Regularization of sweeping process: old and new

Florent Nacry (Laboratoire IRMAR - INSA de Rennes)
Joint work with Lionel Thibault (Institut Montpelliérain Alexander Grothendieck)

SEMINAIRE XLIM-MATHIS/MOD, UNIVERSITE DE LIMOGES

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An introduction to Moreau’s sweeping process

Notation

- The letter $\mathscr{H}$ stands for a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.

- $I := [0, T]$ is a compact interval of $\mathbb{R}$ for some given real $T > 0$.

- $C : I \rightrightarrows \mathscr{H}$ is a given multimapping with nonempty closed values (="moving set").

- Given $S \subset \mathscr{H}$, ψ_S is the indicator function of S, i.e., $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ otherwise.

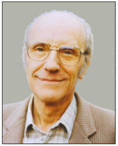


Figure 1: Jean Jacques Moreau (1923-2014)

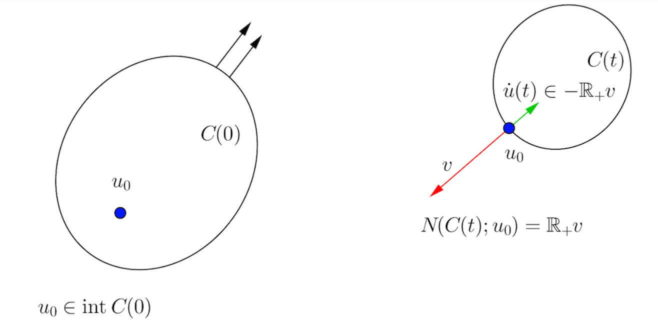
"Given $a \in C(0)$, the problem is that of finding a (single-valued) mapping $u : I \to \mathscr{H}$ absolutely continuous on I such that $u(0) = a$ and that

$$-\frac{du}{dt} \in \partial \psi_{C(t)}(u(t)) \quad \text{a.e. } t \in I. \tag{1}$$

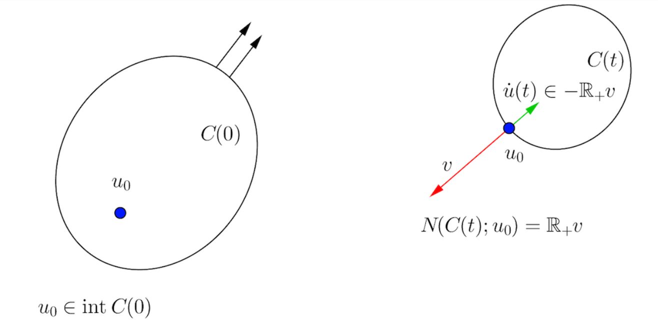
[...] The evolution process defined by condition (1) may be depicted in a mechanical language, especially clear if $C(t)$ possesses a nonempty interior. The moving point $t \mapsto u(t)$ remains at rest as long as it happens to lie in this interior; when caught up by the boundary of the moving set, it can only proceed in an inward direction, as if pushed by this boundary, so as to go on belonging to $C(t)$." [1]

[1] J.J. Moreau **Evolution Problem Associated with a Moving Convex Set in a Hilbert space** (Journal of Differential Equations, 26, 347-374, (1977))

Mechanical view



Mechanical view



Let $u_0 \in C(0)$. The latter problem can be rewritten as: find **absolutely continuous** mappings $u : I \to \mathscr{H}$ satisfying

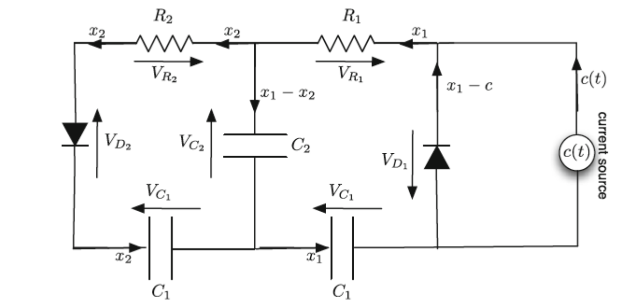
$$\begin{cases} -\dot{u}(t) \in \partial_{?} \psi_{C(t)}(u(t)) := N^{?}(C(t); u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) & \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

$$a \ni C \subset \mathscr{H} \text{ convex} : \partial \psi_C(a) := N(C; a) := \{\zeta \in \mathscr{H} : \langle \zeta, x - a \rangle \leq 0, \forall x \in C\}.$$

