REGULARIZATION OF SWEEPING PROCESS: OLD AND NEW

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Abstract. The paper surveys in great generality several fundamental contributions in the literature on regularization of sweeping processes under the control of the moving set via the Hausdorff-Pompeiu distance. In addition, a large complete new study is provided for the regularization of prox-regular sweeping processes in the significantly weaker situation when merely a suitable truncated Hausdorff distance is involved for the control of the moving set.

Dedicated to Boris Mordukhovich
on the occasion of his seventieth birthday

Contents

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1. Introduction

Sweeping process has been introduced by J.J. Moreau in his 1971 paper [36] as the mathematical model of various elastoplastic mechanical systems. Its formulation is the differential inclusion

$$\begin{cases} -\frac{du}{dt}(t) \in \mathcal{N}(C(t); u(t)) \\ u(T_0) = a \in C(T_0), \end{cases} \tag{1.1}$$

where $C(t)$ is a nonempty closed set in a Hilbert space $\mathcal{H}$ for every $t \in I := [T_0, T]$, and $\mathcal{N}(C(t); x)$ is the normal cone (in a given sense) to $C(t)$ at $x \in \mathcal{H}$, with $\mathcal{N}(C(t); x) = \emptyset$ whenever $x \in \mathcal{H} \setminus C(t)$. The name "sweeping process" coined by J.J. Moreau came from the understanding that the point $u(t)$ is swept by the moving set $C(t)$. The situation when external forces are present as well as certain other models (in economy, electrical circuit, etc.) require the addition of a mapping depending on the state to the second member of (1.1). So, the new differential inclusion (which is an *extended sweeping process*) takes the form

$$\begin{cases} -\frac{du}{dt}(t) \in \mathcal{N}(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = a \in C(T_0), \end{cases} \tag{1.2}$$

where $f : I \times \mathcal{H} \to \mathcal{H}$ is a mapping Lebesgue measurable with respect to the time-variable and Lipschitz with respect to the state-variable. Both differential inclusions (1.1) and (1.2) are clearly differential inclusions with constraints, say $u(t) \in C(t)$ for all $t \in I$. The concept of solution of either (1.1) or (1.2) generally depends on the way that the set $C(t)$ moves with respect to time $t \in I$. Among those concepts, "absolutely continuous solution" is one at the heart of the theory. A mapping $u : I \to \mathcal{H}$ is called an *absolutely continuous solution* of (1.2) provided that $u(\cdot)$ is (of course) absolutely continuous on I with $u(t) \in C(t)$ for all $t \in I$ and the inclusion

$$-\frac{du}{dt}(t) \in \mathcal{N}(C(t); u(t)) + f(t, u(t))$$

is satisfied for Lebesgue almost every $t \in I$. In fact, it should be noted that the latter inclusion property for Lebesgue almost every $t \in I$ plus the continuity of $t \mapsto d(x, C(t))$ (for each $x \in \mathcal{H}$) entails the aforementioned required inclusion $u(t) \in C(t)$ for all $t \in I$. Indeed, fixing any $t \in I$ and taking a sequence $(t_n)_n$ in I tending to t with $u(t_n) \in C(t_n)$ for all integers n, we see from the inequality

$$|d(u(t_n), C(t_n)) - d(u(t), C(t))| \le \|u(t_n) - u(t)\| + |d(u(t), C(t_n)) - d(u(t), C(t))|$$

that $d(u(t_n), C(t_n)) \to d(u(t), C(t))$, hence $d(u(t), C(t)) = 0$, or equivalently $u(t) \in C(t)$. For the notion of *solution with bounded variation* we refer to Section 4.

Actually, diverse approaches for existence of solutions of (1.1) and (1.2) are available in the literature: Catching-up method (see, e.g., [39]), regularization procedure (see, e.g., [36]), reduction to unconstrained differential inclusion (see, e.g., [47]), etc. Putting $\varphi_t(x) := (1/2)d^2_{C(t)}(x)$ for all $(t,x) \in I \times \mathcal{H}$, the function $\varphi_t(\cdot)$ is of class $C^{1,1}$ on $\mathcal{H}$ (resp. on $\{x \in \mathcal{H} : d_{C(t)}(x) < r\}$) if $C(t)$ is convex (resp. if $C(t)$ is r-prox-regular, where $r \in]0,+\infty]$; see Subsection 2.4 for the definition). For each real $\lambda > 0$, the classical differential equation (under Lebesgue measurability in time and appropriate growth condition)

$$\begin{cases} \frac{du_\lambda}{dt}(t) = -\frac{1}{\lambda}\nabla\varphi_t(u_\lambda(t)) - f(t,u_\lambda(t)) \\ u_\lambda(T_0) = a \end{cases}$$

admits a unique solution $u_\lambda(\cdot)$. The *regularization procedure* then consists in showing (when possible) that $(u_\lambda(\cdot))_{\lambda>0}$ converges in a certain sense as $\lambda \downarrow 0$ to a mapping $U(\cdot)$ on an interval $J := [T_0, \overline{T}] \subset I$, and that $U(\cdot)$ or another mapping $u(\cdot)$ easily related to $U(\cdot)$ is a solution on J of the extended sweeping process. Error estimates are also generally investigated. Even when the existence of a solution is obtained via any approach, it is of great interest in theoretical and practical/numerical point of view to know whether this solution is a certain limit of suitable regularized classical differential equations associated to either (1.1) or (1.2). This clearly offers the privilege to take advantage of the very large knowledge of theoretical and practical/numerical features in the literature on classical differential equations.

The aim of the present paper is twofold. On the one hand we survey in great generality several developments in the literature on regularization of sweeping processes under the control of the moving set $C(t)$ via the Hausdorff-Pompeiu distance, and on the other hand we provide a complete new study of the regularization in the significantly weaker situation when merely a suitable truncated Hausdorff distance (see Subsection 2.3 for definition) is involved for the control of the moving set $C(t)$. Section 2 contains all the preliminaries on variational notions as: normal cones of sets and subdifferentials of functions, variation and variation along ρ-truncation of multimappings, bounded variation along ρ-truncation and connections with measure theory, prox-regularity of sets. Section 3 reviews Moreau's basic result in [36] on regularization of (1.1) under the convexity of the sets $C(t)$ and the absolute continuity of the multimapping $C(\cdot)$ with respect to the Hausdorff-Pompeiu distance. The deep works [30, 31] by M.D.P. Monteiro Marques, concerning the case when the sets $C(t)$ are convex and ball-compact but the multimapping $C(\cdot)$ is merely of bounded variation, are analyzed and developed in Section 4. Concerning the situation of nonconvex sets $C(t)$, we begin by recalling in Section 5 the results of L. Thibault ([48]) for (1.1) and M. Sene and L. Thibault ([46]) for (1.2), both dealing with the context where the sets $C(t)$ are r-prox-regular and the multimapping $C(\cdot)$ is still absolutely continuous with respect to the Hausdorff-Pompeiu distance. The long Section 6, which is completely new, studies the regularization procedure of (1.2) with r-prox-regular sets $C(t)$ under the significantly weaker assumption of absolute continuity of $C(\cdot)$ with respect to the more flexible ρ-truncated Hausdorff distance for a suitable real $\rho > 0$. The paper finishes with the recent nice regularization of A. Jourani and E. Vilches ([28])

for (1.2) with ball-compact α-far sets $C(t)$ (see the definition in the same section).

2. Preliminaries

Throughout the paper, $I := [T_0, T]$ is an interval of $\mathbb{R}$ with $T_0 < T$ and $\mathcal{H}$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. The closed (resp. open) ball of $\mathcal{H}$ centered at $x \in \mathcal{H}$ of radius $r \in]0, +\infty]$ is denoted by $B[x, r]$ (resp. $B(x, r)$), and we will use the notation $\mathbb{B}$ for the closed unit ball centered at zero, that is, $\mathbb{B} := B[0, 1]$. By convention, we will set $r\mathbb{B} = \mathcal{H}$ when $r = +\infty$. Let S be a subset of $\mathcal{H}$. As usual, d_S (or $d(\cdot, S)$) is the distance function from S, i.e.,

$$d_S(x) :=: d(x, S) := \inf_{y \in S} \|x - y\| \quad \text{for all } x \in \mathcal{H},$$

and the convex hull (resp. closed convex hull) of S is denoted by $\operatorname{co} S$ (resp. $\overline{\operatorname{co}}\, S$). The support function of S is the function from $\mathcal{H}$ into $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ given by

$$\sigma_S(h) :=: \sigma(h; S) := \sup_{x \in S} \langle x, h \rangle \quad \text{for all } h \in \mathcal{H}.$$

The multimapping (which is obviously graph-closed in $\mathcal{H}^2$ endowed with the usual Hilbert norm) $\operatorname{Proj}_S : \mathcal{H} \rightrightarrows \mathcal{H}$ of nearest points in S is defined by

$$\operatorname{Proj}_S(x) := \{y \in S : \|x - y\| = d_S(x)\} \quad \text{for all } x \in \mathcal{H}.$$

Whenever the latter set is reduced to a singleton for some $\overline{x} \in \mathcal{H}$, that is $\operatorname{Proj}_S(\overline{x}) = \{\overline{y}\}$, the vector $\overline{y} \in S$ is denoted by $\operatorname{proj}_S(\overline{x})$ or $P_S(\overline{x})$. One says that S is (strongly) ball-compact if the intersection of S with any closed ball of $\mathcal{H}$ is compact. In such a case, the set S is obviously closed along with $\operatorname{Proj}_S(x) \neq \emptyset$ for every $x \in \mathcal{H}$ whenever $S \neq \emptyset$.

Given an extended real-valued function $f : \mathcal{H} \to \overline{\mathbb{R}}$, the effective domain of f is denoted by

$$\operatorname{dom} f := \{x \in \mathcal{H} : f(x) < +\infty\}.$$

2.1. **Proximal and Mordukhovich normal cones.** Let S be a closed subset of $\mathcal{H}$. A vector $\zeta \in \mathcal{H}$ is a *proximal normal vector to the set* $S \subset \mathcal{H}$ *at a point* $x \in S$ provided that there exists a real $r > 0$ such that $x \in \operatorname{Proj}_S(x + r\zeta)$. The set $N^P(S; x)$ (which is a convex cone containing zero but not necessarily closed) of all proximal normal vectors of S at x is called the proximal normal cone of S at x. By convention, if $x \in \mathcal{H} \setminus S$, we will put $N^P(S; x) := \emptyset$. It is worth pointing out that for each $v \in \mathcal{H}$ with $\operatorname{Proj}_S(v) \neq \emptyset$,

$$v - w \in N^P(S; w) \quad \text{for all } w \in \operatorname{Proj}_S(v). \tag{2.1}$$

Following [33], a vector $\zeta \in \mathcal{H}$ belongs to the *Mordukhovich limiting* (or *basic*) normal cone $N^L(S; x)$ to the closed set S at $x \in S$ provided that there are a sequence $(x_n)_{n \in \mathbb{N}}$ in S with $x_n \to x$ and a sequence $(\zeta_n)_{n \in \mathbb{N}}$ in

$\mathcal{H}$ with $\zeta_n \overset{w}{\to} \zeta$ such that $\zeta_n \in N^P(S; x_n)$ for every $n \in \mathbb{N}$. Here and below, the letter $\mathbb{N}$ denotes the set of integers starting from 1 and $\overset{w}{\to}$ stands for the weak convergence in $\mathcal{H}$.

