1, the equality analogous to (29) for an r-prox-regular set S is given by

$$d_S(x)\Big(1 - \frac{d_S(x)}{2r}\Big) = q_{x_\star,r}(x) - \phi_{S,r}(x_\star),$$

where $\phi_{S,r}(x_\star) := \sup_{u\in S} q_{x_\star,r}(u)$. A result similar to the latter equality is furnished in the next theorem for the farthest distance function from R-strongly convex sets.

Theorem 4.7. *Let C be an R-strongly convex subset of X for some real $R > 0$ and let $x \in X$ with $\mathrm{dfar}_C(x) > R$. Then, there exists one and only one $x^\star \in X$ with $\|x^\star - R^{-1}x\| = R^{-1}\mathrm{dfar}_C(x) - 1$ (namely, $x^\star := \Upsilon_{C,R}(x)$ as in (17)) such that*

$$\mathrm{dfar}_C(x)\Big(1 - \frac{\mathrm{dfar}_C(x)}{2R}\Big) = q_{x^\star,R}(x) - \Phi_{C,R}(x^\star),$$

where $q_{x^\star,R}$ and $\Phi_{C,R}$ are as defined in (16) and (22), respectively.

Proof. Set $v := \mathrm{far}_C(x)$ by (c) in Theorem 2.2 and set also $\delta := \mathrm{dfar}_C(x)$. The proof is divided into two parts.

Existence. Putting together the inclusion $v - x \in N(C; v)$ (see (1)) and (b) in Theorem 2.2, we see that for every $c \in C$

$$\langle v - x, c - v\rangle \le -\frac{\delta}{2R}\|c - v\|^2,$$

or equivalently,
$$\frac{\delta}{2R}\|c-v\|^2 + \langle v - x, c - x\rangle \le \|v - x\|^2 = \delta^2.$$

This inequality and the inclusion $v \in C$ easily give

$$\delta^2 \le \sup_{c\in C}\Big(\frac{\delta}{2R}\|c-v\|^2 + \langle v-x, c-x\rangle\Big) \le \delta^2,$$

that is,

$$\delta = \sup_{c\in C}\Big(\frac{\|c-v\|^2}{2R} + \delta^{-1}\langle v-x, c-x\rangle\Big).$$

Keeping in mind the definition of $x^\star := \Upsilon_{C,R}(x)$ in (17) and the definition of $q_{x^\star,R}(x)$ in (16), it remains to write

$$\begin{aligned}
\delta - \frac{\delta^2}{2R} &= \sup_{c\in C}\Big(\frac{1}{2R}(\|c-v\|^2 - \|x-v\|^2) + \delta^{-1}\langle v-x, c-x\rangle\Big)\\
&= \big\langle \delta^{-1}(v-x) - R^{-1}v, -x\big\rangle - \frac{\|x\|^2}{2R} + \sup_{c\in C}\Big(\big\langle \delta^{-1}(v-x) - R^{-1}v, c\big\rangle + \frac{\|c\|^2}{2R}\Big)\\
&= q_{x^\star,R}(x) - \inf_{c\in C}\Big(\langle x^\star, c\rangle - \frac{\|c\|^2}{2R}\Big) = q_{x^\star,R}(x) - \Phi_{C,R}(x^\star).
\end{aligned}$$

Uniqueness. Let $x_1^\star, x_2^\star \in X$ be such that for each $i \in \{1,2\}$

$$\Big\|x_i^\star - \frac{x}{R}\Big\| = \frac{\delta}{R} - 1 \quad \text{and} \quad \delta\big(1-(2R)^{-1}\delta\big) = q_{x_i^\star,R}(x) - \Phi_{C,r}(x_i^\star).$$

It is readily seen with $u^\star := 2^{-1}(x_1^\star + x_2^\star)$ that

$$\big\|u^\star - R^{-1}x\big\| \le 2^{-1}\big\|x_1^\star - R^{-1}x\big\| + 2^{-1}\big\|x_2^\star - R^{-1}x\big\| = \frac{\delta}{R} - 1. \tag{30}$$

