Truncated Moreau’s sweeping process

Florent Nacry - *Institut Elie Cartan de Lorraine*

joint works with

Lionel Thibault - *Institut Montpelliérain Alexander Grothendieck*

Journées annuelles du GdR MOA,
Université de Pau et des Pays de l’Adour, 17-19 Octobre 2018

An introduction to Moreau’s sweeping process

Notation

- The letter $\mathscr{H}$ stands for a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.
- $I := [0, T]$ is a compact interval of $\mathbb{R}$ for some given real $T > 0$.
- $C : I \rightrightarrows \mathscr{H}$ is a given multimapping with nonempty closed values (="moving set").
- Distance from $A \subset \mathscr{H}$ to $x \in \mathscr{H}$ is $d(x, A) := \inf_{a \in A} \|x - a\|$.

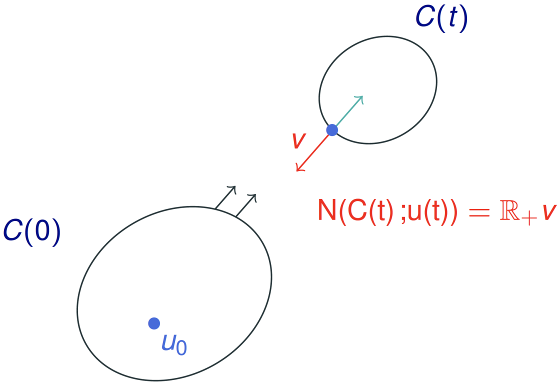
Mechanical point of view

Moreau's sweeping process: find absolutely continuous mappings $u : I = [0, T] \to \mathscr{H}$ satisfying for a given $u_0 \in C(0)$

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) := \{v \in \mathscr{H} : \langle v, x - u(t) \rangle \leq 0, \forall x \in C(t)\} & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

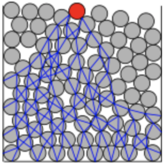
Moreau's sweeping process: find absolutely continuous mappings $u : I = [0, T] \to \mathscr{H}$ satisfying for a given $u_0 \in C(0)$

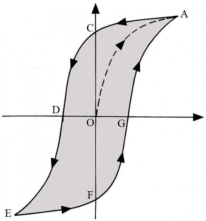
$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) := \{v \in \mathscr{H} : \langle v, x - u(t) \rangle \leq 0, \forall x \in C(t)\} & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

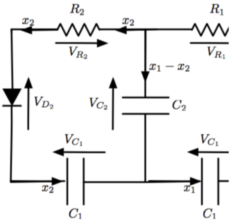


Applications

- Granular material
- Planning procedure
- Non-regular electrical circuits
- Crowd motion
- Hysteresis
- Evolution of sandpiles

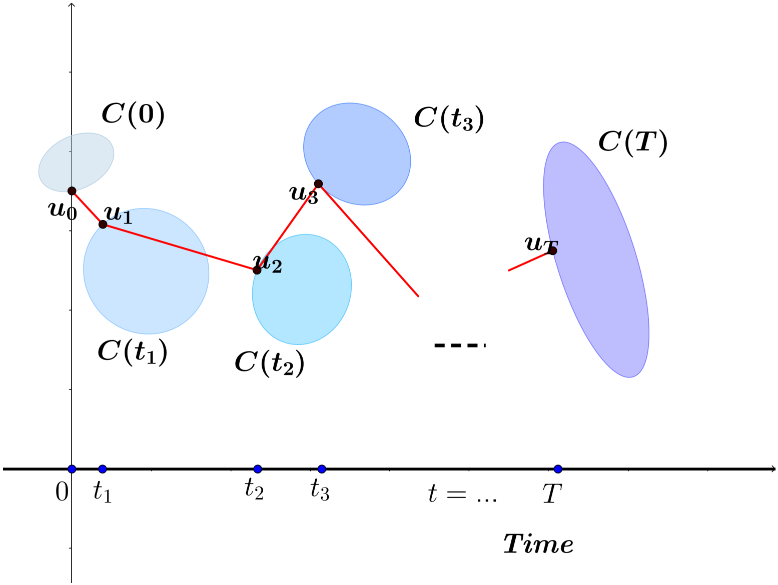






Variants

- Large number of variants:
  - Stochastic (1973);
  - State-dependent (1987/1998);
  - Nonconvex (1988);
  - With perturbations (1984);
  - In Banach spaces framework (2010);
  - Second order (Schatzman's sense (1978), Castaing's sense (1988));
  - Controlled (2015).