The closed convex hull of $N^L(S;x)$ (see [34]) is known to coincide with the Clarke normal cone $N(S;x)$, that is,

$$N(S;x) = \overline{\mathrm{co}}\big(N^L(S;x)\big).$$

Of course, $N^L(S;x)$ and $N(S;x)$ are empty whenever $x \in \mathcal{H} \setminus S$. It will be convenient sometimes, to write $N^P_S(x)$ for the proximal, $N^L_S(x)$ for the Mordukhovich limiting and $N_S(x)$ for the Clarke normal cone to S at x.

It is worth pointing out that the above concepts of normal cones are local, in the sense that for any neighborhood V in $\mathcal{H}$ of $x \in S$

$$\mathcal{N}(S \cap V; x) = \mathcal{N}(S; x), \tag{2.2}$$

where $\mathcal{N}$ stands for any of N^P, N^L and N. If S is convex, it is known (and easily seen) that the three normal cones $N^P(S;x)$, $N^L(S;x)$ and $N(S;x)$ coincide with the normal cone in the sense of convex analysis, that is,

$$N^P(S;x) = N^L(S;x) = N(S;x) = \{\zeta \in \mathcal{H} : \langle \zeta, y - x\rangle \le 0, \forall y \in S\}. \tag{2.3}$$

Given a lower semicontinuous function $f : \mathcal{H} \to \overline{\mathbb{R}}$ and $x \in \mathcal{H}$ with $|f(x)| < +\infty$, the proximal $\partial_P f(x)$, Mordukhovich limiting $\partial_L f(x)$ and Clarke $\partial f(x)$ subdifferential of f at x is defined as the set of $\zeta \in \mathcal{H}$ such that $(\zeta, -1)$ lies in the proximal, Mordukhovich limiting, and Clarke normal cone respectively of the epigraph of f at $(x, f(x))$. By convention, these subdifferentials at x are empty whenever $|f(x)| = +\infty$. Clearly, $\partial_P f(x) \subset \partial_L f(x) \subset \partial f(x)$. The subdifferential $\partial_L f(x)$ can also be derived more directly from the proximal subdifferential: A vector $\zeta \in \partial_L f(x)$ if and only if there are sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $(x_n, f(x_n)) \to (x, f(x))$ and $(\zeta_n)_{n\in\mathbb{N}}$ converging weakly to ζ such that $\zeta_n \in \partial_P f(x_n)$ for all $n \in \mathbb{N}$.

It is known that, for the closed set $S \subset \mathcal{H}$ and $x \in S$

$$\partial_P d_S(x) = \mathbb{B} \cap N^P_S(x),\ \mathbb{R}_+ \partial_L d_S(x) = N^L_S(x),\ \mathrm{cl}\big(\mathbb{R}_+ \partial d_S(x)\big) = N_S(x);$$

see [34] for the middle equality and [16] for the others. When $x \notin S$, the equality

$$\partial_P d_S(x) = N^P(S_r; x) \cap \{\zeta \in \mathcal{H} : \|\zeta\| = 1\} \tag{2.4}$$

holds for the closed set S (see [9]), where $r := d(x, S)$ and S_r denotes the *closed r-enlargement of S*, that is, $S_r := \{x \in \mathcal{H} : d_S(x) \le r\}$; such a property is not true in general for $\partial_L d_S(x)$ and $\partial d_S(x)$.

When f is Lipschitz near $x \in \mathcal{H}$ (in particular finite at x), the Clarke subdifferential $\partial f(x)$ can be described analytically by

$$\partial f(x) = \{\zeta \in \mathcal{H} : \langle \zeta, h\rangle \le f^o(x; h),\ \forall h \in \mathcal{H}\},$$

where $f^o(x;h) := \limsup\limits_{t\downarrow 0, u\to x} t^{-1}[f(u + th) - f(u)]$, so $\partial f(x)$ is a nonempty weakly compact convex set in $\mathcal{H}$ and the continuous sublinear function

$f^o(x;\cdot)$ is its support function, that is,

$$f^o(x;h) = \sigma(h;\partial f(x)) \quad \text{for all } h \in \mathcal{H}.$$

From this, it is easily seen for the local Lipschitz function f near x that on the one hand

$$\partial(-f)(x) = -\partial f(x),$$

and on the other hand that the multimapping $x' \mapsto \partial f(x')$ is scalarly upper semicontinuous at x, in the sense that $x' \mapsto \sigma(h;\partial f(x'))$ is upper $\|\cdot\|$-semicontinuous at x, which is also equivalent to the property that $x' \mapsto \partial f(x')$ is upper semicontinuous at x from $(\mathcal{H},\|\cdot\|)$ into $(\mathcal{H}, w(\mathcal{H},\mathcal{H}))$ in the sense of set-valued analysis, where as usual $w(\mathcal{H},\mathcal{H})$ denotes the weak topology on $\mathcal{H}$. Under the Lipschitz property of f near x one also has (see [34, Theorem 3.57])

$$\partial f(x) = \overline{\mathrm{co}}\left(\partial_L f(x)\right). \tag{2.5}$$

For the function f Lipschitz near x, if $f^o(x;\cdot)$ coincides with the usual directional derivative $f'(x;\cdot)$, one says that the function f is *tangentially regular at* x. We recall that the directional derivative $f'(x;\cdot)$, when it exists, is given by

$$f'(x;h) := \lim_{t\downarrow 0}[f(x+th) - f(x)] \quad \text{for all } h \in \mathcal{H}.$$

For the closed subset S of $\mathcal{H}$, we know by J.M. Borwein, S. Fitzpatrick and J. Giles [7, Theorem 8] that the function $-d_S$ is tangentially regular on $\mathcal{H}\setminus S$. From this, it is not difficult to derive that the function $-d_S^2$ is tangentially regular on $\mathcal{H}$. This can also be seen from the fact (see, [26, Theorem 3.2]) that the opposite of the Moreau envelope $\varphi\Box\frac{1}{2}\|\cdot\|^2$ of any proper lower semicontinuous function $\varphi : \mathcal{H} \to \mathbb{R}\cup\{+\infty\}$ bounded from below is tangentially regular on $\mathcal{H}$ (i.e., at each point in $\mathcal{H}$), where

$$(\varphi\Box\frac{1}{2}\|\cdot\|^2)(x) := \inf_{y\in\mathcal{H}}[\varphi(y) + \frac{1}{2}\|x-y\|^2] \quad \text{for all } x \in \mathcal{H}.$$

Indeed, taking φ as the indicator function ψ_S of the closed set S, defined by $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ otherwise, gives the tangential regularity of $-d_S^2$. We state this property in the following proposition.

Proposition 2.1. *For any nonempty closed set S in the Hilbert space $\mathcal{H}$, the function $-d_S^2$ is tangentially regular on $\mathcal{H}$.*

Let S be a nonempty closed set of the Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}\setminus S$. One knows (see, e.g., [17, Lemma 5]) that, for any $x' \in \mathcal{H}\setminus S$ with $\partial_P d_S(x') \neq \emptyset$, the unique nearest point $P_S(x')$ exists and $\partial_P((1/2)d_S^2)(x') = \{x' - P_S(x')\}$.

Let $\zeta \in \partial_L((1/2)d_S^2)(x)$. Taking any sequences $(\zeta_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $\zeta_n \in \partial_P((1/2)d_S^2)(x_n)$ for every $n \in \mathbb{N}$ and $x_n \in \mathcal{H}\setminus S$ along with $x_n \to x$ and $\zeta_n \xrightarrow{w} \zeta$, we have $\zeta_n = x_n - P_S(x_n)$ for all $n \in \mathbb{N}$. So, $(P_S(x_n))_{n\in\mathbb{N}}$ converges weakly to $x - \zeta$. Now assume in addition that the nonempty set S is strongly ball-compact. Then we have $\|P_S(x_n) - (x-\zeta)\| \to 0$ according to the fact that the topology induced on the compact set $S \cap r\mathbb{B}$ by the weak and strong topologies coincide, where $r := \sup_{n\in\mathbb{N}}(\|x_n\| + d_S(x_n))$. The closedness (for $\mathcal{H} \times \mathcal{H}$ endowed with its natural norm) of the graph of the multimapping Proj_S entails that $x - \zeta \in \mathrm{Proj}_S(x)$. From this we see that $\partial_L((1/2)d_S^2)(x) \subset x - \mathrm{Proj}_S(x)$.

Now, let any $u \in \mathcal{H}$ where $P_S(u) =: v$ exists, and let us still keep S as strongly ball-compact. Fixing any $h \in \mathcal{H}$ we note that for every real $t > 0$

$$\frac{1}{2t}[d_S^2(u+th) - d_S^2(u)] \le \frac{1}{2t}[\|u+th-v\|^2 - \|u-v\|^2] = \langle u-v, h\rangle + \frac{t}{2}\|h\|^2.$$

Further, for each real $t > 0$ choosing $v_t \in \mathrm{Proj}_S(u+th)$, it is easily seen from the ball-compactness of S that $v_t \to v$ as $t \downarrow 0$, and we also observe that for every real $t > 0$,

$$\begin{aligned}\frac{1}{2t}[d_S^2(u+th) - d_S^2(u)] &= \frac{1}{2t}[\|u+th-v_t\|^2 - d_S^2(u)]\\ &= \langle u-v_t, h\rangle + \frac{t}{2}\|h\|^2 + \frac{1}{2t}[\|u-v_t\|^2 - d_S^2(u)]\\ &\ge \langle u-v_t, h\rangle + \frac{t}{2}\|h\|^2.\end{aligned}$$

It results that $(1/2)d_S^2$ is Gâteaux differentiable at u with $u - v = u - P_S(u)$ as Gâteaux gradient at u. Then, considering any $y \in \mathrm{Proj}_S(x)$ and noting (as in [45, Example 8.53]), for every $t \in]0,1[$ that $y = P_S(x_t)$, where $x_t := x + t(y-x) \notin S$, we obtain that $(1/2)d_S^2$ is Gâteaux differentiable at x_t, with $x_t - y$ as its Gâteaux gradient at x_t. Since $x_t \to x$ as $t \downarrow 0$, it ensues that $x - y \in \partial_L((1/2)d_S^2)(x)$ (see, e.g., (3.62) in [34, Theorem 3.59], keeping in mind that $\mathcal{H}$ is a Hilbert space). This yields $x - \mathrm{Proj}_S(x) \subset \partial((1/2)d_S^2)(x)$.

Taking also (2.5) into account, we have then proved the following:

Proposition 2.2. *Let S be a nonempty strongly ball-compact set of the Hilbert space $\mathcal{H}$, which holds in particular whenever $\mathcal{H}$ is finite dimensional and S is nonempty and closed. For any $x \in \mathcal{H} \setminus S$ one has*

$$\partial_L((1/2)d_S^2)(x) = x - \mathrm{Proj}_S(x) \quad \textit{and} \quad \partial((1/2)d_S^2)(x) = x - \overline{\mathrm{co}}\,(\mathrm{Proj}_S(x))\,.$$

We also mention [33, Proposition 2.7] and [45, Example 8.53] for the finite-dimensional version of Proposition 2.2.

For further properties of the above normal cones and subdifferentials, we refer the reader to the monographs [33, 34] for the proximal and limiting objects and to [16] for the proximal and Clarke ones.

2.2. **Vector measures.** In all the paper, any positive measure μ (resp. vector measure m with values in $\mathcal{H}$) on $I = [T_0, T]$ will be a Radon measure, in particular $\mu(I) < +\infty$. When a property holds on the complement of a Lebesgue negligible set of I, we merely say that it holds for almost every (a.e., for short) $t \in I$, in particular we do not specify the Lebesgue measure.