Setting $q_i := q_{x_i^\star,R}$ for each $i \in \{1,2\}$, it is also straightforward to check that

$$q_{u^\star,R}(x) = 2^{-1}[q_1(x) + q_2(x)].$$

We deduce from what precedes

$$\begin{aligned}
q_{u^\star,R}(x) - \Phi_{C,R}(u^\star) &= 2^{-1}\big[q_1(x) + q_2(x) - \inf_{c\in C}\big(q_1(c) + q_2(c)\big)\big]\\
&\le 2^{-1}\big[q_1(x) + q_2(x) - \inf_{c\in C} q_1(c) - \inf_{c\in C} q_2(c)\big]\\
&= 2^{-1}\big(q_1(x) - \Phi_{C,R}(x_1^\star) + q_2(x) - \Phi_{C,R}(x_2^\star)\big)\\
&= \delta\big(1 - \frac{\delta}{2R}\big).
\end{aligned} \tag{31}$$

Combining (31), the definition of $q_{u^\star,R}$ and Proposition 4.4(b), we obtain

$$\begin{aligned}
\delta(2R-\delta) &\ge 2R\big(q_{u^\star,R}(x) - \Phi_{C,R}(u^\star)\big)\\
&= 2Rq_{u^\star,R}(x) - R^2\|u^\star\|^2 + R^2\|u^\star\|^2 - 2R\Phi_{C,R}(u^\star)\\
&= -\|Ru^\star - x\|^2 + \mathrm{dfar}_C^2(Ru^\star).
\end{aligned} \tag{32}$$

Using (32), the 1-Lipschitz property of dfar_C, the inequality $\delta - \|Ru^\star - x\| \ge 0$ and (30), we get

$$\begin{aligned}
\delta(2R-\delta) &\ge -\|Ru^\star - x\|^2 + \big(\delta - \|Ru^\star - x\|\big)^2\\
&= \delta^2 - 2\delta\|Ru^\star - x\|\\
&\ge \delta^2 - 2\delta(\delta - R) = \delta(2R-\delta).
\end{aligned} \tag{33}$$

Through (33), we then see that $\|Ru^\star - x\| = \delta - R$, or equivalently,

$$\|u^\star - \frac{x}{R}\| = \frac{\delta}{R} - 1 =: \kappa.$$

We conclude that $x_1^\star, x_2^\star, u^\star$ lie on the sphere $R^{-1}x + \kappa\mathbb{S}_X$ with $u^\star = (x_1^\star + x_2^\star)/2$. It remains to invoke the strict convexity of the Hilbert norm $\|\cdot\|$ to obtain $x_1^\star = x_2^\star$. The proof is complete. □

Next we provide, in addition to the preceding theorem, a proposition describing (for a strongly convex set C) the farthest distance $\mathrm{dfar}_C(x)$ in terms of the unique function $q_{x^\star,R}$. The expression also reverses appropriate supremum and minimum.

Proposition 4.8. *Let C be an R-strongly convex subset of X for some real $R > 0$, $x \in X$ with $\delta := \mathrm{dfar}_C(x) > R$ and $\mathcal{L} := \{y^\star \in X : \|y^\star - R^{-1}x\| = R^{-1}\delta - 1\}$. Then, one has with $q_{y^\star,R}$ as defined in* (16)

$$\delta\big(1 - \frac{\delta}{2R}\big) = \min_{y^\star\in\mathcal{L}} \sup_{c\in C} \big(q_{y^\star,R}(x) - q_{y^\star,R}(c)\big) = \sup_{c\in C} \min_{y^\star\in\mathcal{L}} \big(q_{y^\star,R}(x) - q_{y^\star,R}(c)\big).$$

Proof. For every $y^\star \in X$, set

$$\mu(y^\star) := \sup_{c\in C} \Big(\frac{\|c\|^2 - \|x\|^2}{2R} + \langle y^\star, x - c\rangle\Big) = \sup_{c\in C} \big(q_{y^\star,R}(x) - q_{y^\star,R}(c)\big).$$

Thanks to Theorem 4.7, we have with $x^\star := \Upsilon_{C,R}(x)$

$$\mu(x^\star) = \sup_{c\in C} \big(q_{x^\star,R}(x) - q_{x^\star,R}(c)\big) = q_{x^\star,R}(x) - \Phi_{C,R}(x^\star) = \delta\big(1 - \frac{\delta}{2R}\big). \tag{34}$$