$C(0)$
$C(t_3)$
$C(T)$
u_3
u_0
u_1
u_T
u_2
$C(t_1)$
$C(t_2)$
0
t_1
t_2
t_3
$t = \dots$
T
Time

Handling sweeping process: the catching-up algorithm

- **Step 1: Time discretization** $t_i^n := i\frac{T}{2^n}$ and **iterations** $u_{i+1}^n \in \mathrm{Proj}_{C(t_i^n)}(u_i^n) \neq \emptyset$.

$\hookrightarrow$ Assumption on $C(\cdot)$ is needed here: convex-valued? ball-compact?...

Handling sweeping process: the catching-up algorithm

- **Step 1: Time discretization** $t_i^n := i\frac{T}{2^n}$ and **iterations** $u_{i+1}^n \in \mathrm{Proj}_{C(t_i^n)}(u_i^n) \neq \emptyset$.

$\hookrightarrow$ Assumption on $C(\cdot)$ is needed here: convex-valued? ball-compact?...

- **Step 2: Construction of step mappings**

$$I := [0, T] \ni t \mapsto u_n(t) := u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n).$$

Handling sweeping process: the catching-up algorithm

- **Step 1: Time discretization** $t_i^n := i\frac{T}{2^n}$ and **iterations** $u_{i+1}^n \in \mathrm{Proj}_{C(t_i^n)}(u_i^n) \neq \emptyset$.

$\hookrightarrow$ Assumption on $C(\cdot)$ is needed here: convex-valued? ball-compact?...

- **Step 2: Construction of step mappings**

$$I := [0,T] \ni t \mapsto u_n(t) := u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n).$$

- **Step 3: Convergence** of $(u_n(\cdot))_n$ to $u(\cdot) : [0,T] \to \mathscr{H}$.

$\hookrightarrow$ What kind of convergence? Assumption on the behavior of $C(\cdot)$ is needed here:

$$\exists L > 0, \forall s,t \in I, \sup_{x\in\mathscr{H}} \left|d_{C(t)}(x) - d_{C(s)}(y)\right| \leq L\left|t-s\right|.$$

Handling sweeping process: the catching-up algorithm

- **Step 1: Time discretization** $t_i^n := i\frac{T}{2^n}$ and **iterations** $u_{i+1}^n \in \mathrm{Proj}_{C(t_i^n)}(u_i^n) \neq \emptyset$.

$\hookrightarrow$ Assumption on $C(\cdot)$ is needed here: convex-valued? ball-compact?...

- **Step 2: Construction of step mappings**

$$I := [0,T] \ni t \mapsto u_n(t) := u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n).$$

- **Step 3: Convergence** of $(u_n(\cdot))_n$ to $u(\cdot) : [0,T] \to \mathscr{H}$.

$\hookrightarrow$ What kind of convergence? Assumption on the behavior of $C(\cdot)$ is needed here:

$$\boxed{\exists L > 0, \forall s,t \in I, \sup_{x \in \mathscr{H}} \left| d_{C(t)}(x) - d_{C(s)}(y) \right| \leq L\,|t-s|\,.}$$

- **Step 4:** $u(\cdot)$ is a solution of the Moreau's sweeping process.