In order to deal with differential measure inclusions (namely, bounded variation sweeping process) some preliminaries on vector measure theory are necessary. Throughout this subsection, ν and $\hat{\nu}$ are positive Radon measures on $I = [T_0, T]$. For each $t \in I$, $r \in]0, +\infty[$, one sets

$$I(t,r) := I \cap [t-r, t+r]\,, I^+(t,r) := I \cap [t, t+r] \text{ and } I^-(t,r) := I \cap [t-r, t].$$

For a subset A of I, we denote by $\mathbf{1}_A$ the *characteristic function* (in the sense of measure theory) of A relative to I, i.e., for all $t \in I$, $\mathbf{1}_A(t) = 1$ if $t \in A$ and $\mathbf{1}_A(t) = 0$ otherwise. Let X be a Hilbert space and J be a subinterval of I. For any real $p \geq 1$, $L^p(J, X, \nu)$ stands for the real space of (classes of) Bochner ν-measurable mappings from J to X for which the p-th power of their norm value is ν-integrable on J. If ν is the Lebesgue measure on J, we denote $L^p(J, X)$ instead of $L^p(J, X, \nu)$ and if in addition $X = \mathbb{R}$, we merely write $L^p(J)$.

With the convention $\frac{0}{0} = 0$, the *derivative of the measure $\hat{\nu}$ with respect to ν* is defined as the following limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\hat{\nu}(I(t,r))}{\nu(I(t,r))} \tag{2.6}$$

which exists for ν-almost every $t \in I$. Further, it is worth mentioning that $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a nonnegative Borel function. Coming back to a general Radon measure $\hat{\nu}$ on I, it is known that the measure $\hat{\nu}$ is absolutely continuous with respect to ν if and only if $\hat{\nu} = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$ (i.e., $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a density relative to ν). If the latter equality holds, a mapping $u(\cdot) : I \to \mathcal{H}$ is $\hat{\nu}$-integrable on I if and only if $u(\cdot)\frac{d\hat{\nu}}{d\nu}(\cdot)$ is ν-integrable on I. In such a case, one has

$$\int_I u(t) d\hat{\nu}(t) = \int_I u(t) \frac{d\hat{\nu}}{d\nu}(t) d\nu(t).$$

If the two Radon measures ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other one, one says that ν and $\hat{\nu}$ are *absolutely continuously equivalent.*

Now, let us consider a Radon vector measure m on I with values in the real Hilbert space $\mathcal{H}$. The *variation measure* $|m|$ of m is defined for any Borel set $A \subset I$ by

$$|m|\,(A) := \sup_{(B_n)_{n \in \mathbb{N}} \in \mathcal{B}} \sum_{n=1}^{+\infty} \|m(B_n)\|\,,$$

where $\mathcal{B}$ is the set of all sequences $(B_n)_{n\in\mathbb{N}}$ of Borel mutually disjoint subsets of I such that $A = \bigcup\limits_{n \in \mathbb{N}} B_n$. The vector measure m is said to be *absolutely continuous with respect to ν* whenever the positive measure $|m|$ is absolutely continuous with respect to ν. Since $\mathcal{H}$ has the Radon-Nikodým property, under such an absolute continuity assumption, the vector measure m has a density $\zeta : I \to \mathcal{H}$ relative to ν, i.e., $m = \zeta(\cdot)\nu$ (or equivalently, $\zeta(\cdot) \in$

$L^1(I, \mathcal{H}, \nu)$ and for all Borel sets $A \subset I$,

$$m(A) = \int_A \zeta(t) d\nu(t)).$$

In the rest of this section, we focus on mappings with bounded variation. Let $u : I \to \mathcal{H}$ be a mapping. Any $\sigma = (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that $T_0 = t_0 < \ldots < t_k = T$ is called a *subdivision* σ of $[T_0, T] = I$ and to such a subdivision σ, one associates the real $S_\sigma := \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$. If $\mathcal{S}$ denotes the set of all subdivisions of I, one defines the *variation* of u as the extended real

$$\operatorname{var}(u; I) := \sup_{\sigma \in \mathcal{S}} S_\sigma.$$

The mapping u is said to be of *bounded variation on* I if $\operatorname{var}(u; I) < +\infty$. It is well-known that $u(\cdot)$ has one sided limits at each point of I whenever it is of bounded variation on I. In such a case, one defines the mappings $u^-, u^+ : I := [T_0, T] \to \mathcal{H}$ by

$$u^-(\tau) := \lim_{t \uparrow \tau} u(t) \text{ for all } t \in]T_0, T] \text{ and } u^+(\tau) := \lim_{t \downarrow \tau} u(t) \text{ for all } \tau \in [T_0, T[,$$

with the conventions $u^-(T_0) := u(T_0)$, $u^+(T) := u(T)$. The mapping u^+ (resp. u^-) is easily seen to be of bounded variation and right-continuous (resp. left-continuous) on I, and it is called the *right-continuous* (resp. *left-continuous*) *with bounded variation envelope of* u. Further, it is known (and not difficult to see) that

$$u^+(\tau) = (u^+)^+(\tau) \quad \text{and} \quad u^-(\tau) = (u^-)^-(\tau) \quad \text{for all } \tau \in [T_0, T]$$

and

$$(u^-)^+(\tau) = u^+(\tau) \ \forall \tau \in [T_0, T[, \text{ and } (u^+)^-(\tau) = u^-(\tau) \ \forall \tau \in]T_0, T]. \tag{2.7}$$

Assume that $u(\cdot)$ is of bounded variation on I. The *differential measure* du associated to $u(\cdot)$ (see, e.g., [22]) is defined through the equality

$$\int_{[s,t]} du = u^+(t) - u^-(t) \quad \text{for all } s, t \in I \text{ with } s \le t.$$

In particular, the following relation holds for every $\tau \in I$ and every $s, t \in I$ with $s \le t$,

$$\int_{\{\tau\}} du = u^+(\tau) - u^-(\tau) \quad \text{and} \quad \int_{]s,t]} du = u^+(t) - u^+(s).$$

For the differential measure $d(u^+)$, abbreviated as du^+ as usual, we can write (by what precedes) for all $s, t \in I$ with $s \le t$

$$\int_{[s,t]} du^+ = (u^+)^+(t) - (u^+)^-(s) = u^+(t) - (u^+)^-(s).$$

Using the aforementioned equality $(u^+)^-(s) = u^-(s)$ for all $s \in]T_0, T]$, we see that $\int_{[s,t]} du^+ = \int_{[s,t]} du$ for all $s, t \in I$ with $s \le t$, if and only if $(u^+)^-(T_0) =$

$u^-(T_0)$, or equivalently $u^+(T_0) = u(T_0)$. This and the similar property for $du^- := d(u^-)$ mean that

$$du = du^+ \ \Leftrightarrow\ u^+(T_0) = u(T_0), \quad \text{and} \quad du = du^- \ \Leftrightarrow\ u^-(T) = u(T). \tag{2.8}$$

On the other hand, if there is a ν-integrable mapping $\hat{u} : I \to \mathcal{H}$ on I satisfying

$$u(t) = u(T_0) + \int_{]T_0,t]} \hat{u}(t) d\nu(t) \quad \text{for all } t \in I,$$

then $u(\cdot)$ is of bounded variation and right continuous on I. In such a case, one has

$$|du|\,(]s,t]) = \int_{]s,t]} \|\hat{u}(\tau)\|\, d\nu(\tau) \quad \text{for all } s,t \in I \text{ with } s \le t$$

and du is absolutely continuous with respect to ν and has $\hat{u}(\cdot)$ as a density relative to ν, i.e.,

$$du = \hat{u}(\cdot) d\nu.$$

According to J.J. Moreau and M.Valadier ([41]), for ν-almost every $t \in I$, the following limits exists in $\mathcal{H}$,

$$\hat{u}(t) = \frac{du}{d\nu}(t) := \lim_{r\downarrow 0} \frac{du(I(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{du(I^+(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{du(I^-(t,r))}{\nu(I(t,r))}. \tag{2.9}$$

Consider again a mapping of bounded variation $u : I \to \mathcal{H}$, the Radon-Nikodym property of the Hilbert space $\mathcal{H}$ ensures that du has a density with respect to $|du|$, denoted as $\frac{du}{|du|}$, that is $du = \frac{du}{|du|}(\cdot)\,|du|$. A vector-valued mapping $\phi : I \to \mathcal{H}$ is known to be du-integrable on I if and only if it is Bochner $|du|$-integrable on I, and in this case its du-integral (relative to the inner product of $\mathcal{H}$) on a subinterval $J \subset I$, denoted as $\int_J \phi \cdot du$, is given by

$$\int_J \phi \cdot du = \int_J \langle \phi(t), \frac{du}{|du|}(t)\rangle\, |du|(t).$$

Concerning this integral, we have the following convergence result for which we refer to M.D.P. Monteiro Marques [31, Chapter 0, Theorem 2.1(ii-iii)]; in fact this follows from the proof of (ii) in Theorem 2.1 in [31, Chapter 0].

Proposition 2.3. *Let $I := [T_0, T]$ and $u, u_n : I \to \mathcal{H}$, $n \in \mathbb{N}$, be mappings of bounded variation into the Hilbert space $\mathcal{H}$ such that the sequence $(u_n)_{n\in\mathbb{N}}$ converges pointwise to u for $\mathcal{H}$ endowed with the weak topology, that is, for every $t \in I$*

$$u_n(t) \xrightarrow{w} u(t) \quad \text{as } n \to +\infty.$$

(a) If the mappings u and u_n, $n \in \mathbb{N}$, are left-continuous on I, then for any mapping $\phi : I \to \mathcal{H}$ which is either continuous on I, or right-continuous with bounded variation on I, one has for all $s,t \in I$ with $s < t$,

$$\int_{[s,t[} \phi \cdot du_n \to \int_{[s,t[} \phi \cdot du \quad \text{as } n \to +\infty.$$

(b) If the mappings u and u_n, $n \in \mathbb{N}$, are right-continuous on I, then for any mapping $\phi : I \to \mathcal{H}$ which is either continuous on I, or left-continuous with bounded variation on I, one has for all $s, t \in I$ with $s < t$,

$$\int_{]s,t]} \phi \cdot du_n \to \int_{]s,t]} \phi \cdot du \quad \text{as } n \to +\infty.$$

Vector-valued mappings with bounded variation enjoys the following Helly-type compactness property for which we refer to V. Barbu and T. Precupanu [6, Theorem 1.126] and to M.D.P. Monteiro Marques [31, Chapter 0, Theorem 2.1(i)].

Theorem 2.4. *Let $I := [T_0, T]$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of mappings of bounded variation from I into the Hilbert space $\mathcal{H}$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in I} \|u_n(t)\| < +\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \operatorname{var}(u_n; I) < +\infty.$$

Then, the sequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{s(n)})_{n\in\mathbb{N}}$ converging pointwise, for $\mathcal{H}$ endowed with the weak topology, to some mapping $u : I \to \mathcal{H}$ of bounded variation, that is, for every $t \in I$

$$u_{s(n)}(t) \overset{w}{\to} u(t) \quad \text{as } n \to +\infty,$$

and in addition $\operatorname{var}(u; I) \leq \sup_{n\in\mathbb{N}} \operatorname{var}(u_n; I)$.