We obviously have for every $y^\star \in \mathcal{L}$

$$\begin{aligned}
&\sup_{c\in C} \Big(\frac{\|c\|^2 - \|x\|^2}{2R} + \langle y^\star, x - c\rangle\Big) = \sup_{c\in C} \Big(\Big\langle y^\star - \frac{x}{R}, x - c\Big\rangle + \frac{\|x - c\|^2}{2R}\Big)\\
&\geq \sup_{c\in C} \Big(-\|y^\star - \frac{x}{R}\|\|x - c\| + \frac{\|x - c\|^2}{2R}\Big)\\
&= \sup_{c\in C} \Big(\big(1 - \frac{\delta}{R}\big)\|x - c\| + \frac{\|x - c\|^2}{2R}\Big) =: \kappa,
\end{aligned}$$

so $\inf_{y^\star\in\mathcal{L}} \mu(y^\star) \geq \kappa$. On the other hand, by (c) in Theorem 2.2 put $c_0 := \mathrm{far}_C(x)$. It is clear that

$$\big(1 - \frac{\delta}{R}\big)\|x - c_0\| + \frac{\|x - c_0\|^2}{2R} \leq \kappa,$$

hence by (34)

$$\mu(x^\star) = \delta\big(1 - \frac{\delta}{2R}\big) = \big(1 - \frac{\delta}{R}\big)\delta + \frac{\delta^2}{2R} \leq \kappa \leq \inf_{y^\star\in\mathcal{L}} \mu(y^\star).$$

This and the inclusion $x^\star \in \mathcal{L}$ ensure that

$$\mu(x^\star) = \delta\big(1 - \frac{\delta}{2R}\big) = \kappa = \min_{y^\star\in\mathcal{L}} \mu(y^\star). \tag{35}$$

The left desired equality of the proposition is then established. Regarding the right equality, let us first write

$$\inf_{y^\star\in\mathcal{L}}\Big(\frac{\|c\|^2-\|x\|^2}{2R}+\langle y^\star,x-c\rangle\Big)=\frac{\|c\|^2-\|x\|^2}{2R}+\inf_{y^\star\in\mathcal{L}}\langle y^\star,x-c\rangle$$
$$=\frac{\|c\|^2-\|x\|^2}{2R}+\Big\langle\frac{x}{R},x-c\Big\rangle+\inf_{y^\star\in\mathcal{L}}\Big\langle y^\star-\frac{x}{R},x-c\Big\rangle,$$

which yields

$$\inf_{y^\star\in\mathcal{L}}\Big(\frac{\|c\|^2-\|x\|^2}{2R}+\langle y^\star,x-c\rangle\Big)=\frac{\|x-c\|^2}{2R}+\inf_{\|y^\star\|=R^{-1}\delta-1}\langle y^\star,x-c\rangle$$
$$=\frac{\|x-c\|^2}{2R}-\big(\frac{\delta}{R}-1\big)\sup_{\|y^\star\|=1}\langle y^\star,c-x\rangle=\frac{\|x-c\|^2}{2R}+\big(1-\frac{\delta}{R}\big)\|x-c\|.$$

Therefore, we have for every $c\in C$,

$$\inf_{y^\star\in\mathcal{L}}\Big(\frac{\|c\|^2-\|x\|^2}{2R}+\langle y^\star,x-c\rangle\Big)=\min_{y^\star\in\mathcal{L}}\Big(\frac{\|c\|^2-\|x\|^2}{2R}+\langle y^\star,x-c\rangle\Big)$$
$$=\frac{\|x-c\|^2}{2R}+\big(1-\frac{\delta}{R}\big)\|x-c\|,$$

where the above minimum is justified by reflexivity of X since the first term is the infimum of a continuous affine functional over the sphere $\mathcal{L}$ of X. We deduce that

$$\sup_{c\in C}\min_{y^\star\in\mathcal{L}}\Big(\frac{\|c\|^2-\|x\|^2}{2R}+\langle y^\star,x-c\rangle\Big)=\kappa,$$

and this finishes the proof according to (35). □

Remark 4.9. Let C be an R-strongly convex subset of X for some real $R>0$. Consider any real $\rho>R$ and any nonempty convex set V with $V\subset\mathcal{E}_\rho(C)$. According to Theorem 3.3, we know for $\sigma:=(\rho-R)^{-1}$ that the function $-\mathrm{dfar}_C$ is $\sigma-$semiconvex on the set V, or equivalently, $-\mathrm{dfar}_C+\psi_V$ is σ-semiconvex on the whole space X. Proceeding as in [2, Remark 3.2], we can easily establish that

$$-\mathrm{dfar}_C(x)=\sup_{x^\star\in X}\inf_{y\in V}\big(q_{x^\star,\rho-R}(x)-q_{x^\star,\rho-R}(y)-\mathrm{dfar}_C(y)\big)\ \text{for all } x\in C.$$ □