Applications

- Sweeping process appears in (non-exhaustive list):
  - Planning procedure;
  - Crowd motion;
  - Elasto-plastic models;
  - Non-regular electrical circuits;
  - Evolution of sandpiles;
  - Quasistatic evolution problems with friction, micromechanical damage models for iron materials.

Two resistors R_1, R_2 with voltage/current laws $V_{R_k} = R_k x_k$.
Two capacitors C_1, C_2 with voltage/current laws $V_{C_k} = C_k^{-1} \int x_k(t)dt$.
Two ideal diodes with characteristics $0 \leq -V_{D_k} \perp x_k \geq 0$.



Through electrical circuits

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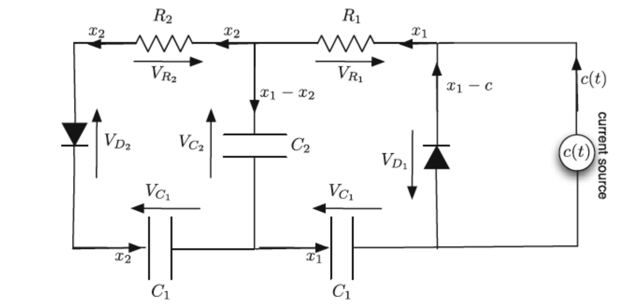


$\Rightarrow$ Using Kirchhoff's laws ($\dot{q}_i = x_i$ and $C(t) = [c(t), +\infty[\times [0, +\infty[$)

$$\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} \\ -\frac{1}{C_2} & \frac{1}{C_1} + \frac{1}{C_2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in -N(C(t); \dot{q}(t))$$

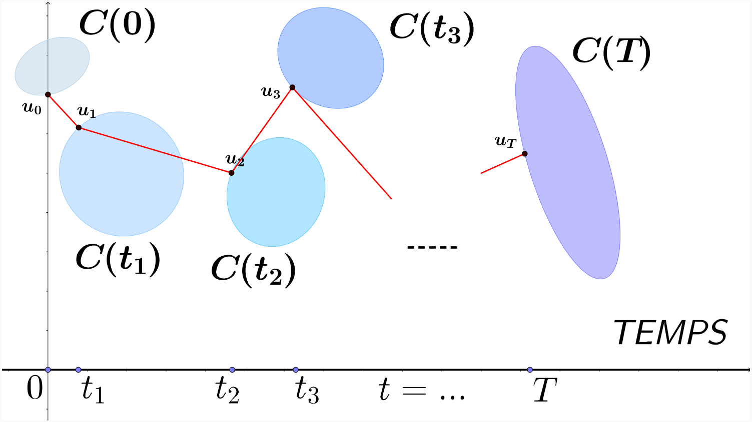
Variants

- Large number of variants:
  - Stochastic (1973);
  - State-dependent (1987/1998);
  - Nonconvex (1988);
  - With perturbations (1984);
  - In Banach spaces framework (2010);
  - Second order (Schatzman's sense (1978), Castaing's sense (1988));
  - In Wasserstein space (2016).

How to handle Moreau's sweeping processes?

Catching-up algorithm

Time discretization $0 = t_0^n < \ldots < t_{p(n)}^n = T$ + iterations of the form $u_i^n = \mathrm{proj}_{C(t_i^n)}(u_{i-1}^n)$ (with $u_0^n := u_0$ where u_0 is the initial condition) + suitable interpolation $\Rightarrow$ Sequence of mappings $(u_n(\cdot))$ $\Rightarrow$ Convergence to a solution $u(\cdot)$.