$\hookrightarrow$ It requires a closedness property: $\forall t_n \downarrow t, \forall C(t_n) \ni x_n \to x$,

$$\limsup_{n \to +\infty} \sigma(z, \partial d_{C(t_n)}(x_n)) \leq \sigma(z, \partial d_{C(t)}(x)).$$

Handling sweeping process: regularization

To solve the differential inclusion

$$(SP)\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

Handling sweeping process: regularization

To solve the differential inclusion

$$(SP)\begin{cases} -\dot{u}(t) \in N(C(t);u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

- **Step 1:** Find a family or ordinary differential equation

$$(E_j)\begin{cases} -\dot{u}_j(t) = f_j(t, u_j(t)), \\ u_j(0) = u_0. \end{cases}$$

To solve the differential inclusion

$$
(SP)\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}
$$

- **Step 1:** Find a family or ordinary differential equation

$$
(E_j)\begin{cases} -\dot{u}_j(t) = f_j(t, u_j(t)), \\ u_j(0) = u_0. \end{cases}
$$

- **Step 2:** Established a convergence

$$
u_j(\cdot) \xrightarrow{?} u(\cdot).
$$

To solve the differential inclusion

$$(SP)\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

- **Step 1:** Find a family or ordinary differential equation

$$(E_j)\begin{cases} -\dot{u}_j(t) = f_j(t, u_j(t)), \\ u_j(0) = u_0. \end{cases}$$

- **Step 2:** Established a convergence

$$u_j(\cdot) \xrightarrow{?} u(\cdot).$$

- **Step 3:** Show that $u(\cdot)$ is a solution of (SP).

Handling sweeping process: reduction

Assume that there is a nondecreasing absolutely continuous mapping $v : I \to \mathbb{R}_+$ such that

$$\mathrm{haus}(C(s), C(t)) := \sup_{x \in \mathscr{H}} \left| d(x, C(t)) - d(x, C(s)) \right| \leq v(t) - v(s) \quad \text{for all } s \leq t.$$

Handling sweeping process: reduction

Assume that there is a nondecreasing absolutely continuous mapping $v : I \to \mathbb{R}_+$ such that

$$\mathrm{haus}(C(s), C(t)) := \sup_{x \in \mathscr{H}} \left| d(x, C(t)) - d(x, C(s)) \right| \le v(t) - v(s) \quad \text{for all } s \le t.$$

Idea: The following constrained differential inclusion is equivalent (under assumptions!)

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) & \text{for all } t \in I, \\ u(0) = u_0, \end{cases}$$

to the unconstrained one

$$\begin{cases} -\dot{u}(t) \in \dot{v}(t) \partial d(u(t), C(t)) & \lambda\text{-a.e. } t \in I, \\ u(0) = u_0. \end{cases}$$

Sweeping process with truncated variation

First existence result

$\operatorname{exc}(A, B) := \sup_{x \in A} d(x, B)$.

Theorem (Moreau (1971))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is **nonempty closed convex** valued and

$$\operatorname{exc}(C(s), C(t)) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

for some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}_+$.

First existence result

$\mathrm{exc}(A, B) := \sup_{x \in A} d(x, B)$.

Theorem (Moreau (1971))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is **nonempty closed convex** valued and

$$\mathrm{exc}(C(s), C(t)) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

for some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}_+$.

Then, **there exists one and only one** absolutely continuous mapping $u : [0, T] \to \mathscr{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Let S, S' be **nonempty** subsets of $\mathscr{H}$.

One defines the Hausdorff-Pompeiu distance as

$$\mathrm{haus}(S, S') = \max\left\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\right\},$$

where

$$\mathrm{exc}(S, S') = \sup_{x \in S} d(x, S').$$

Let S, S' be **nonempty** subsets of $\mathscr{H}$.