Another important result, which can be seen as a geometrical characterization of the farthest distance from a strongly convex set, is presented in Theorem 4.10. It is the analog for strongly convex sets of a similar result for convex sets [20, Chapter 6] recently extended to prox-regular sets in [1, Theorem 7]. It tells us that the farthest distance of a given point x from a strongly convex set is the minimum of the farthest distance from x to suitable closed balls separating the set and the point x.

Theorem 4.10. *Let C be an R-strongly subset of X for some real $R\in]0,+\infty[$ and let $x\in X$ with $\delta:=\mathrm{dfar}(x,C)>2R$. Then, one has with $L_{y^\star,R,\alpha}$ as defined above in* (22)

$$\delta=\min\left\{\mathrm{dfar}(x,L_{y^\star,R,\alpha}):(y^\star,\alpha)\in X\times\mathbb{R},C\subset L_{y^\star,R,\alpha},x\notin L_{y^\star,R,\alpha}\right\}. \tag{36}$$

The minimum is attained at $(x^\star,\beta)$ with $x^\star:=\Upsilon_{C,R}(x)$ and $\beta:=\Phi_{C,R}(x^\star)$ (see (17) *and* (22)*).*

Further, for all $y^\star \in X$ with $\|y^\star - R^{-1}x\| = R^{-1}\delta - 1$ and all $\alpha \in \mathbb{R}$, one has the following implication

$$\left.\begin{array}{l}\delta = d(x, L_{y^\star,r,\alpha}),\\ C \subset L_{y^\star,r,\alpha}, x \notin L_{y^\star,r,\alpha}\end{array}\right\} \Rightarrow (y^\star, \alpha) = \big(x^\star, \Phi_{C,R}(x^\star)\big).$$

Proof. Set $v := \mathrm{far}_C(x)$. Thanks to Lemma 4.5, we have

$$\inf\{\mathrm{dfar}(x, L_{y^\star,R,\alpha}) : (y^\star, \alpha) \in X \times \mathbb{R}, C \subset L_{y^\star,R,\alpha}, x \notin L_{y^\star,R,\alpha}\} \geq \delta. \tag{37}$$

On the other hand, Theorem 4.3 gives some real κ such that $C \subset L_{x^\star,R,\kappa}$ and $x \notin L_{x^\star,R,\kappa}$. Lemma 4.5 again says that $C \subset L_{x^\star,R,\beta} =: L$ and $x \notin L_{x^\star,R,\beta}$ with $\beta := \Phi_{C,R}(x^\star)$. According to Proposition 4.6(c), we have $L = B[Rx^\star, R]$, hence (see Proposition 4.6(a))

$$\mathrm{dfar}(x, L) = R + \|Rx^\star - x\| = R + (\delta - R) = \delta. \tag{38}$$

The desired equality (36) directly follows from (37) and (38).

Fix any $y^\star \in X$ satisfying $\|y^\star - R^{-1}x\| = R^{-1}\delta - 1$. Let $t \in \mathbb{R}$ be chosen such that $\mathrm{dfar}(x, C) = \mathrm{dfar}(x, L_{y^\star,R,t})$ along with

$$C \subset L_{y^\star,R,t} \quad \text{and} \quad x \notin L_{y^\star,R,t} =: L_t. \tag{39}$$

According to Lemma 4.5, we have with $\theta := \Phi_{C,R}(y^\star)$

$$C \subset L_{y^\star,R,\theta} \subset L_t \quad \text{and} \quad x \notin L_{y^\star,R,\theta} =: L_\theta.$$

Hence, we get $$\delta \leq \mathrm{dfar}(x, L_\theta) \leq \mathrm{dfar}(x, L_t) = \delta.$$

From Proposition 4.4, we then see that $\rho := R^2\|y^\star\|^2 - 2R\theta = \mathrm{dfar}^2_C(Ry^\star)$ along with

$$\delta = \mathrm{dfar}(x, L_\theta) = \mathrm{dfar}\big(x, B[Ry^\star, \sqrt{\rho}]\big) = \sqrt{\rho} + \|Ry^\star - x\|.$$