Reduction to unconstrained differential inclusion

Assume that there is a Lipschitz mapping $\zeta : I \to \mathbb{R}$ wich allows to control the moving set $C(\cdot)$ in the following way:

$$|d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |\zeta(t) - \zeta(s)|$$

Idea: The following constrained differential inclusion is equivalent (under assumptions!)

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0, \end{cases}$$

to the unconstrained one

$$\begin{cases} -\dot{u}(t) \in \dot{\zeta}(t)\partial_C d(u(t), C(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(0) = u_0. \end{cases}$$

Regularization of the normal cone

To solve the differential inclusion

$$(SP)\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0, \end{cases}$$

Step 1: Find a family or ordinary differential equation

$$(E_j)\begin{cases} -\dot{u}_j(t) = f(t, u_j(t)) \\ u_j(0) = u_0. \end{cases}$$

Step 2: Established a convergence

$$u_j(\cdot) \xrightarrow{?} u(\cdot).$$

Step 3: Show that $u(\cdot)$ is a solution of (SP).

Regularization: AC & BV convex cases

Yosida approximation

Let $A : \mathscr{H} \to \mathscr{H}$ be a maximal monotone operator, i.e., A is monotone

$$\langle x^\star - y^\star, x - y \rangle \geq 0 \quad \text{for all } x^\star \in A(x), y^\star \in A(y)$$

and no enlargement of its graph is possible in $\mathscr{H}^2$ without destroying monotonicity.

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Consider the Evolution Equation associated with Maximal Monotone Operator: find $u : [0, +\infty[\to \mathscr{H}$ such that

$$\boxed{\dot{u}(t) + Au(t) \ni 0 \quad \text{with initial condition} \quad u(0) = u_0. \qquad (EEMMO)}$$

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Yosida approximation of A with parameter $\lambda > 0$: $A_\lambda := (\lambda I + A^{-1})^{-1}$. It is maximal monotone, single-valued and Lipschitz continuous with λ^{-1}-Lipschitz constant.

To solve (EEMMO): Follow a regularization procedure with the family of ODE:

$$\boxed{\dot{u}_\lambda(t) + A_\lambda u_\lambda(t) = 0 \quad \text{with initial condition} \quad u_\lambda(0) = u_0.}$$

Let us go back to the Moreau's sweeping process with $A_t(\cdot) := N(C(t);\cdot)$:

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- $f \in \Gamma_0(\mathscr{H}) \Rightarrow \partial f : \mathscr{H} \rightrightarrows \mathscr{H}$ defined by

$$\partial f(x) := \left\{ \zeta \in \mathscr{H} : \left\langle \zeta, x' - x \right\rangle \leq f(x') - f(x) \quad \forall x' \in \mathscr{H} \right\}.$$

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is a maximal monotone operator.

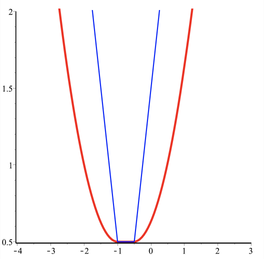
• For each $t > 0$, $A_t(\cdot) = \partial \psi_{C(t)}(\cdot)$ is a maximal monotone operator (convex subdifferential of a proper lower semi-continuous function !)

How to compute Yosida approximation of A_t?

Moreau envelope

Let $f : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ a function, $\lambda > 0$.
The λ-Moreau envelope of f is the function $e_\lambda f : \mathscr{H} \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$e_\lambda f(x) := \inf_{y \in \mathscr{H}} (f(y) + \frac{1}{2\lambda} \|y - x\|^2) \quad \text{for all } x \in \mathscr{H}.$$



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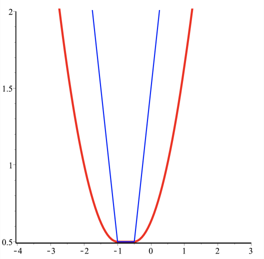

- J.J. Moreau, **Fonctions convexes duales et points proximaux dans un espace hilbertien**, CRAS 1962.
- Term coined by R.T. Rockafellar and R.J.B. Wets in **Variational Analysis**, 1998.