One defines the Hausdorff-Pompeiu distance as

$$\mathrm{haus}(S, S') = \max\left\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\right\},$$

where

$$\mathrm{exc}(S, S') = \sup_{x \in S} d(x, S').$$

One has the following equalities

$$\mathrm{exc}(S, S') = \sup_{x \in X} \left(d(x, S') - d(x, S)\right)$$

and

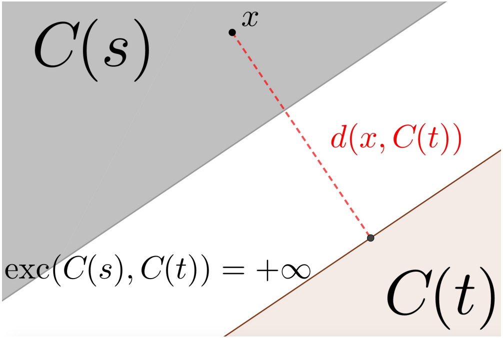
$$\mathrm{haus}(S, S') = \sup_{x \in X} \left|d(x, S') - d(x, S)\right|.$$

Let $\zeta : I \to \mathscr{H}$ and $\beta : I \to \mathbb{R}$ be two mappings. Consider the **moving hyperplane**

$$C(t) := \{x \in \mathscr{H} : \langle \zeta(t), x \rangle - \beta(t) \leq 0\}.$$

Let $\zeta : I \to \mathscr{H}$ and $\beta : I \to \mathbb{R}$ be two mappings. Consider the **moving hyperplane**

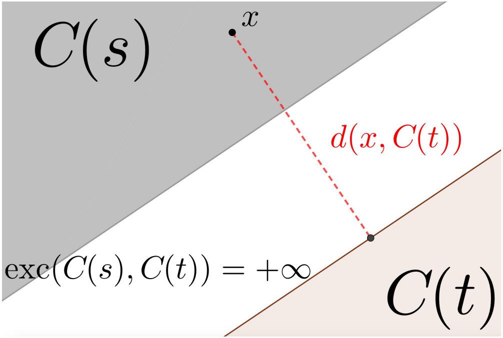
$$C(t) := \{x \in \mathscr{H} : \langle \zeta(t), x\rangle - \beta(t) \leq 0\}.$$



Hyperplane case

Let $\zeta : I \to \mathscr{H}$ and $\beta : I \to \mathbb{R}$ be two mappings. Consider the **moving hyperplane**

$$C(t) := \{x \in \mathscr{H} : \langle \zeta(t), x\rangle - \beta(t) \leq 0\}.$$



$\hookrightarrow$ The Hausdorff-Pompeiu excess $\mathrm{exc}(\cdot,\cdot)$ is not suitable to handle unbounded sweeping process.

Truncated Hausdorff-Pompeiu distance

$$\rho \in]0,+\infty];\ \mathbb{B} := \{x \in \mathscr{H} : \|x\| \leq 1\}.$$

Truncated Hausdorff-Pompeiu distance

$$\rho \in]0,+\infty];\ \mathbb{B} := \{x \in \mathscr{H} : \|x\| \le 1\}.$$

- The ρ-pseudo Hausdorff-Pompeiu distance is

$$\mathrm{haus}_\rho(S,S') := \max\left\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S)\right\},$$

with

$$\boxed{\mathrm{exc}_\rho(S,S') := \sup_{x \in S \cap \rho\mathbb{B}} d(x,S').}$$

Truncated Hausdorff-Pompeiu distance

$$\rho \in]0,+\infty]; \mathbb{B} := \{x \in \mathscr{H} : \|x\| \leq 1\}.$$

- The ρ-pseudo Hausdorff-Pompeiu distance is

$$\mathrm{haus}_\rho(S,S') := \max\left\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S)\right\},$$

with

$$\boxed{\mathrm{exc}_\rho(S,S') := \sup_{x\in S\cap\rho\mathbb{B}} d(x,S').}$$

- The ρ-Hausdorff-Pompeiu distance is defined as

$$\widehat{\mathrm{haus}}_\rho(S,S') := \sup_{x\in\rho\mathbb{B}} \left|d(x,S') - d(x,S)\right| = \max\left\{\widehat{\mathrm{exc}}_\rho(S,S'), \widehat{\mathrm{exc}}_\rho(S',S)\right\},$$

where

$$\boxed{\widehat{\mathrm{exc}}_\rho(S,S') := \sup_{x\in\rho\mathbb{B}} \left(d(x,S') - d(x,S)\right)^+.}$$

Existence under truncated excess

$\mathscr{H} = \mathbb{R}^n$, $m(t) := \mathrm{proj}_{C(t)}(0)$, $K(t) := C(t) - m(t)$.