This and the equality $\|Ry^\star - x\| = \delta - R$ furnish

$$R^2\|y^\star\|^2 - 2R\theta = \rho = \big(\delta - \|Ry^\star - x\|\big)^2 = R^2,$$

or equivalently, $$\theta = \frac{R}{2}\big(\|y^\star\|^2 - 1\big).$$

From the definitions of $q_{y^\star,R}(x)$ and θ, we obtain that

$$\begin{aligned} q_{y^\star,R}(x) - \Phi_{C,R}(y^\star) &= q_{y^\star,R}(x) - \theta = \langle y^\star, x\rangle - \frac{\|x\|^2}{2R} - \frac{R}{2}\big(\|y^\star\|^2 - 1\big)\\ &= -\frac{R}{2}\|y^\star - \frac{x}{R}\|^2 + \frac{R}{2} = -\frac{R}{2}(R^{-1}\delta - 1)^2 + \frac{R}{2} = \delta\big(1 - \frac{\delta}{2R}\big).\end{aligned}$$

The equality above and Theorem 4.7 yield $y^\star = x^\star = \Upsilon_{C,R}(x)$. It remains to show that $t = \Phi_{C,R}(y^\star)$. Note that the inclusion in (39) and the definition of $\Phi_{C,R}$ in (22) easily give $\Phi_{C,R}(y^\star) \geq t$.

If $t < \Phi_{C,R}(y^\star)$, then Proposition 4.4 and Proposition 4.6(c) would entail

$$\begin{aligned}\delta = \mathrm{dfar}(x, L_{x^\star,R,t}) &= \sqrt{R^2\|x^\star\|^2 - 2Rt} + \|Rx^\star - x\| \\ &> \sqrt{R^2\|x^\star\|^2 - 2R\Phi_{C,R}(x^\star)} + \|Rx^\star - x\| \\ &= R + \|Rx^\star - x\| = \mathrm{dfar}(x, L) = \delta,\end{aligned}$$

which obviously leads to a contradiction. We conclude that $t = \Phi_{C,R}(y^\star)$. The proof is then complete. □

References

[1] S. Adly, F. Nacry, L. Thibault: *New metric properties for prox-regular sets*, Math. Programming Ser. B 189 (2021) 7–36.

[2] S. Adly, F. Nacry, L. Thibault: *Prox-regular sets and Legendre-Fenchel transform related to separation properties*, Optimization 71 (2022) 2097–2129.

[3] G. Allaire, C. Dapogny, P. Frey: *Shape optimization with a level set based mesh evolution method*, Comput. Methods Appl. Mech. Engrg. 282 (2014) 22–53.

[4] L. Ambrosio: *Geometric evolution problems, distance function and viscosity solutions*, in: *Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory*, Pisa 1996, G. Buttazzo et al. (eds.), Springer, Berlin (2000) 5–93.

[5] J.-P. Aubin, A. Cellina: *Differential Inclusions. Set-Valued Maps and Viability Theory*, Grundlehren der Mathematischen Wissenschaften 264, Springer, Berlin (1984).

[6] M. V. Balashov: *Weak convexity of the distance function*, J. Convex Analysis 20 (2013) 93–106.

[7] M. V. Balashov, M. O. Golubev: *Weak concavity of the antidistance function*, J. Convex Analysis 21 (2014) 951–964.

[8] M. V. Balashov, G. E. Ivanov: *On farthest points of sets*, Math. Notes 80 (2006) 159–166.

[9] M. V. Balashov, G. E. Ivanov: *Weakly convex and proximally smooth sets in Banach spaces*, Izv. Math. 73 (2009) 455–499.

[10] H. H. Bauschke, P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert spaces*, CMS Books in Mathematics, Springer, New York (2011).

[11] F. Bernard, L. Thibault, N. Zlateva: *Prox-regular sets and epigraphs in uniformly convex Banach spaces: various regularities and other properties.* Trans. Amer. Math. Soc. 363 (2011) 2211–2247.

[12] A. Canino: *On p-convex sets and geodesics*, J. Diff. Equations 75 (1988) 118–157.

[13] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[14] C. Castaing, L. Thibault: *Various perturbations and relaxations of sweeping process*, J. Convex Analysis (2023).

[15] F. H. Clarke: *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience Publication, New York (1983).