Few words about Moreau envelope

The λ-Moreau envelope of f is given by $e_\lambda f = f \Box \frac{1}{2\lambda} \|\cdot\|^2$ where $\Box$ is the inf-convolution/epi-sum (J.J. Moreau, 1963)

$$(g \Box h)(x) := \inf_{y \in \mathscr{H}} (g(y) + h(x - y)).$$

First properties of $e_\lambda f$

Let $f \in \Gamma_0(\mathscr{H})$, $x \in \mathscr{H}$. Then, one has:

(*i*) $\forall \lambda > 0$, $e_\lambda f$ is convex, real-valued, continuous and exact at only one point.

(*ii*) The net $(e_\lambda f(x))_{\lambda > 0}$ is decreasing.

(*iii*) $e_\lambda f(x) \uparrow f(x)$ as $\lambda \downarrow 0$ and $e_\lambda f(x) \downarrow \inf f(\mathscr{H})$ as $\lambda \uparrow +\infty$.

(*iv*) $e_\lambda f$ is differentiable on $\mathscr{H}$ with gradient λ^{-1}-Lipschitz.

Aim: Compute Yosida approximation $(A_t)_\lambda$ where

$$A_t(\cdot) := \partial \psi_{C(t)}(\cdot) = N(C(t); \cdot).$$

(Moreau envelope-Yosida regularization)

Let $f \in \Gamma_0(\mathscr{H})$. Then, the Yosida regularization of ∂f for any $\lambda > 0$ is the gradient mapping $\nabla e_\lambda f$ associated with the Moreau envelope $e_\lambda f$.

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$f = \psi_{C(t)}$ where $C(t) \subset \mathscr{H}$ is nonempty closed and convex, $\lambda > 0$:

$$\begin{aligned} e_\lambda \psi_{C(t)}(x) &= \inf_{y \in X} \left(\psi_{C(t)}(y) + \frac{1}{2\lambda} \|x - y\|^2\right) \\ &= \inf_{y \in C(t)} \frac{1}{2\lambda} \|x - y\|^2 \\ &= \frac{1}{2\lambda} d^2_{C(t)}(x). \end{aligned}$$

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$$\boxed{(A_t)_\lambda(x) := \frac{1}{2\lambda}\nabla d^2_{C(t)}(x)}$$

Hausdorff-Pompeiu distance

Hausdorff-Pompeiu distance between nonempty subsets $S, S' \subset \mathscr{H}$,

$$\operatorname{haus}(S, S') := \sup_{x \in \mathscr{H}} \left| d(x, S) - d(x, S') \right| = \max \left\{ \operatorname{exc}(S, S'), \operatorname{exc}(S', S) \right\}$$

where $\operatorname{exc}(S, S') := \sup_{x \in S} d(x, S')$.

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Distance property

The Hausdorff-Pompeiu distance $\mathrm{haus}(\cdot, \cdot)$ is a semi-distance (resp. a distance) on the nonempty subsets (resp. the nonempty bounded subsets) of $\mathscr{H}$.

Variation of a moving set

The variation of the multmapping (= the moving set) $C(\cdot)$ on I is given by

$$\mathrm{var}(C;I) := \sup\left\{\sum_i \mathrm{haus}(C(t_i), C(t_{i+1})) : (t_i)_i \text{subdivision of } I\right\}.$$

$C(\cdot)$ is of bounded variation $\Leftrightarrow \mathrm{var}(C;I) < +\infty$.

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$C(\cdot)$ is of bounded variation $\Leftrightarrow \mathrm{var}(C;I) < +\infty$.

Existence of solutions for sweeping process $\Rightarrow$ We must control the variation function $t \mapsto \mathrm{var}(C;[0,t])$.