When the mapping $u : I \to \mathcal{H}$ is absolutely continuous, denoting as usual by $\dot{u}$ its derivative defined Lebesgue almost everywhere and denoting by $\mathcal{L}$ the Lebesgue measure on I, we have

$$du = \frac{du}{d\mathcal{L}}(\cdot) d\mathcal{L} = \dot{u}(\cdot) d\mathcal{L}.$$

So, for the above integral with respect to the differential measure du we obtain for $\phi : I \to \mathcal{H}$ Bochner $\mathcal{L}$-integrable on I

$$\int_J \phi \cdot du = \int_J \langle \phi(t), \dot{u}(t) \rangle \, dt \tag{2.10}$$

for any subinterval $J \subset I$.

2.3. **Bounded variation along ρ-truncation.** Let $\rho \in]0, +\infty]$ be a given extended real and let S and S' be nonempty subsets of $\mathcal{H}$.

One defines the *ρ-pseudo excess of S over S'* (also called the *pseudo excess of the ρ-truncation of S over S'*) as the extended real

$$\operatorname{exc}_\rho(S, S') := \sup_{x \in S \cap \rho\mathbb{B}} d(x, S').$$

If $\rho = +\infty$, remembering the convention $\rho\mathbb{B} = \mathcal{H}$, we see in this case that the ρ-pseudo excess of S over S' is the usual *excess of S over S'*, that is,

$$\operatorname{exc}_\infty(S, S') = \sup_{x\in S} d(x, S') =: \operatorname{exc}(S, S').$$

It is readily seen that for every $x' \in \mathcal{H}$,

$$d(x', S') \le d(x', x) + \mathrm{exc}_\rho(S, S') \quad \text{for all } x \in S \cap \rho\mathbb{B},$$

i.e.,

$$d(x', S') \le d(x', S \cap \rho\mathbb{B}) + \mathrm{exc}_\rho(S, S') \quad \text{for all } x' \in \mathcal{H}.$$

With the above concept at hand, one can define the *Hausdorff ρ-pseudo distance between S and S'* as

$$\mathrm{haus}_\rho(S, S') := \max\left\{\mathrm{exc}_\rho(S, S'), \mathrm{exc}_\rho(S', S)\right\}.$$

If $\rho = +\infty$, $\mathrm{haus}_\rho(S, S')$ coincides with $\mathrm{haus}(S, S')$, the usual *Hausdorff-Pompeiu distance between S and S'*, i.e.,

$$\mathrm{haus}_\infty(S, S') = \max\left\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\right\} =: \mathrm{haus}(S, S').$$

It is worth pointing out that

$$\mathrm{haus}_\rho(S, S') \le \sup_{x \in \rho\mathbb{B}} \left|d(x, S) - d(x, S')\right| =: \widehat{\mathrm{haus}}_\rho(S, S'). \tag{2.11}$$

Further, for any extended real ρ' such that $\rho' \ge 2\rho + \max\{d_S(0), d_{S'}(0)\}$, one has

$$\widehat{\mathrm{haus}}_\rho(S, S') \le \mathrm{haus}_{\rho'}(S, S').$$

Before recalling the variations of multimappings, let us state the following lemma on the Hölder property with exponent $1/2$ of metric projections onto convex sets with respect to the Hausdorff-Pompeiu distance $\mathrm{haus}(\cdot,\cdot)$ over those sets. The statement is exactly the one by J.J. Moreau in [39, Inequality (2.17) in Lemma p. 362]. A previous result for the Hölder continuity with respect to $\mathrm{haus}(\cdot,\cdot)$ has been established by J. W. Daniel [20, Theorem 2.2]. The exponent in the result in [20] is also $1/2$ but the Hölder constant therein is less accurate than the one in [39]. The proof below follows the one by M.D.P. Monteiro Marques [31, Proposition 4.7].

Lemma 2.5. *Let S, S' be two nonempty closed convex subsets of $\mathcal{H}$ and $x, x' \in \mathcal{H}$. Then, one has*

$$\begin{aligned}\left\|\mathrm{proj}_S(x) - \mathrm{proj}_{S'}(x')\right\|^2 &\le \|x - x'\|^2 + 2d_S(x)\mathrm{exc}\,(S', S) + 2d_{S'}(x')\mathrm{exc}(S, S')\\ &\le \left\|x - x'\right\|^2 + 2\left(d_S(x) + d_{S'}(x')\right)\mathrm{haus}(S, S').\end{aligned}$$

Proof. The inequality valid for all $a, b \in \mathcal{H}$, $\|a\|^2 - \|b\|^2 \le 2\langle a, a - b\rangle$ entails, with $p := \mathrm{proj}_S(x)$ and $p' := \mathrm{proj}_{S'}(x')$, that

$$\begin{aligned}\|p - p'\|^2 - \|x - x'\|^2 &\le 2\langle p - p', p - p' - x + x'\rangle\\ &= 2\langle p' - p, x - p\rangle + 2\langle p - p', x' - p'\rangle.\end{aligned}$$

On the other hand, noting for $q := \mathrm{proj}_S(p')$ that $\langle q - p, x - p\rangle \le 0$ (since $p = \mathrm{proj}_S(x)$ and $q \in S$), we also have

$$\langle p' - p, x - p\rangle \le \langle p' - q, x - p\rangle \le \|x - p\|\, d_S(p') \le \|x - p\|\, \mathrm{exc}(S', S).$$

Interchanging, we also have $\langle p - p', x' - p'\rangle \le \|x' - p'\|\, \mathrm{exc}(S, S')$, which combined with what precedes finishes the proof. □

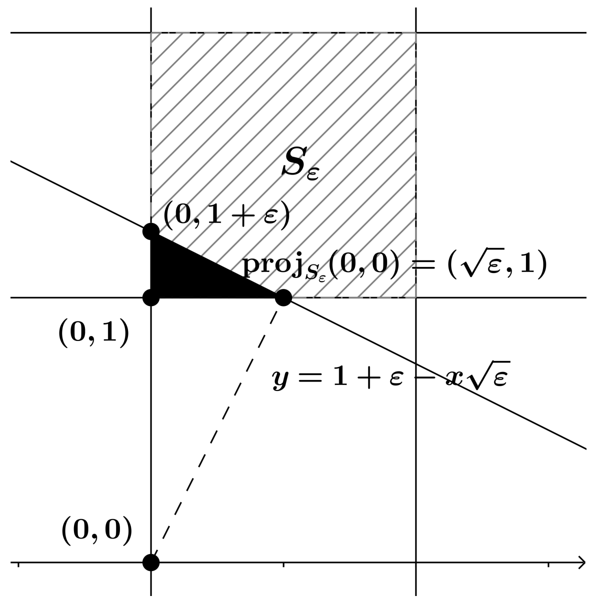

FIGURE 1. Daniel's example

The following example is due to J.W. Daniel [20, p. 235].

Example 2.6. The exponent 1/2 in the above Hölder property is sharp. Indeed, let $S := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 1 \le y \le 2\}$ and $S_\varepsilon := \{(x,y) \in S : x\sqrt{\varepsilon} + y \ge 1 + \varepsilon\}$ for $\varepsilon \in]0,1[$ (see Figure 1). It is easily seen that for any $\varepsilon \in]0,1[$, $\mathrm{proj}_S(0,0) = (0,1)$, $\mathrm{proj}_{S_\varepsilon}(0,0) = (\sqrt{\varepsilon}, 1)$, so $\|\mathrm{proj}_S(0,0) - \mathrm{proj}_{S_\varepsilon}(0,0)\| = \sqrt{\varepsilon}$, whereas $\mathrm{haus}(S_\varepsilon, S) = \varepsilon/\sqrt{1+\varepsilon}$. $\square$

Now, let us consider an extended real $\rho \in]0, +\infty]$ and a multimapping $C : I = [T_0, T] \rightrightarrows \mathcal{H}$. To each subdivision $\sigma_0 = (t_0, \dots, t_k)$ of I (with $k \in \mathbb{N}$), one associates the extended real

$$h_{\sigma_0,\rho} := \sum_{i=0}^{k-1} \mathrm{haus}_\rho(C(t_i), C(t_{i+1})).$$

The *ρ-pseudo variation* (or the *pseudo variation along ρ-truncation*) of $C(\cdot)$ on $I = [T_0, T]$ is defined as the extended real

$$\mathrm{var}_\rho(C; [T_0, T]) := \sup_{\sigma \in \mathcal{S}} h_{\sigma,\rho},$$

where $\mathcal{S}$ is the set of all subdivisions of I. When $\mathrm{var}_\rho(C; I) < +\infty$, one says that $C(\cdot)$ is of *pseudo bounded ρ-variation* (or *pseudo bounded variation along ρ-truncation*) on I. Assume for a moment that there is a positive Radon measure μ on I such that

$$\mathrm{haus}_\rho(C(s), C(t)) \le \mu(]s,t]) \quad \text{for all } s, t \in I. \tag{2.12}$$

It is then readily seen that $C(\cdot)$ is of pseudo bounded ρ-variation since

$$\mathrm{var}_\rho(C; I) \leq \mu(]T_0, T]) < +\infty.$$

Furthermore, for every $\bar{t} \in [T_0, T[$ the inequalities

$$0 \leq \mathrm{var}_\rho(C; [T_0, t]) - \mathrm{var}_\rho(C; [T_0, \bar{t}]) \leq \mu(]\bar{t}, t]) \quad \text{for all } t \in]\bar{t}, T]$$

say in particular that $t \mapsto \mathrm{var}_\rho(C; [T_0, t])$ is right-continuous at $\bar{t}$.

Conversely, assume that $C(\cdot)$ has a pseudo bounded variation on I along ρ-truncation and that the function $\mathrm{var}_\rho(C; [T_0, \cdot])$ is right-continuous on I. Since the latter function is nondecreasing on I, it is of bounded variation on I. So, if we denote by $\mu_{C,\rho}$ the differential Radon measure associated with it, we have

$$\mathrm{var}_\rho(C; [T_0, t]) - \mathrm{var}_\rho(C; [T_0, s]) = \mu_{C,\rho}(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t,$$

which entails in a straightforward way the inequality (2.12) with $\mu := \mu_{C,\rho}$.

2.4. **Prox-regular sets in Hilbert spaces.** In addition to the assumption (2.12), that is, the inequality

$$\mathrm{haus}_\rho(C(s), C(t)) \leq \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t,$$

for a given positive Radon measure μ on I and an extended real $\rho > 0$, the multimapping $C(\cdot)$ will be assumed to be uniformly prox-regular valued. Let us thus give the definition of prox-regular sets.

Definition 2.7. Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0, +\infty]$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P(S; x) \cap \mathbb{B}$ and for every real $t \in]0, r]$, one has $x \in \mathrm{Proj}_S(x + tv)$.

The following theorem recalls some useful characterizations and properties of uniform prox-regular sets (see, e.g., [42, 17]). Before stating it, define for any extended real $r > 0$, the *r-open enlargement* of a subset S of $\mathcal{H}$ as

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}.$$

Theorem 2.8. *Let S be a nonempty closed subset of $\mathcal{H}$. Consider the following assertions.*
(a) The set S is r-prox-regular.
(b) For all $x, x' \in S$, for all $v \in N(S; x)$, one has

$$\langle v, x' - x \rangle \leq \frac{1}{2r} \|v\| \|x - x'\|^2.$$

(c) For all $x, x' \in S$, for all $v \in N(S; x) \cap r\mathbb{B}$, for all $v' \in N(S; x') \cap r\mathbb{B}$,

$$\langle v' - v, x' - x \rangle \geq - \|x' - x\|^2.$$

(d) For any real $\gamma \in]0, 1[$, for all $x, x' \in U_{r\gamma}(S)$,

$$\|\mathrm{proj}_S(x) - \mathrm{proj}_S(x')\| \leq \frac{1}{1 - \gamma} \|x - x'\|.$$

(e) For all $u \in U_r(S) \setminus S$, one has (with $x = \mathrm{proj}_S(u)$)

$$x = \mathrm{proj}_S\Big(x + t\frac{u-x}{\|u-x\|}\Big) \quad \text{for all}\, t \in [0, r[.$$

(f) The function d_S^2 is $C^{1,1}$ on $U_r(S)$ and

$$\nabla d_S^2(x) = 2(x - \mathrm{proj}_S(x)) \quad \text{for all}\, x \in U_r(S).$$

(g) The set S is normally regular in the sense that

$$N^P(S;x) = N^L(S;x) = N(S;x) \quad \text{for all}\, x \in \mathcal{H};$$

further, $\partial_P d_S(x) = \partial_L d_S(x) = \partial d_S(x)$ for all $x \in U_r(S)$.