Theorem (Colombo, Henrion, Hoang, Mordukhovich (2015))

Let $u_0 \in C(0)$. Assume that $C(\cdot)$ is **nonempty closed convex valued**. Assume also that $m(\cdot)$ is absolutely continous on $[0, T]$ and that for all real $\rho > 0$, there exists a nondecreasing absolutely continuous mapping $v_\rho : [0, T] \to \mathbb{R}_+$ such that

$$\mathrm{exc}_\rho(K(s), K(t)) \leq v_\rho(t) - v_\rho(s) \quad \text{for all } 0 \leq s \leq t \leq T.$$

Existence under truncated excess

$\mathscr{H} = \mathbb{R}^n$, $m(t) := \mathrm{proj}_{C(t)}(0)$, $K(t) := C(t) - m(t)$.

Theorem (Colombo, Henrion, Hoang, Mordukhovich (2015))

Let $u_0 \in C(0)$. Assume that $C(\cdot)$ is **nonempty closed convex valued**. Assume also that $m(\cdot)$ is absolutely continous on $[0, T]$ and that for all real $\rho > 0$, there exists a nondecreasing absolutely continuous mapping $v_\rho : [0, T] \to \mathbb{R}_+$ such that

$$\mathrm{exc}_\rho(K(s), K(t)) \leq v_\rho(t) - v_\rho(s) \quad \text{for all } 0 \leq s \leq t \leq T.$$

Then, **there exists one and only one** absolutely continuous mapping $u : [0, T] \to \mathscr{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Existence under truncated excess

Theorem (Thibault (2016))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is **nonempty closed convex valued**. Assume also that there exist a real $\rho_0 \geq \|u_0\|$, a real $\rho > \rho_0$ and some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}_+$ satisfying

$$\mathrm{exc}_\rho\big(C(s), C(t)\big) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

and such that for all $t_1 < \ldots < t_k$ in I

$$\big\|\big(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\big)(u_0)\big\| \leq \rho_0.$$

Existence under truncated excess

Theorem (Thibault (2016))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is **nonempty closed convex valued**. Assume also that there exist a real $\rho_0 \geq \|u_0\|$, a real $\rho > \rho_0$ and some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}_+$ satisfying

$$\operatorname{exc}_\rho\big(C(s), C(t)\big) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

and such that for all $t_1 < \ldots < t_k$ in I

$$\big\|\big(\operatorname{proj}_{C(t_k)} \circ \ldots \circ \operatorname{proj}_{C(t_1)}\big)(u_0)\big\| \leq \rho_0.$$

Then, **there exists one and only one** absolutely continuous mapping $u : [0, T] \to \mathscr{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Some consequences

If we assume one of the two following conditions, we can remove the assumption

$$\left\| \left(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\right)(a) \right\| \leq \rho_0.$$

Some consequences

If we assume one of the two following conditions, we can remove the assumption

$$\left\| \left(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\right)(a) \right\| \leq \rho_0.$$

- *I.* Time dependence on ρ_0:

$$\boxed{\rho_0 \geq \|u_0\| + v(T) - v(T_0).}$$

If we assume one of the two following conditions, we can remove the assumption

$$\left\| \left(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\right)(a) \right\| \leq \rho_0.$$

- *I.* Time dependence on ρ_0:

$$\boxed{\rho_0 \geq \|u_0\| + v(T) - v(T_0).}$$

- *II.* Bounded variation of projection mapping:

$$\exists a \in \mathscr{H}, W := \mathrm{var}(\mathrm{proj}_{C(\cdot)}(a); [0, T]) := \sup \sum_{i=1}^{n} \left\| \mathrm{proj}_{C(t_{i+1})}(a) - \mathrm{proj}_{C(t_i)}(a) \right\| < +\infty$$

and

$$\boxed{\rho_0 \geq \|u_0 - a\| + W + \sup_{t \in I} \left\| \mathrm{proj}_{C(t)}(a) \right\|.}$$