Standard assumption in sweeping process theory: $C(\cdot)$ is Lipschitz relative to $\mathrm{haus}(\cdot,\cdot)$, i.e.,

$$\boxed{\exists L > 0, \forall t, s \in I, \mathrm{haus}(C(t), C(s)) \leq L\,|t-s|}$$

First result of the theory

Theorem (J.J. Moreau (1971))

Let $a \in C(0)$. Assume that $C(\cdot)$ is nonempty closed convex valued and that $t \mapsto \mathrm{var}(C;[0,t])$ of C is absolutely continuous on I with $\dot{v} \in L^2(I)$. For each real $\lambda > 0$, let u_λ be the unique absolutely continuous solution of the differential equation

$$\dot{u}_\lambda(t) = -\nabla\left(\frac{1}{2\lambda} d^2_{C(t)}\right)(u_\lambda(t)) \quad \text{with initial condition } u_\lambda(0) = a.$$

Then, the family $(u_\lambda)_{\lambda>0}$ converges uniformly on I as $\lambda \downarrow 0$ to an absolutely continuous function $u : I \to \mathscr{H}$ of the sweeping differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) & \text{for all } t \in I, \\ u(0) = a. \end{cases}$$

In addition, one has: $\dot{u}_\lambda(\cdot) \xrightarrow{L^2} \dot{u}(\cdot)$ as $\lambda \downarrow 0$.

Theorem (M.D.P. Monteiro Marques (1987))

Let $a \in C(0)$. Assume that $C(\cdot)$ is nonempty ball-compact convex valued and that C has a bounded variation on I and $t \mapsto \mathrm{var}(C;[0,t])$ is right-continuous on I and continuous at T.

Then, for any real $\lambda > 0$, the (classical) differential equation over $I = [0, T]$

$$\dot{u}_\lambda(t) = -\nabla\left(\frac{1}{2\lambda} d^2_{C(t)}\right)(u_\lambda(t)) \quad \text{with initial condition } u_\lambda(0) = a$$

has a unique absolutely continuous solution $u_\lambda(\cdot)$ on I and the family $(u_\lambda(\cdot))_{\lambda>0}$ converges pointwise on I as $\lambda \downarrow 0$ to a mapping $u : I \to \mathscr{H}$ solution of the BV sweeping process

$$\begin{cases} -du^+ \in N(C(t); u(t)) \\ u^+(t) \in C(t) \quad \text{for all } t \in I, \\ u^+(0) = a, \end{cases}$$

where $u^+(t) := \lim_{\tau \downarrow t} u(\tau)$.

Prox-regular and α-far regularization

Definition

The Clarke tangent cone (resp. Clarke normal cone) to a subset $S \subset \mathscr{H}$ at $x \in S$ is the set

$$T^C(S;x) := \{h \in \mathscr{H} : \forall S \ni x_n \to x, \forall t_n \downarrow 0, \exists \mathscr{H} \ni h_n \to h, x_n + t_n h_n \in S, \forall n \in \mathbb{N}\}$$

(resp.

$$N^C(S;x) := \left\{v \in \mathscr{H} : \langle v, h \rangle \leq 0, \forall h \in T^C(S;x)\right\}).$$

For $x \in \mathscr{H}$, U an open neighborhood of x and $f : U \to \overline{\mathbb{R}}$ an extended real-valued function finite at x, the Clarke subdifferential of f at x is defined as the set

$$\partial_C f(x) := \left\{v \in \mathscr{H} : (v, -1) \in N^C(\mathrm{epi}\, f; (x, f(x)))\right\}.$$

Definition of uniform prox-regularity

For any $S \subset \mathscr{H}$ and any $r \in]0, +\infty]$, one sets $U_r(S) := \{x \in \mathscr{H} : d_S(x) < r\}$.

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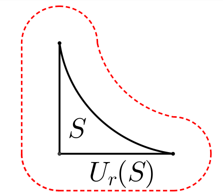

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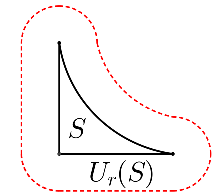

Definition

Let S be a nonempty closed subset of $\mathscr{H}$ and $r \in]0, +\infty]$ be an extended real. One says that S is *r-prox-regular* (or *uniformly prox-regular with constant r*) whenever the mapping $\mathrm{proj}_S : U_r(S) \to \mathscr{H}$ is well-defined and norm-to-norm continuous.