Then, the assertions (a), (b), (c), (d), (e) and (f) are pairwise equivalent and each one implies (g).

3. Regularization of convex sweeping process under absolute continuity

Let $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $f^* : \mathcal{H} \to \overline{\mathbb{R}}$ be its Legendre-Fenchel conjugate defined by $f^*(x^*) := \sup\limits_{x\in\mathcal{H}}[\langle x^*, x\rangle - f(x)]$ for all $x^* \in \mathcal{H}$. Since f is proper and lower semicontinuous, it is known that f^* takes its values in $\mathbb{R} \cup \{+\infty\}$ and is also proper and lower semicontinuous. For each $z \in \mathcal{H}$, the function $x \mapsto \Phi(x) := \frac{1}{2}\|z - x\|^2 + f(x)$ is weakly lower semicontinuous with $\Phi(x) \to +\infty$ as $\|x\| \to +\infty$, hence Φ attains its minimum. In fact, by strict convexity Φ posseses one and only one minimizer denoted by $\mathrm{prox}_f(z)$ and called by Moreau [38, 3.b Notation] the *proximal point of z relative to f*. Clearly, for a nonempty closed convex set S of $\mathcal{H}$ the proximal point of z relative to the indicator function ψ_S of S coincides with the projection of z onto S, that is, $\mathrm{prox}_{\psi_S}(z) = \mathrm{proj}_S(z)$. Given $x, y, z \in \mathcal{H}$ it is known by [38, Proposition 4.a] that

$$\big(x = \mathrm{prox}_f(z) \text{ and } y = \mathrm{prox}_{f^*}(z)\big) \;\Leftrightarrow\; \big(z = x + y \text{ and } f(x) + f^*(y) = \langle x, y\rangle\big).$$

By [38, Proposition 5.b] the mapping $\mathrm{prox}_f : \mathcal{H} \to \mathcal{H}$ is *nonexpansive* (like the mapping proj_S with S nonempty closed convex) in the sense that $\|\mathrm{prox}_f(z) - \mathrm{prox}_f(z')\| \le \|z - z'\|$ for all $z, z' \in \mathcal{H}$. We also note that for the function Φ above $\inf\limits_{x\in\mathcal{H}} \Phi(x) = -\Phi^*(0)$ (by the definition of Φ^*). Fix for a moment any $z \in \mathcal{H}$. Denoting $q := (1/2)\|\cdot\|^2$ and $q_z := q(\cdot - z)$, by the continuity of q_z we have $\Phi^* = (q_z^* \Box f^*)$ (see, e.g., [35]). Since $q_z^*(x^*) = q^*(x^*) + \langle x^*, z\rangle$ for every $x^* \in \mathcal{H}$ (as known and easily seen for the conjugate of $h(\cdot - z)$), we deduce

$$\Phi^*(0) = \inf_{x^*\in\mathcal{H}}[q_z^*(-x^*) + f^*(x^*)] = \inf_{x^*\in\mathcal{H}}[q^*(-x^*) - \langle x^*, z\rangle + f^*(x^*)].$$

Using the well-known equality $q^* = q$ (see [38, Proposition 9.a]), it ensues that

$$\left(\frac{1}{2}\|\cdot\|^2 \Box f\right)(z) = -\inf_{x^*\in\mathcal{H}}\left[\frac{1}{2}\|x^*\|^2 + f^*(x^*) - \langle x^*, z\rangle\right],$$

so in the particular case of the indicator function $f = \psi_S$ of a nonempty closed convex set S of $\mathcal{H}$ we obtain

$$\frac{1}{2}d_S^2(z) = -\inf_{x\in\mathcal{H}}\left[\frac{1}{2}\|x\|^2 + \sigma_S(x) - \langle z, x\rangle\right]. \tag{3.1}$$

Let be given a multimapping $C : [T_0, T] \rightrightarrows \mathcal{H}$ with nonempty closed convex values. Throughout this section, we assume that the multimapping C is absolutely continuous, that is, the variation function $I \ni t \mapsto \mathrm{var}(C; [T_0, t]) := \mathrm{var}_\infty(C; [T_0, t])$ is absolutely continuous on $I := [T_0, T]$. For convenience, we will denote $v(t) := \mathrm{var}(C; [T_0, t])$ for all $t \in I$. Consider the sweeping differential inclusion

$$\dot{u}(t) \in -N_{C(t)}(u(t)) \quad \text{with initial condition } u(T_0) = a,$$

where $\dot{u}(\cdot) := \frac{du}{dt}(\cdot)$. Given any pair of absolutely continuous solutions u, w, if any, by the monotonicity of $N_{C(t)}(\cdot)$, putting $\xi(t) := (1/2)\|u(t)-w(t)\|^2$ for all $t \in I$, we have $\dot{\xi}(t) = \langle \dot{u}(t) - \dot{w}(t), u(t) - w(t)\rangle \le 0$ for almost every $t \in I$. The function $\xi(\cdot)$ is then nonincreasing on $I = [T_0, T]$ with $\xi(T_0) = 0$ and $\xi(\cdot) \ge 0$, so $\xi(t) = 0$ for all $t \in I$, hence the sweeping differential inclusion with initial condition $u(T_0) = a$ has at most one absolutely continuous solution.

Now, let the function $\varphi : I \times \mathcal{H} \to \mathbb{R}$ be defined by

$$\varphi(t, x) := \frac{1}{2}d^2_{C(t)}(x) \quad \text{for all } (t, x) \in I \times \mathcal{H}.$$

In some places it will be convenient, for each $t \in I$, to denote as usual by φ_t the function $\varphi(t, \cdot)$, that is, $\varphi_t(x) := \varphi(t, x)$ for all $x \in \mathcal{H}$. For each $t \in I$, by [38, Proposition 7.d] (see also Theorem 2.8(f) with $r = +\infty$) the function $\varphi(t, \cdot)$ is of class C^1 on $\mathcal{H}$ with

$$\nabla\varphi_t(x) = x - \mathrm{proj}_{C(t)}(x) \quad \text{for all } x \in \mathcal{H}.$$

We also notice from the latter equality that for each $t \in I$ the mapping $x \mapsto \nabla\varphi_t(x)$ is Lipschitz on $\mathcal{H}$, and for each $x \in \mathcal{H}$ the mapping $t \mapsto \nabla\varphi_t(x)$ is continuous according to the Hölder property of $S \mapsto \mathrm{proj}_S(x)$ with respect to $\mathrm{haus}(\cdot, \cdot)$ on the space of nonempty closed convex subsets of $\mathcal{H}$ (see Lemma 2.5). Furthermore, for any absolutely continuous mapping $z : I \to \mathcal{H}$, writing

$$|d_{C(t)}(z(t)) - d_{C(s)}(z(s))| \le \|z(t) - z(s)\| + |v(t) - v(s)| \quad \text{for all } s, t \in I,$$

we see that $t \mapsto d_{C(t)}(z(t))$, and hence also $t \mapsto \varphi(t, z(t))$, is absolutely continuous on I.

Let S, S' be nonempty closed convex subsets of $\mathcal{H}$. It is known that

$$\mathrm{exc}(S, S') = \sup_{u\in\mathbb{B}\cap\mathrm{dom}\,\sigma_{S'}} \big(\sigma_S(u) - \sigma_{S'}(u)\big).$$

If $\mathrm{exc}(S, S') < +\infty$, it then follows that $\mathbb{B} \cap \mathrm{dom}\,\sigma_{S'} \subset \mathrm{dom}\,\sigma_S$, hence $\mathrm{dom}\,\sigma_{S'} \subset \mathrm{dom}\,\sigma_S$ because $\mathrm{dom}\,\sigma_S$ is a cone. Since $\mathrm{haus}(C(s), C(t)) < +\infty$

for any $s, t \in I$, by the absolute continuity of $C(\cdot)$, it results that

$$\operatorname{dom} \sigma_{C(s)} = \operatorname{dom} \sigma_{C(t)} =: D \quad \text{for all } s, t \in I. \tag{3.2}$$

We then see that for every $x \in D$ and any $s, t \in I$,

$$|\sigma_{C(t)}(x) - \sigma_{C(s)}(x)| \leq \|x\| \operatorname{haus}(C(s), C(t)) \leq \|x\| \, |v(t) - v(s)|.$$

This implies for each $x \in D$ that $t \mapsto \sigma_{C(t)}(x)$ is absolutely continuous and

$$\left| \frac{d}{dt} \sigma_{C(t)}(x) \right| \leq \|x\| \, \dot{v}(t) \quad \text{a.e. } t \in I. \tag{3.3}$$

The proofs of the next lemma and of Theorem 3.2 *follow the development of Moreau* [38].

Lemma 3.1. *For any absolutely continuous mapping $z : I \to \mathcal{H}$ the function $g : I \to \mathbb{R}$, defined by $g(t) := \varphi(t, z(t)) = (1/2)d^2_{C(t)}(z(t))$ for all $t \in I$ is absolutely continuous on I and*

$$|\dot{g}(t) - \langle \dot{z}(t), \nabla \varphi_t(z(t)) \rangle| \leq \dot{v}(t) \, d_{C(t)}(z(t)) \quad \textit{a.e. } t \in I.$$

Proof. By what precedes the lemma the function g is absolutely continuous on I. On the other hand, by (3.1) we can write for every $t \in I$,

$$\begin{aligned} \frac{1}{2} d^2_{C(t)}(z(t)) &= \left(\frac{1}{2} \| \cdot \|^2 \square \psi_{C(t)} \right)(z(t)) = -\inf_{x \in \mathcal{H}} \left[\frac{1}{2}\|x\|^2 + \sigma_{C(t)}(x) - \langle z(t), x \rangle \right] \\ &= \frac{1}{2}\|z(t)\|^2 - \inf_{x \in \mathcal{H}} \left[\frac{1}{2}\|z(t) - x\|^2 + \sigma_{C(t)}(x) \right], \end{aligned}$$

and by the definition of prox_f both latter infima are attained at the unique point

$$y(t) := \operatorname{prox}_{\sigma_{C(t)}}(z(t)) = z(t) - \operatorname{proj}_{C(t)}(z(t)) = \nabla \varphi_t(z(t)),$$

where the second equality is due to the equality valid for any $w \in \mathcal{H}$ $\operatorname{prox}_f(w) + \operatorname{prox}_{f^*}(w) = w$ recalled above. The mapping $y(\cdot)$ being continuous on I by what precedes the lemma, for each fixed $t \in I$ and any real $\varepsilon > 0$ there exists a real $\eta > 0$ such that for any reals $r, s \in I \cap [t - \eta, t + \eta]$ with $r \leq s$ and for $G(t, \varepsilon) := D \cap B(y(t), \varepsilon)$ (with D as in (3.2)) one has $y(r) \in G(t, \varepsilon)$ and $y(s) \in G(t, \varepsilon)$, hence in particular

$$\begin{aligned} g(s) &= -\inf_{x \in G(t,\varepsilon)} \left[\frac{1}{2}\|x\|^2 + \sigma_{C(s)}(x) - \langle z(s), x \rangle \right] \\ &= \sup_{x \in G(t,\varepsilon)} \left[-\frac{1}{2}\|x\|^2 - \sigma_{C(s)}(x) + \langle z(s), x \rangle \right] \end{aligned}$$

along with a similar equality for $g(r)$. We derive that for r, s as above

$$g(s) - g(r) \leq \sup_{x \in G(t,\varepsilon)} \left[\sigma_{C(r)}(x) - \sigma_{C(s)}(x) + \langle z(s) - z(r), x \rangle \right].$$