Half-space and hyperplane moving set

Let $\zeta : I \to \mathscr{H}$ and $\beta : I \to \mathbb{R}$ be **absolutely continuous** mappings on I. Set

$$C_1(t) := \{x \in \mathscr{H} : \langle \zeta(t), x \rangle = \beta(t)\} \quad \text{and} \quad C_2(t) := \{x \in \mathscr{H} : \langle \zeta(t), x \rangle \leq \beta(t)\}.$$

Assume the following normalization condition

$$\|\zeta(t)\| = 1 \quad \text{for all } t \in I = [0, T].$$

Half-space and hyperplane moving set

Let $\zeta : I \to \mathscr{H}$ and $\beta : I \to \mathbb{R}$ be **absolutely continuous** mappings on I. Set

$C_1(t) := \{x \in \mathscr{H} : \langle \zeta(t), x\rangle = \beta(t)\}$ and $C_2(t) := \{x \in \mathscr{H} : \langle \zeta(t), x\rangle \leq \beta(t)\}$.

Assume the following normalization condition

$$\|\zeta(t)\| = 1 \quad \text{for all } t \in I = [0, T].$$

Proposition

Let $i \in \{1,2\}$. The mapping $\mathrm{proj}(0, C_i(\cdot))$ is of **absolutely continuous on I**. Further, one has for every real $\rho > 0$,

$$\mathrm{exc}_\rho(C_i(s), C_i(t)) \leq v(t) - v(s) \quad \text{for all } s, t \in I \text{ with } s \leq t,$$

where $v(t) := \int_{T_0}^{t} \rho \left\|\dot{\zeta}(\tau)\right\| + \left|\dot{\beta}(\tau)\right| d\tau$ for every $t \in I$.

Some variants

Few words on second order theory

In order to obtain a trajectory $u(\cdot)$ satisfying

$$\begin{cases} -\ddot{u}(t) \in N\big(C(t, u(t)); \dot{u}(t)\big) \\ \dot{u}(t) \in C(t, u(t)) \\ u(0) = u_0, \dot{u}(0) = v_0 \end{cases}$$

S. Adly and B.K. Le (2016) required that

$$L(t, x, s, y) \leq L(|t - s| + \|x - y\|),$$

Few words on second order theory

In order to obtain a trajectory $u(\cdot)$ satisfying

$$\begin{cases} -\ddot{u}(t) \in N\big(C(t, u(t)); \dot{u}(t)\big) \\ \dot{u}(t) \in C(t, u(t)) \\ u(0) = u_0, \dot{u}(0) = v_0 \end{cases}$$

S. Adly and B.K. Le (2016) required that

$$L(t, x, s, y) \leq L(|t - s| + \|x - y\|),$$

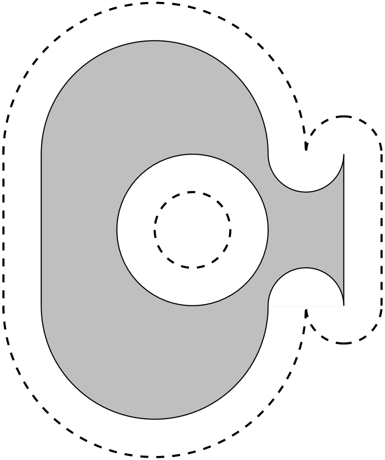
where

$$L(t, x, s, y) := \begin{cases} \mathrm{haus}_\rho(C(t, x), C(s, y)) & \text{if } C(t, x) \cap \rho\mathbb{B} \neq \emptyset, C(s, y) \cap \rho\mathbb{B} \neq \emptyset \\ \mathrm{exc}_\rho(C(t, x), C(s, y)) & \text{if } C(t, x) \cap \rho\mathbb{B} \neq \emptyset, C(s, y) \cap \rho\mathbb{B} = \emptyset \\ 0 & \text{if } C(t, x) \cap \rho\mathbb{B} = \emptyset, C(s, y) \cap \rho\mathbb{B} = \emptyset \end{cases}$$

with $\rho := 1 + \|u_0\| + \|v_0\| + LT + e^{(L+1)T}$.