[16] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, P. R. Wolenski: *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer, New York (1998).

[17] F. H. Clarke, R. J. Stern, P. R. Wolenski: *Proximal smoothness and the lower-C^2 property*, J. Convex Analysis 2 (1995) 117–144.

[18] G. Colombo, L. Thibault: *Prox-regular sets and applications*, in: *Handbook of Nonconvex Analysis and Applications*, D. Y. Gao et al. (eds.), International Press, Somerville (2010) 99–182.

[19] M. C. Delfour, J.-P. Zolésio: *Shape analysis via oriented distance functions*, J. Funct. Analysis 123 (1994) 129–201.

[20] F. Deutsch: *Best Approximation in Inner Product Spaces*, CMS Books in Mathematics 7, Springer, New York (2001).

[21] H. Federer: *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[22] H. Frankowska, C. Olech: *R-convexity of the integral of the set-valued functions*, Contributions to analysis and geometry, Suppl. Am. J. Math. (1981) 117–129.

[23] V. V. Goncharov, G. E. Ivanov: *Strong and weak convexity of closed sets in a Hilbert space*, in: *Operations Research, Engineering, and Cyber Security. Trends in Applied Mathematics and Technology*, Nea Peramos 2015, N. Daras et al. (eds.), Springer Optimization and Its Applications 113, Springer, Cham (2017) 259–297.

[24] D. R. Luke: *Finding best approximation pairs relative to a convex and prox-regular set in a Hilbert space*, SIAM J. Optimization 19 (2008) 714–739.

[25] A. D. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematics, Springer, Cham (2017).

[26] G. E. Ivanov: *Weakly Convex Sets and Functions: Theory and Applications (Russian)*, Fizmatlit, Moscow (2006).

[27] G. E. Ivanov: *Farthest points and strong convexity of sets*, Math. Notes 87 (2020) 355–366.

[28] G. E. Ivanov, E. S. Polovinkin: *On strongly convex differential games*, Diff. Equations 31 (1995) 1603–1612.

[29] E. S. Levitin, B. T. Polyak: *Constrained minimization methods*, U.S.S.R. Comput. Math. Math. Phys. 6 (1966) 787–823.

[30] B. Maury, J. Venel: *A discrete contact model for crowd motion*, ESAIM Math. Model. Numer. Analysis 45 (2011) 145–168.

[31] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Grundlehren der mathematischen Wissenschaften 330, Springer, Berlin (2006).

[32] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. II: Applications*, Grundlehren der mathematischen Wissenschaften 331, Springer, Berlin (2006).

[33] J. J. Moreau: *Rafle par un convexe à courbure minorée: questions asymptotiques*, Sém. d'Analyse Convexe, Exposé no 2, Université de Montpellier (1978).

[34] F. Nacry, L. Thibault: *Regularization of sweeping process: old and new*, Pure Appl. Funct. Analysis 4 (2019) 59–117.

[35] F. Nacry, L. Thibault: *Distance function associated to a prox-regular set*, Set-Valued Var. Analysis 30 (2022) 731–750.

[36] J.-P. Penot: *Calculus Without Derivatives*, Graduate Texts in Mathematics 266, Springer, New York (2013).

[37] R. A. Poliquin, R. T. Rockafellar, L. Thibault: *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[38] E. S. Polovinkin, M. V. Balashov: *Elements of Convex and Strongly Convex Analysis (Russian)*, Fizmatlit, Moscow (2004).

[39] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. 7 (1966) 72–75.

[40] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der mathematischen Wissenschaften 317, Springer, Berlin (1998).

[41] A. Shapiro: *Existence and differentiability of metric projections in Hilbert spaces*, SIAM J. Optimization 4 (1994) 130–141.

[42] I. Singer: *Duality for Nonconvex Approximation and Optimization*, CMS Books in Mathematics 24, Springer, New York (2006).

[43] L. Thibault: *Sweeping process with regular and nonregular sets*, J. Diff. Equations 193 (2003) 1–26.

[44] L. Thibault: *Unilateral Variational Analysis in Banach Spaces*, World Scientific, Singapore (2023), to appear.

[45] J. P. Vial: *Strong and weak convexity of sets and functions*, Math. Oper. Res. 8 (1983) 231–259.

[46] A. Weber, G. Reissig: *Local characterization of strongly convex sets*, J. Math. Analysis Appl. 400 (2013) 743–750.