On the other hand, for each $x \in G(t,\varepsilon)$ we saw that $\tau \mapsto \sigma_{C(\tau)}(x)$ is absolutely continuous on I, so by (3.3) we have for r,s as above

$$\begin{aligned}\sigma_{C(r)}(x) - \sigma_{C(s)}(x) &= -\int_r^s \frac{d}{d\tau}\sigma_{C(\tau)}(x)\,d\tau \\ &\leq \|x\| \int_r^s \dot v(\tau)\,d\tau \leq (\|y(t)\|+\varepsilon)\int_r^s \dot v(\tau)\,d\tau\end{aligned}$$

along with $\langle z(s)-z(r), x-y(t)\rangle \leq \varepsilon \int_r^s \|\dot z(\tau)\|\,d\tau$. It follows that for r,s as above

$$\begin{aligned}\int_r^s [\dot g(\tau) - \langle \dot z(\tau), y(t)\rangle]\,d\tau &= g(s)-g(r)-\langle z(s)-z(r), y(t)\rangle \\ &\leq \|y(t)\| \int_r^s \dot v(\tau)\,d\tau + \varepsilon \int_r^s (\|\dot z(\tau)\| + \dot v(\tau))\,d\tau.\end{aligned}$$

This being true for every $\varepsilon > 0$ it readily ensues that for almost every $t \in I$

$$E(t) := \dot g(t) - \langle \dot z(t), y(t)\rangle \leq \|y(t)\|\,\dot v(t).$$

The above development with $s < r$ also gives for almost every $t \in I$ that $-E(t) \leq \|y(t)\|\,\dot v(t)$, thus $|E(t)| \leq \|y(t)\|\,\dot v(t)$ for almost every $t \in I$, which finishes the proof. □

Theorem 3.2 (Moreau). *Let $C : I = [T_0,T] \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed convex values and let $a \in C(T_0)$. Assume that the variation $t \mapsto v(t) := \mathrm{var}(C;[T_0,t])$ of C is absolutely continuous on I with $\dot v \in L^2(I)$. For each real $\lambda > 0$, let u_λ be the unique C^1-solution of the differential equation*

$$\dot u_\lambda(t) = -\nabla(\frac{1}{2\lambda} d^2_{C(t)})(u_\lambda(t)) \quad \textit{with initial condition } u_\lambda(T_0) = a.$$

Then, the family $(u_\lambda)_{\lambda>0}$ converges uniformly on I as $\lambda \downarrow 0$ to the absolutely continuous solution u of the sweeping differential inclusion

$$\dot u(t) \in -N_{C(t)}(u(t)) \quad \textit{with initial condition } u(T_0) = a.$$

Futher, the family of derivatives $(\dot u_\lambda)_{\lambda>0}$ converges strongly as $\lambda \downarrow 0$ to $\dot u$ in $L^2(I,\mathcal{H})$.

Proof. Fix for a moment any real $\lambda > 0$. Remembering the notation $\varphi(t,x) := (1/2)d^2_{C(t)}(x)$ and the equality $\nabla\varphi_t(x) = x - \mathrm{proj}_{C(t)}(x)$ for every $(t,x) \in I \times \mathcal{H}$, the mapping $t \mapsto \nabla\varphi_t(x)$ is continuous on I for each $x \in \mathcal{H}$, and for each $t \in I$ the function $\nabla\varphi_t(\cdot)$ is Lipschitz on $\mathcal{H}$. The differential equation in the statement then has a unique C^1 solution u_λ, so the equality $\dot u_\lambda(t) = -\lambda^{-1}\nabla\varphi_t(u_\lambda(t))$ holds for all $t \in I$. To simplify notation put $h_\lambda(t) := \|\nabla\varphi_t(u_\lambda(t))\| = [2\varphi(t,u_\lambda(t))]^{1/2}$ for all $t \in I$. By Lemma 3.1 the function $g_\lambda := (1/2)h_\lambda^2$ is absolutely continuous on I with

$$|\dot g_\lambda(t) - \langle \dot u_\lambda(t), \nabla\varphi_t(u_\lambda(t))\rangle| \leq h_\lambda(t)\,\dot v(t) \quad \text{a.e. } t \in I,$$

which (by the equality $h_\lambda(t) := \|\nabla\varphi_t(u_\lambda(t))\|$ valid for all $t \in I$) is equivalent to

$$|\dot g_\lambda(t) + \frac{1}{\lambda}(h_\lambda(t))^2| \le h_\lambda(t)\,\dot v(t) \quad \text{a.e. } t \in I.$$

Since $g_\lambda(T_0) = (1/2)d^2_{C(T_0)}(a) = 0$, the latter inequality entails that

$$g_\lambda(T) + \lambda^{-1}\int_{T_0}^T (h_\lambda(t))^2\,dt \le \int_{T_0}^T h_\lambda(t)\,\dot v(t)\,dt.$$

Let us denote $\|\cdot\|_{L^2_{\mathcal H}}$ (resp. $\|\cdot\|_{L^2}$) the usual norm on $L^2(\mathcal H, I)$ (resp. $L^2(I)$). The latter inequality and the inequality $g_\lambda(T) \ge 0$ yield $\lambda^{-1}\|h_\lambda\|^2_{L^2} \le \|h_\lambda\|_{L^2}\|\dot v\|_{L^2}$, which gives

$$\|h_\lambda\|_{L^2} \le \lambda\,\|\dot v\|_{L^2} \quad \text{and} \quad \|\dot u_\lambda\|_{L^2_{\mathcal H}} \le \|\dot v\|_{L^2}. \tag{3.4}$$

Fix any reals $\mu, \nu > 0$ and any $t \in I$, and note that

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}\|u_\mu - u_\nu\|^2)(t) &= \langle u_\mu(t) - u_\nu(t), \dot u_\mu(t) - \dot u_\nu(t)\rangle \\ &= -\langle u_\mu(t) - u_\nu(t), \frac{1}{\mu}\nabla\varphi_t(u_\mu(t)) - \frac{1}{\nu}\nabla\varphi_t(u_\nu(t))\rangle. \qquad (3.5)\end{aligned}$$

On the other hand, the equality $u_\mu(t) - \nabla\varphi_t(u_\mu(t)) = \mathrm{proj}_{C(t)}(u_\mu(t))$ assures us that $\mu^{-1}\nabla\varphi_t(u_\mu(t)) \in N_{C(t)}\left(u_\mu(t) - \nabla\varphi_t(u_\mu(t))\right)$, which entails by the monotonicity of $N_{C(t)}(\cdot)$ that

$$\left\langle u_\mu(t) - \nabla\varphi_t(u_\mu(t)) - u_\nu(t) + \nabla\varphi_t(u_\nu(t)), \frac{1}{\mu}\nabla\varphi_t(u_\mu(t)) - \frac{1}{\nu}\nabla\varphi_t(u_\nu(t))\right\rangle \ge 0,$$

or equivalently

$$\begin{aligned}&\left\langle u_\mu(t) - u_\nu(t), \frac{1}{\mu}\nabla\varphi_t(u_\mu(t)) - \frac{1}{\nu}\nabla\varphi_t(u_\nu(t))\right\rangle \\ \ge &\left\langle \nabla\varphi_t(u_\mu(t)) - \nabla\varphi_t(u_\nu(t)), \frac{1}{\mu}\nabla\varphi_t(u_\mu(t)) - \frac{1}{\nu}\nabla\varphi_t(u_\nu(t))\right\rangle.\end{aligned}$$

Combining this with (3.5) and integrating on I we obtain

$$\begin{aligned}&\frac{1}{2}\|u_\mu(T) - u_\nu(T)\|^2 \\ \le &-\int_{T_0}^T \left\langle \nabla\varphi_t(u_\mu(t)) - \nabla\varphi_t(u_\nu(t)), \frac{1}{\mu}\nabla\varphi_t(u_\mu(t)) - \frac{1}{\nu}\nabla\varphi_t(u_\nu(t))\right\rangle dt.\end{aligned}$$

It follows with the inner product $\langle\cdot,\cdot\rangle_{L^2_{\mathcal H}}$ in $L^2(I, \mathcal H)$ that

$$\langle \mu\,\dot u_\mu - \nu\,\dot u_\nu, \dot u_\mu - \dot u_\nu\rangle_{L^2_{\mathcal H}} \le 0. \tag{3.6}$$

This combined with the second inequality in (3.4) entails by Lemma 3.3(b) (with $\mu \preceq \nu$ if $\nu \ge \mu$ in $J :=]0, +\infty[$) that the family $(\dot u_\lambda)_{\lambda>0}$ converges strongly in $L^2(I, \mathcal H)$ to some $\zeta(\cdot)$ as $\lambda \downarrow 0$. Defining $u : I \to \mathcal H$ by $u(t) :=$

$a + \int_{T_0}^{t} \zeta(s)\, ds$ for all $t \in I$, we see that $u(\cdot)$ is absolutely continuous on I and $\dot{u} = \zeta$ almost everywhere. Further, writing for any $t \in I$

$$\|u(t) - u_\mu(t)\| = \left\| \int_{T_0}^{t} (\dot{u}(s) - \dot{u}_\mu(s))\, ds \right\| \le \sqrt{T - T_0}\, \|\dot{u} - \dot{u}_\mu\|_{L^2_{\mathcal{H}}},$$

shows that the family $(u_\lambda)_{\lambda>0}$ converges uniformly on I to u as $\lambda \downarrow 0$.

Now take any sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $]0, +\infty[$ tending to 0 and for all $n \in \mathbb{N}$ and $t \in I$ put $w_n(t) := u_{\lambda_n}(t)$ and $p_n(t) := \mathrm{proj}_{C(t)}(w_n(t))$, so

$$w_n(t) - p_n(t) = \nabla\varphi_t(w_n(t)) = -\lambda_n \dot{w}_n(t) \text{ and } -\dot{w}_n(t) \in N_{C(t)}(p_n(t)),$$

for all $n \in \mathbb{N}$ and all $t \in I$. The equality $w_n - p_n = -\lambda_n\, \dot{w}_n$ and the inequality $\|\dot{w}_n\|_{L^2_{\mathcal{H}}} \le \|\dot{v}\|_{L^2}$ for all $n \in \mathbb{N}$ in (3.4) ensure that $(p_n)_{n\in\mathbb{N}}$ converges strongly to u in $L^2(I, \mathcal{H})$. Then there is a Lebesgue negligible set $N \subset I$ such that for every $t \in I \setminus N$ the sequences $(p_n(t))_{n\in\mathbb{N}}$ and $(\dot{w}_n(t))_{n\in\mathbb{N}}$ converge strongly in $\mathcal{H}$ to $u(t)$ and $\dot{u}(t)$ respectively, so in particular $u(t) \in C(t)$. Fix any $t \in I \setminus N$ and $n \in \mathbb{N}$. For any $x \in C(t)$ writing $\langle -\dot{w}_n(t), x - p_n(t)\rangle \le 0$ by the inclusion in $-\dot{w}_n(t) \in N_{C(t)}(p_n(t))$, and passing to the limit we obtain $\langle -\dot{u}(t), x - u(t)\rangle \le 0$. This tells us that $-\dot{u}(t) \in N_{C(t)}(u(t))$, and the proof is complete. □

We now prove for completeness the result related to (3.6) used in the proof of Theorem 3.2. This result as well as its proof below are due to M. G. Crandall and A. Pazy [19, Lemma 2.4].