Definition

Let S be a nonempty closed subset of $\mathscr{H}$, $r \in]0, +\infty]$. The set S is r-**prox-regular** if the mapping $\mathrm{proj}_S : U_r(S) := \{x \in \mathscr{H} : d_S(x) < r\} \to \mathscr{H}$ is well-defined and continuous.



Facts on prox-regularity

- S is convex $\Leftrightarrow$ S is ∞-prox-regular.
- S is r-prox-regular $\Rightarrow$ S is r'-prox-regular for every $0 < r' < r$
- S is r-prox-regular $\Leftrightarrow$ $d_S^2(\cdot)$ is $C^{1,1}$ on $U_r(S)$.
- S is r-prox-regular if and only if for all $x, x' \in S$ and $x^\star \in N(S;x)$ one has

$$\left\langle x^\star, x' - x \right\rangle \leq \frac{1}{2r} \|x^\star\| \left\|x' - x\right\|^2$$

- S is r-prox-regular $\Rightarrow$ S is tangentially and normally regular.

Existence under truncated excess

Theorem (N., Thibault (2018))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is r**-prox-regular** valued. Assume also that there exist a real $\rho_0 \geq \|u_0\|$, a real $\rho > \rho_0$ and some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}$ satisfying

$$\mathrm{haus}_\rho\big(C(s), C(t)\big) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

and such that for all $t_1 < \ldots < t_k$ in I

$$\left\|\left(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\right)(u_0)\right\| \leq \rho_0$$

whenever $\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}$ is **well-defined.**

Existence under truncated excess

Theorem (N., Thibault (2018))

Let $u_0 \in C(0)$. Assume that the multimapping $C(\cdot)$ is r**-prox-regular** valued. Assume also that there exist a real $\rho_0 \geq \|u_0\|$, a real $\rho > \rho_0$ and some nondecreasing absolutely continuous mapping $v : [0, T] \to \mathbb{R}$ satisfying

$$\mathrm{haus}_\rho\big(C(s), C(t)\big) \leq v(t) - v(s) \quad \text{for all } 0 \leq s \leq t \leq T,$$

and such that for all $t_1 < \ldots < t_k$ in I

$$\left\|\left(\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}\right)(u_0)\right\| \leq \rho_0$$

whenever $\mathrm{proj}_{C(t_k)} \circ \ldots \circ \mathrm{proj}_{C(t_1)}$ is **well-defined.**

Then, **there exists one and only one** absolutely continuous mapping $u : [0, T] \to \mathscr{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

State-dependent

Now, we focus on state-dependent sweeping process, i.e., we assume that the moving set depends on both time t and state x. The problem can be be written as

$$\begin{cases} -\dot{u}(t) \in N\big(C(t, u(t)); u(t)\big) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t, u(t)) & \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

To construct a solution, we consider the following implicit scheme

$$x_i^n = \operatorname{proj}(x_{i-1}^n, C(t_i^n, x_i^n)).$$

The well-posedness is based on the existence for each y of a fixed point x_y for

$$x \mapsto \operatorname{proj}(y, C(t, x))$$

along with an uniform upper bound

$$\|x_y - y\| \le M.$$

Nonconvex, state-dependent

Theorem (N. (2018))

Let $C : I \times \mathscr{H} \rightrightarrows \mathscr{H}$ be a multimapping with r**-prox-regular values** for some $r \in]0, +\infty]$, $u_0 \in \mathscr{H}$ with $u_0 \in C(0, u_0)$, $\rho_0 \in] \left\| u_0 \right\|, +\infty[$. Assume that:

Nonconvex, state-dependent

Theorem (N. (2018))