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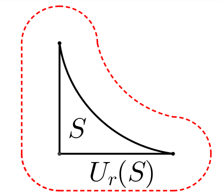

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- Notable contributors: H. Federer (1957); J.-P. Vial (1983); A. Canino (1988); A. Shapiro (1994); F.H. Clarke, R.L. Stern, P.R. Wolenski (1995); R.A. Poliquin, R. T. Rockafellar, L. Thibault (2000).

Characterizations and properties of uniform prox-regular sets

Let $r \in]0, +\infty]$. Convention: $\frac{1}{r} = 0$ whenever $r = +\infty$.

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Theorem (R.A. Poliquin, R.T. Rockafellar, L. Thibault (2000))

Let S be a nonempty closed subset of $\mathscr{H}$, $r \in]0,+\infty]$ be an extended real.

Consider the following assertions.

(a) S is r-prox-regular.

(b) For all $x_1, x_2 \in S$, for all $i \in \{1,2\}$, for all $v_i \in N^C(S;x_i) \cap \mathbb{B}_{\mathscr{H}}$, one has

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2 .$$

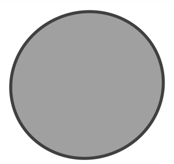
(c) The function d_S^2 is $C^{1,1}$ on $U_r(S)$.

(d) $\partial_P d_S(x) \neq \emptyset$ (resp., $\partial_F d_S(x) \neq \emptyset$) for all $x \in U_r(S)$.

(e) the mapping $\mathrm{proj}_S : U_r(S) \to \mathscr{H}$ is well-defined.

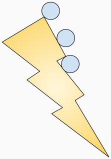
Then, one has $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Rightarrow (e)$. If in addition S is weakly closed, then $(a) \Leftrightarrow (e)$.

Prox-regular sets - examples and counter-examples

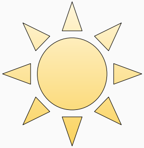


Nonempty closed convex

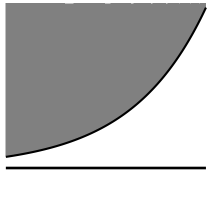
$\Leftrightarrow$ ∞-prox-regular



Lack of prox-regularity ("angle")



Disconnected prox-regular set



Lack of prox-regularity ("crushing")

A.C. prox-regular regularization

Theorem (L. Thibault (2008))

Let $a \in C(0)$. Assume that $C(\cdot)$ has r-prox-regular values for some $r \in]0,+\infty]$ and there exists a real $\kappa \geq 0$ such that

$$\mathrm{haus}(C(s), C(t)) \leq \kappa |s - t| \quad \text{for all } s, t \in I.$$

Let θ be a positive real number such that $\theta < r/(3\kappa)$.

Then, for any $\lambda \in]0, \kappa^{-1} r[$, the (classical) differential equation over $[0, \theta] \times B(a, \frac{r}{3})$

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda} \nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(0) = a \end{cases}$$

is well defined and has a unique solution $u_\lambda(\cdot)$ on $[0, \theta]$, and the family $(u_\lambda(\cdot))_{0<\lambda<\kappa^{-1}r}$ converges uniformly on $[0, \theta]$ as $\lambda \downarrow 0$ to a solution of the differential inclusion sweeping process

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \text{a.e. } t \in I \\ u(t) \in C(t) \quad \text{for all } t \in I \\ u(T_0) = a \in C(T_0). \end{cases}$$

Further results

- Global existence on $[0, T]$ through the classical way:
  - $u_0(\cdot)$ solution on $[0, T_1]$ of
  $$-\dot{u}_0(t) \in N(C(t); u_0(t)) \quad \text{and} \quad u_0(0) = a$$
  - $u_1(\cdot)$ solution on $[T_1, T_2]$ of
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  - $u_p(\cdot)$ solution on $[T_p, T]$ of
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- Absolutely continuous control on the moving set (Moreau's reduction).
- Single-valued perturbation of normal cone f.

Definition of α-farness (T. Haddad, A. Jourani, L. Thibault (2008))

We always have

$$\partial_C d_S(x) \subset N^C(S + d_S(x)\mathbb{B}; x) \cap \mathbb{B} \ni 0.$$

$\Longrightarrow$ We want to control/quantify how 0 is far from $\partial_C d_S(x)$.