Lemma 3.3 (Crandall-Pazy [19])**.** *Let X be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle_X$ and its associated norm $\|\cdot\|_X$. Let $(J, \preceq)$ be a directed set, $(r_j)_{j\in J}$ and $(z_j)_{j\in J}$ be nets in $]0, +\infty[$ and X respectively. Assume that*

$$\langle z_j - z_i, r_j z_j - r_i z_i\rangle_X \ge 0 \quad \textit{for all } i, j \in J.$$

(a) If the net $(r_j)_{j\in J}$ is increasing, then the net $(\|z_j\|_X)_{j\in J}$ is nonincreasing and the net $(z_j)_{j\in J}$ strongly converges in X.
(b) If the net $(r_j)_{j\in J}$ is decreasing and the net $(\|z_j\|_X)_{j\in J}$ is bounded, then the net $(z_j)_{j\in J}$ srongly converges in X.

Proof. Writing for any $i, j \in J$

$$0 \ge 2\langle z_j - z_i, r_j z_j - r_i z_i\rangle_X = (r_j + r_i)\|z_j - z_i\|_X^2 + (r_j - r_i)(\|z_j\|_X^2 - \|z_i\|_X^2),$$

we derive on the one hand that the net $(\|z_j\|_X)_{j\in J}$ is nonincreasing (resp. nondecreasing) whenever the net $(r_j)_{j\in J}$ is increasing (resp. decreasing). On the other hand, we also derive that for any $i, j \in J$

$$\|z_j - z_i\|_X^2 \le \frac{r_j - r_i}{r_j + r_i}(\|z_i\|_X^2 - \|z_j\|_X^2) = \frac{|r_j - r_i|}{r_j + r_i}|\, \|z_j\|_X^2 - \|z_i\|_X^2 \,| \le |\, \|z_j\|_X^2 - \|z_i\|_X^2 \,|.$$

This entails that $(z_j)_{j\in J}$ is a Cauchy net in X, since in either (a) or (b) the net $(\|z_j\|_X)_{j\in J}$ converges in $\mathbb{R}$. □

4. Regularization of convex sweeping process under continuous variation

In this section, we are concerned with the regularization of Moreau's sweeping process with bounded variation. Let us first recall the concept of solutions for such a problem.

Let $C : I \rightrightarrows \mathcal{H}$ be an r-prox-regular valued multimapping with $r \in]0, +\infty]$ and for which there exists a positive Radon measure μ on $I := [T_0, T]$ such that (2.12) holds for some extended real $\rho > 0$. Given $u_0 \in C(T_0)$, a mapping $u : I \to \mathcal{H}$ is a solution of the measure differential inclusion

$$(\mathcal{P}) \begin{cases} -du \in N(C(t); u(t)) \\ u(T_0) = u_0, \end{cases}$$

whenever:
(a) the mapping $u(\cdot)$ is of bounded variation on I, right-continuous on I and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;
(b) there exists a positive Radon measure ν on I, absolutely continuously equivalent to μ and with respect to which the differential measure du of u is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1(I, \mathcal{H}, \nu)$-density and

$$\frac{du}{d\nu}(t) \in -N(C(t); u(t)) \quad \nu\text{-a.e. } t \in I. \tag{4.1}$$

It is known (see, e.g., [49]) that the concept of solution does not depend on the measure ν, i.e., a mapping $u(\cdot) : I \to \mathcal{H}$ satisfying (a) above is a solution of $(\mathcal{P})$ if and only if (4.1) holds for any positive Radon measure ν which is absolutely continuously equivalent to μ.

The first existence result for such a differential inclusion is due to J.J. Moreau and can be stated in the following form.

Theorem 4.1 (Moreau [39])**.** *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a nonempty closed convex valued multimapping for which there exists a positive Radon measure μ on I such that*

$$d(y, C(t)) \le d(y, C(s)) + \mu(]s, t]) \quad \textit{for all}\, s, t \in I \textit{ with}\, s \le t. \tag{4.2}$$

Then, for each $u_0 \in C(T_0)$, the measure differential sweeping process

$$\begin{cases} -du \in N(C(t); u(t)) \\ u(T_0) = u_0 \end{cases}$$

admits one and only one right-continuous with bounded variation solution.

We derive from the well-posedness of right-continuous bounded variation Moreau's sweeping process the following result concerned with selections of multimappings.

Corollary 4.2. *Let $\tau_0, \tau_1 \in \mathbb{R}$ with $\tau_0 < \tau_1$ and let $C : [\tau_0, \tau_1] \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed convex values satisfying* (4.2)*. Then, for every $x_0 \in C(\tau_0)$, there exists a right-continuous with bounded variation selection $\phi(\cdot)$ of $C(\cdot)$ satisfying the initial condition $\phi(\tau_0) = x_0$.*

Although $\mathcal{H}$ is possibly infinite-dimensional, (as Theorem 3.2) Theorem 4.1 does not require any compactness assumption on the sets $C(t)$. The proof of the latter Moreau's result is no longer based on a regularization technique but on the so-called *Moreau's catching-up algorithm* which consists in a time discretization $(t_i^n)_i$ of $I := [T_0, T]$ setting (with $u_0^n := u_0$)

$$u_i^n := \mathrm{proj}_{C(t_i^n)}(u_{i-1}^n).$$

Many variants, extensions and applications of the latter Moreau's result have been developed over the years in various contexts (see, e.g., [31, 23, 32, 1, 27, 49, 28, 43, 52] and the references therein).

Later, in [30], M.D.P. Monteiro Marques showed that a regularization technique can also be used to handle measure differential inclusions as $(\mathcal{P})$ in certain situations. Observe that the development of such a regularization must differ from Section 3 concerning both approaches and hypotheses. Indeed, it is clear that in the right-continuous bounded variation case, no uniform convergence for $(u_\lambda(\cdot))_{\lambda>0}$ as $\lambda \downarrow 0$ to $u(\cdot)$ could be expected (otherwise the solution mapping $u(\cdot)$ would be continuous). The crucial assumptions here will be on one hand the continuity of $\mathrm{var}(C; [T_0, \cdot])$ at the endpoint T of $I = [T_0, T]$ and on the other hand the strongly ball-compact values of the moving set $C(\cdot)$.

The following result is a partial form of the main result of [30]. Almost all the *proof given below follows the one of M.D.P. Monteiro Marques in* [30]. The approach in this section uses essentially Measure Theory arguments.

Theorem 4.3 (Monteiro Marques [30])**.** *Let $C : I \rightrightarrows \mathcal{H}$ be a multimapping with nonempty ball-compact convex values and $a \in C(T_0)$. Assume that C has a bounded variation on I and* $\mathrm{var}(C; [T_0, \cdot])$ *is right-continuous on I and continuous at T.*

Then, for any real $\lambda > 0$, the (classical) differential equation over I

$$\begin{cases} \dot{U}_\lambda(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(U_\lambda(t)) & a.e.\ t \in I \\ U_\lambda(T_0) = a \end{cases}$$

has a unique absolutely continuous solution $U_\lambda(\cdot)$ on I and the family $(U_\lambda(\cdot))_{\lambda>0}$ converges pointwise on I as $\lambda \downarrow 0$ to a mapping $U : I \to \mathcal{H}$ of bounded variation whose right-continuous envelope $u := U^+$ is the right-continuous with bounded variation solution of the differential inclusion sweeping process

$$\begin{cases} -du \in N(C(t); u(t)) \\ u(T_0) = a. \end{cases} \tag{4.3}$$

Furthermore, the pointwise limit $U(\cdot)$ of $(U_\lambda(\cdot))_{\lambda>0}$ is the left-continuous envelope of the solution $u(\cdot)$, that is, $u(t) = U^-(t)$ for all $t \in I$.

Proof. The uniqueness of right-continuous with bounded variation solution for (4.3) can be seen as a direct consequence of Theorem 4.1. We also refer to [49, Proposition 3.16] for a general result of uniqueness of solution of this type for perturbed right-continuous bounded variation sweeping process described by a prox-regular moving set. Let us define $v(\cdot) : I \to \mathbb{R}_+$ by

$$v(t) := \mathrm{var}(C; [T_0, t]) \quad \text{for all}\, t \in I = [T_0, T].$$

We may and do suppose that $v([T_0, T]) > 0$, otherwise $C(t) = C(T_0)$ for all $t \in I$, and everything is obvious.

Denote by $\mu := \mu_C$ the differential measure associated to $v(\cdot)$, hence (see Section 2) we have

$$\mathrm{haus}(C(s), C(t)) \leq \mu(]s, t]) \quad \text{for all}\, s, t \in I. \tag{4.4}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $\varepsilon_n \downarrow 0$. Choose for each integer $n \geq 1$, an integer $q_n \geq 1$ and $0 = M_0^n < M_1^n < \ldots < M_{q_n}^n = M =: v(T)$ satisfying the two following conditions:
(a) for all $j \in \{0, \ldots, q_n - 1\}$, $M_{j+1}^n - M_j^n \leq \varepsilon_n$;
(b) for all integer $k \geq 1$, $\left\{M_0^k, \ldots, M_{q_k}^k\right\} \subset \left\{M_0^{k+1}, \ldots, M_{q_{k+1}}^{k+1}\right\}$.
For each integer $n \geq 1$, set $M_{1+q_n}^n := M + \varepsilon_n$ and consider the partition $(J_j^n)_{j\in\{0,\ldots,q_n-1\}}$ of I where for each $j \in \{0, \ldots, q_n - 1\}$

$$J_j^n := v^{-1}(\left[M_j^n, M_{j+1}^n\right[) = \{t \in I : M_j^n \leq v(t) < M_{j+1}^n\}.$$

Note that $(J_j^m)_{0\leq j\leq q_m}$ is a refinement of $(J_j^n)_{0\leq j\leq q_n}$ for any integers $1 \leq n \leq m$. Let $n \geq 1$ be an integer. Thanks to the fact that $v(\cdot)$ is nondecreasing and right-continuous on I, we can check that for each $j \in \{0, \ldots q_{n-1}^n\}$, either $J_j^n = \emptyset$ or $J_j^n = [\tau, \tau'[$ with $\tau < \tau'$. Further, we observe that $J_{q_n}^n = \{T\}$. Hence, we have an integer $p(n) \geq 1$ and

$$T_0 =: t_0^n < \ldots < t_{p(n)}^n := T$$

such that for each $i \in \{0, \ldots, p(n) - 1\}$, there is some $j_i \in \{0, \ldots, q_n - 1\}$ satisfying $J_{j_i}^n = \left[t_i^n, t_{i+1}^n\right[$. Without loss of generality, we will assume that $p(n) > 2$ for every integer $n \geq 1$. It follows from what precedes that for all $i \in \{0, \cdots, p(n) - 1\}$ and $t \in \left[t_i^n, t_{i+1}^n\right[$, one has

$$\mu(]t_i^n, t]) = v(t) - v(t_i^n) \leq \varepsilon_n,$$

so in particular

$$\mu(\left]t_i^n, t_{i+1}^n\right[) \leq \varepsilon_n. \tag{4.5}$$

Fix for a moment any integer $n \geq 1$ and any real $\lambda > 0$. Let us consider the multimapping $D_n : I \rightrightarrows \mathcal{H}$ defined by $D_n(T) := C(T)$ and by

$$D_n(t) := C(t_i^n),$$

for all $t \in [t_i^n, t_{i+1}^n[$ where $i \in \{0, \cdots, p(n) - 1\}$. Note by (4.4) and (4.5) that

$$h_n(t) := \mathrm{haus}(D_n(t), C(t)) \leq \varepsilon_n \quad \text{for all}\, t \in I. \tag{4.6}$$

Let any $b \in \mathcal{H}$. According to [10, Théorème 1.4], we know that there is one (and only one) absolutely continuous mapping $V_{\lambda,b}^n(\cdot) : I \to \mathcal{H}$ (resp.