Let $C : I \times \mathscr{H} \rightrightarrows \mathscr{H}$ be a multimapping with ***r*-prox-regular values** for some $r \in]0,+\infty]$, $u_0 \in \mathscr{H}$ with $u_0 \in C(0,u_0)$, $\rho_0 \in]\left\|u_0\right\|,+\infty[$. Assume that:

(i) there exist a real $L_1 \geq 0$, a real $L_2 \in [0,1[$ and an extended real $\rho \geq \rho_0 + L_1 T(1-L_2)^{-1} + r$ such that

$$\mathrm{haus}_\rho(C(t,x),C(\tau,y)) \leq L_1 |t-\tau| + L_2 \|x-y\| \quad \text{for all } t,\tau \in I, x,y \in \mathscr{H};$$

Nonconvex, state-dependent

Theorem (N. (2018))

Let $C : I \times \mathscr{H} \rightrightarrows \mathscr{H}$ be a multimapping with ***r*-prox-regular values** for some $r \in]0,+\infty]$, $u_0 \in \mathscr{H}$ with $u_0 \in C(0,u_0)$, $\rho_0 \in]\,\|u_0\|\,,+\infty[$. Assume that:

(i) there exist a real $L_1 \geq 0$, a real $L_2 \in [0,1[$ and an extended real $\rho \geq \rho_0 + L_1 T(1-L_2)^{-1} + r$ such that

$$\mathrm{haus}_\rho(C(t,x),C(\tau,y)) \leq L_1 |t-\tau| + L_2 \|x-y\| \quad \text{for all } t,\tau \in I, x,y \in \mathscr{H};$$

(ii) there exists a real $\delta > \|u_0\| + L_1 T(1-L_2)^{-1}$ such that for every bounded subset B of $\mathscr{H}$ with $\gamma(B) > 0$,

$$\gamma(C(t,B)) \cap \delta\mathbb{B}) < \gamma(B).$$

Nonconvex, state-dependent

Theorem (N. (2018))

Let $C : I \times \mathscr{H} \rightrightarrows \mathscr{H}$ be a multimapping with r**-prox-regular values** for some $r \in]0,+\infty]$, $u_0 \in \mathscr{H}$ with $u_0 \in C(0,u_0)$, $\rho_0 \in]\,\|u_0\|\,,+\infty[$. Assume that:

(i) there exist a real $L_1 \geq 0$, a real $L_2 \in [0,1[$ and an extended real $\rho \geq \rho_0 + L_1 T(1-L_2)^{-1} + r$ such that

$$\mathrm{haus}_\rho(C(t,x), C(\tau,y)) \leq L_1\,|t-\tau| + L_2\,\|x-y\| \quad \text{for all } t,\tau \in I, x,y \in \mathscr{H};$$

(ii) there exists a real $\delta > \|u_0\| + L_1 T(1-L_2)^{-1}$ such that for every bounded subset B of $\mathscr{H}$ with $\gamma(B) > 0$,

$$\gamma(C(t,B)) \cap \delta\mathbb{B}) < \gamma(B).$$

Then, **there exists** a Lipschitz continuous mapping $u : I \to \mathscr{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t,u(t)); u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t,u(t)) & \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

- S. ADLY, B.K. LE, *Unbounded second-order state-dependent Moreau's sweeping Processes in Hilbert spaces*, J. Optim. Theory Appl. 169 (2016), 407-423.
- G. COLOMBO, R. HENRION, N.D. HOANG, B.S. MORDUKHOVICH, *Discrete approximations of a controlled sweeping process*, Set-Valued Var. Anal. 23 (2015), 69âĂŞ86.
- J.J. MOREAU, *Rafle par un convexe variable I*, Travaux Sém. Anal. Convexe Montpellier, 1971.
- F. NACRY, *Truncated nonconvex state-dependent sweeping process: implicit and semi-implicit adapted Moreau's catching-up algorithms*, J. Fixed Point Theory Appl. 20 (2018)
- F. NACRY, L. THIBAULT, *BV prox-regular sweeping process with bounded truncated variation*, Optimization, doi.org/10.1080/02331934.2018.1514039

Thank you for your attention ! Any questions ?