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Let $S \subset \mathscr{H}$ be nonempty and closed, $\alpha > 0$. One says that S is α-far whenever there exists a real $\rho > 0$ such that for every $x \in U_\rho(S) \setminus S$,

$$\zeta \in \partial_C d_S(x) \Rightarrow \|\zeta\| \geq \alpha.$$

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Otherwise stated:

$$\boxed{S \,\text{is}\, \alpha - \text{far} \Leftrightarrow \exists \rho > 0, \alpha \leq \inf_{x \in U_\rho(S) \setminus S} d(0, \partial_C d(x, S)).}$$

Examples

Consider the (non prox-regular!) subset in $\mathbb{R}^2$

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Representation of Clarke's subgradients: $\partial_C f(x) = \operatorname{co}\{\lim \nabla f(x_n) : x_n \to x\}$

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Among the α-far sets: convex, prox-regular, subsmooth, paraconvex.

α-far regularization

Problem: In general, for an α-far subset $S \subset \mathscr{H}$, the function $d_S^2(\cdot)$ is not differentiable! Then, the standard regularization procedure (convex & prox-regular)

$$\dot{u}_\lambda(t) = -\nabla d_{C(t)}^2(u_\lambda(t)) \quad \text{and} \quad u_\lambda(0) = u_0$$

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is no longer applicable.

Solution: Consider the family of differential inclusions (DI_λ)

$$\boxed{\dot{u}_\lambda(t) = -\partial_C d_{C(t)}^2(u_\lambda(t)) \quad \text{with initial condition} \quad u_\lambda(0) = u_0.}$$

Theorem (A. Jourani, E. Vilches (2017))

Let $a \in C(0)$. Assume that $\mathscr{H}$ is separable and that $C(\cdot)$ be a multimapping with nonempty closed values for which there is a real $\kappa \geq 0$ such that

$$\mathrm{haus}(C(s), C(t)) \leq \kappa|s-t| \quad \text{for all } s, t \in I.$$

Assume that the following conditions (i) and (ii) hold:

(i) there exist $r \in]0, +\infty]$ and a real $\alpha > 0$ such that for every $x \in U_r(C(t)) \setminus C(t)$ one has

$$d(0, \partial_C d_{C(t)}) \geq \alpha.$$

(ii) for each $t \in I$ and each real $\mathfrak{r} > 0$ the set $C(t) \cap \mathfrak{r}\mathbb{B}$ is strongly compact in the space $\mathscr{H}$.

A. Jourani, E. Vilches (2017)

Then the following hold.

(a) For each positive real $\lambda < \alpha^2 r/\kappa$, the differential inclusion (DI_λ) admits at least one absolutely continuous solution.

(b) For each sequence $(\lambda'_n)_{n\in\mathbb{N}}$ in $]0,+\infty[$, there exist a subsequence $(\lambda_n)_{n\in\mathbb{N}}$ and an absolutely continuous solution $u_{\lambda_n}(\cdot)$ of the differential inclusion (DI_{λ_n}) for each $n\in\mathbb{N}$ such that $(u_{\lambda_n}(\cdot))_{n\in\mathbb{N}}$ converges pointwise on I to an absolutely continuous solution $u(\cdot)$ of the sweeping process

$$\dot{u}(t) \in -N(C(t);u(t)) \quad \text{with initial condition } u(T_0) = u_0,$$

and the derivative $\dot{u}(\cdot)$ of this solution satisfies $\|\dot{u}(t)\| \le \alpha^{-2}\kappa$ for almost every $t \in I$.

Sweeping process with truncated variation

Truncated Hausdorff-Pompeiu distance

$$\rho \in]0,+\infty];\ \mathbb{B} := \{x \in \mathscr{H} : \|x\| \leq 1\}.$$

The ρ-Hausdorff-Pompeiu distance is defined as

$$\mathrm{haus}_\rho(S,S') := \sup_{x\in\rho\mathbb{B}} \left| d(x,S') - d(x,S) \right| = \max\left\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S')\right\},$$

where

$$\mathrm{exc}_\rho(S,S') := \sup_{x\in\rho\mathbb{B}} (d(x,S') - d(x,S))^+.$$

The ρ-pseudo Hausdorff-Pompeiu distance is

$$\widehat{\mathrm{haus}}_\rho(S,S') := \max\left\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S')\right\},$$

with

$$\widehat{\mathrm{exc}}_\rho(S,S') := \sup_{x\in S\cap\rho\mathbb{B}} d(x,S').$$

$\Longrightarrow$ $\mathrm{haus}_\rho(\cdot,\cdot)$ semi-distance on nonempty subsets of $\mathscr{H}$.
$\Longrightarrow$ $\widehat{\mathrm{haus}}_\rho(\cdot,\cdot)$ does not fulfill the triangle inequality !

Hyperplane case

Consider the particular moving set with $\zeta : I \to \mathscr{H} \cap \mathbb{S}$ and $\beta : I \to \mathbb{R}$ be γ-Lipschitz mappings

$$C(t) := \{x \in \mathscr{H} : \langle \zeta(t), x \rangle - \beta(t) \leq 0\}.$$

In general, $C(\cdot)$ fails to be Lipschitz with respect to $\mathrm{haus}(\cdot,\cdot)$.

For all $x \in C(s)$,

$$\begin{aligned}
\langle \zeta(t), x \rangle - \beta(t) &= \langle \zeta(t) - \zeta(s), x \rangle + \langle \zeta(s), x \rangle - \beta(t) \\
&= \langle \zeta(t) - \zeta(s), x \rangle + (\langle \zeta(s), x \rangle - \beta(s)) + \beta(s) - \beta(t) \\
&\leq \|\zeta(t) - \zeta(s)\| \, \|x\| + 0 + |\beta(s) - \beta(t)| \\
&\leq \gamma(\|x\| + 1) \, |t - s| .
\end{aligned}$$

$$\boxed{\exists L > 0, \forall t, s \in I, \widehat{\mathrm{haus}}_\rho(C(t), C(s)) \leq L\,|t - s|}$$

Local existence result

Theorem (N., L. Thibault (2018))

Let $a \in C(0)$. Assume that $C(\cdot)$ is r-prox-regular valued for some $r \in]0,+\infty]$.
Assume that there exist $\kappa > 0$ and $\|a\| + r \leq \rho \leq +\infty$ such that

$$\mathrm{haus}_\rho(C(s), C(t)) \leq \kappa\,|t - s| \quad \text{for all } s, t \in [0, T].$$

Then, for any $\theta > 0$ with $\theta \leq T$ and satisfying $\theta < \frac{r}{3\kappa}$, **there exists** a κ-Lipschitz continuous mapping $u : [0, \theta] \to B(a, \frac{r}{3})$ solution of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \text{a.e. } t \in [0, \theta], \\ u(t) \in C(t) & \text{for all } t \in [0, \theta], \\ u(0) = a. \end{cases}$$

Global existence result

Theorem (N., L. Thibault (2018))

Let $a \in C(0)$. Assume that $C(\cdot)$ is r-prox-regular valued for some $r \in]0,+\infty]$. Assume that there exist a function $v : [0,T] \to \mathbb{R}$ which is κ-Lipschitz continuous for some real $\kappa > 0$ on $[0,T]$, a real $\theta \in]0, \min\left\{T, \frac{r}{3\kappa}\right\}[$ and an integer $p \geq 2$ with $p\theta \geq T$ such that

$$\mathrm{haus}_\rho(C(s), C(t)) \leq |v(t) - v(s)|$$

for some extended real $\rho \geq \|a\| + \frac{p+2}{3} r$ and for all $s, t \in I$.

Then, there exists a κ-Lipschitz continuous mapping $u : I \to B(a, \frac{pr}{3})$ solution of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) & \text{a.e. } t \in [0,T], \\ u(t) \in C(t) & \text{for all } t \in [0,T], \\ u(0) = a. \end{cases} \tag{2}$$

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Thank you for your attention ! Any questions ?