$U_{\lambda,b}(\cdot) : I \to \mathcal{H}$) satisfying

$$\begin{cases} \dot{V}^n_{\lambda,b}(t) = -\frac{1}{2\lambda}\nabla d^2_{D_n(t)}(V^n_{\lambda,b}(t)) & \text{a.e. } t \in I \\ V^n_{\lambda,b}(T_0) = b \end{cases} \tag{4.7}$$

(resp.

$$\begin{cases} \dot{U}_{\lambda,b}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(U_{\lambda,b}(t)) & \text{a.e. } t \in I \\ U_{\lambda,b}(T_0) = b). \end{cases} \tag{4.8}$$

It is known and not difficult to check (see, e.g., the proof of [10, Théorème 1.4]) that for every $t \in [T_0, T]$,

$$U_{\lambda,b}(t) = e^{-\frac{t-T_0}{\lambda}} b + \frac{1}{\lambda}\int_{T_0}^t e^{\frac{s-t}{\lambda}} \mathrm{proj}_{C(s)}(U_{\lambda,b}(s))ds \tag{4.9}$$

and

$$V^n_{\lambda,b}(t) = e^{-\frac{t-T_0}{\lambda}} b + \frac{1}{\lambda}\int_{T_0}^t e^{\frac{s-t}{\lambda}} \mathrm{proj}_{D_n(s)}(V^n_{\lambda,b}(s))ds. \tag{4.10}$$

Set $x^n_i := V^n_{\lambda,b}(t^n_i)$ and $y^n_i := \mathrm{proj}_{C(t^n_i)}(x^n_i)$ for every $i \in \{0, \dots, p(n)\}$. In particular, we have $b = V^n_{\lambda,b}(T_0) = x^n_0$. Set also $c := y^n_0 = \mathrm{proj}_{C(T_0)}(b)$.

The use of any element $b \in \mathcal{H}$ and the choice of such an element c as above will be crucial for (4.22) in Lemma 4.8, which is at the heart of the proof of Lemma 4.11. These lemmas are parts of a series of Lemmas constituting the elaboration of the rest of the proof of the theorem.

The first of the series of lemmas is devoted to compute $V^n_{\lambda,b}(\cdot)$ explicitly.

Lemma 4.4. *For every $i \in \{0, \dots, p(n)-1\}$, for every $t \in [t^n_i, t^n_{i+1}[$, one has*

$$V^n_{\lambda,b}(t) = y^n_i + e^{-\frac{t-t^n_i}{\lambda}}(x^n_i - y^n_i).$$

Proof. Fix any $i \in \{0, \dots, p(n)-1\}$. Recall that $D_n(t) = C(t^n_i)$ and define $\theta : [t^n_i, t^n_{i+1}[\to \mathcal{H}$ by

$$\theta(t) := y^n_i + e^{-\frac{t-t^n_i}{\lambda}}(x^n_i - y^n_i) \quad \text{for all } t \in [t^n_i, t^n_{i+1}[.$$

It is clear that $\theta(t) \in [x^n_i, y^n_i]$ for all $t \in [t^n_i, t^n_{i+1}[$. This and the definition of y^n_i ensure that

$$\mathrm{proj}_{D_n(t)}(\theta(t)) = \mathrm{proj}_{C(t^n_i)}(\theta(t)) = y^n_i.$$

Then, it is routine to check that $\theta(\cdot)$ satisfies (4.7) on $[t^n_i, t^n_{i+1}[$. On the other hand, the restriction of $V^n_{\lambda,b}(\cdot)$ to $[t^n_i, t^n_{i+1}[$ also satisfies the Cauchy problem (4.7). It follows that $\theta(\cdot) = V^n_{\lambda,b}(\cdot)$ on $[t^n_i, t^n_{i+1}[$. The proof is then complete. □

Our goal is now to establish that $(V^k_{\lambda,b}(\cdot))_{k\geq 1}$ converges to $U_{\lambda,b}(\cdot)$ as $k \to +\infty$. A lemma is needed first.

Lemma 4.5. *For every $i \in \{0, \dots, p(n)-1\}$ and every $t \in [t_i^n, t_{i+1}^n[$, one has*

$$\left\| V_{\lambda,b}^n(t) - \mathrm{proj}_{D_n(t)}(V_{\lambda,b}^n(t)) \right\| \leq e^{-\frac{t-t_0^n}{\lambda}} \|b-c\| + \sum_{j=1}^{i} e^{-\frac{t-t_j^n}{\lambda}} \mu(]t_{j-1}^n, t_j^n]),$$

where by convention the latter sum is equal to 0 when $i=0$.

Proof. By (finite) induction, let us show that for every $i \in \{1, \dots, p(n)-1\}$,

$$\|x_i^n - y_i^n\| \leq e^{-\frac{t_i^n - t_0^n}{\lambda}} \|b-c\| + \sum_{j=1}^{i} e^{-\frac{t_i^n - t_j^n}{\lambda}} \mu(]t_{j-1}^n, t_j^n]). \tag{4.11}$$

Observe first that for any integer $k \in \{0, \dots, p(n)-1\}$ the continuity of $V_{\lambda,b}^n(\cdot)$ at t_{k+1}^n (as solution of (4.7)) along with Lemma 4.4 ensure that

$$\left\| x_{k+1}^n - y_k^n \right\| = e^{-\frac{t_{k+1}^n - t_k^n}{\lambda}} \|x_k^n - y_k^n\|.$$

It follows that for any $k \in \{1, \dots, p(n)-1\}$ (keeping in mind that $y_{k-1}^n \in C(t_{k-1}^n)$)

$$\begin{aligned} \|x_k^n - y_k^n\| = d_{C(t_k^n)}(x_k^n) &\leq \left\| x_k^n - y_{k-1}^n \right\| + d_{C(t_k^n)}(y_{k-1}^n) \\ &\leq e^{-\frac{t_k^n - t_{k-1}^n}{\lambda}} \left\| x_{k-1}^n - y_{k-1}^n \right\| + \mathrm{haus}(C(t_{k-1}^n), C(t_k^n)) \\ &\leq e^{-\frac{t_k^n - t_{k-1}^n}{\lambda}} \left\| x_{k-1}^n - y_{k-1}^n \right\| + \mu(]t_{k-1}^n, t_k^n]). \end{aligned} \tag{4.12}$$

A particular case of the latter inequality (4.12) is

$$\|x_1^n - y_1^n\| \leq e^{-\frac{t_1^n - t_0^n}{\lambda}} \|b-c\| + \mu(]T_0, t_1^n]).$$

Now, assume that (4.11) holds up to step $i \in \{1, \dots, p(n)-2\}$. Using (4.12), we get

$$\begin{aligned} \left\| x_{i+1}^n - y_{i+1}^n \right\| &\leq e^{-\frac{t_{i+1}^n - t_i^n}{\lambda}} \|x_i^n - y_i^n\| + \mu(]t_i^n, t_{i+1}^n]) \\ &\leq e^{-\frac{t_{i+1}^n - t_0^n}{\lambda}} \|b-c\| + \sum_{j=1}^{i} e^{-\frac{t_{i+1}^n - t_j^n}{\lambda}} \mu(]t_{j-1}^n, t_j^n]) + \mu(]t_i^n, t_{i+1}^n]) \\ &\leq e^{-\frac{t_{i+1}^n - t_0^n}{\lambda}} \|b-c\| + \sum_{j=1}^{i+1} e^{-\frac{t_{i+1}^n - t_j^n}{\lambda}} \mu(]t_{j-1}^n, t_j^n]), \end{aligned}$$

which completes the induction. Now, consider any $i \in \{0, \dots, p(n)-1\}$ and $t \in [t_i^n, t_{i+1}^n[$. Lemma 4.4 says that

$$V_{\lambda,b}^n(t) = y_i^n + e^{-\frac{t-t_i^n}{\lambda}} (x_i^n - y_i^n),$$

in particular $V_{\lambda,b}^n(t) \in [x_i^n, y_i^n]$. This and the equalities $D_n(t) = C(t_i^n)$ and $y_i^n = \mathrm{proj}_{C(t_i^n)}(x_i^n)$ then give

$$\mathrm{proj}_{D_n(t)}(V_{\lambda,b}^n(t)) = \mathrm{proj}_{C(t_i^n)}(V_{\lambda,b}^n(t)) = y_i^n.$$

Consequently, we have

$$\left\| V^n_{\lambda,b}(t) - \mathrm{proj}_{D_n(t)}(V^n_{\lambda,b}(t)) \right\| = e^{-\frac{t-t_i^n}{\lambda}} \left\| x_i^n - y_i^n \right\| .$$

It remains to put together the latter equality and (4.11) to finish the proof. □

As a direct consequence of Lemma 4.5, we have for every $t \in [T_0, T[$

$$\begin{aligned} d_{D_n(t)}(V^n_{\lambda,b}(t)) = \left\| V^n_{\lambda,b}(t) - \mathrm{proj}_{D_n(t)}(V^n_{\lambda,b}(t)) \right\| &\leq e^{-\frac{t-T_0}{\lambda}} \|b-c\| + \mu(]T_0,t]) \\ &\leq \|b-c\| + \mu(]T_0,T]) =: M_1. \end{aligned}$$

Hence, for every $t \in [T_0, T[$,

$$\begin{aligned} d_{D_n(T)}(V^n_{\lambda,b}(T)) &\leq \left\| V^n_{\lambda,b}(t) - V^n_{\lambda,b}(T) \right\| + \mathrm{haus}(D_n(t), D_n(T)) + d_{D_n(t)}(V^n_{\lambda,b}(t)) \\ &\leq \left\| V^n_{\lambda,b}(t) - V^n_{\lambda,b}(T) \right\| + \mu(]t,T]) + M_1 \end{aligned}$$

Using $\mu(\{T\}) = 0$ (thanks to the continuity of $\mathrm{var}(C; [T_0, \cdot])$ at T) and the continuity of $V^n_{\lambda,b}(\cdot)$ at T and letting $t \uparrow T$, we obtain $d_{D_n(T)}(V^n_{\lambda,b}(T)) \leq M_1$. We then have

$$d_{D_n(t)}(V^n_{\lambda,b}(t)) \leq M_1 \quad \text{for all}\, t \in I. \tag{4.13}$$

Keeping in mind that $U_\lambda(\cdot)$ is absolutely continuous on I (hence bounded) it is straightforward to check that

$$\begin{aligned} &\left\| U_{\lambda,b}(t) - \mathrm{proj}_{C(t)}(U_{\lambda,b}(t)) \right\| = d_{C(t)}(U_{\lambda,b}(t)) \\ &\leq \left\| U_{\lambda,b}(t) - \mathrm{proj}_{C(t)}(c) \right\| \\ &\leq \|U_{\lambda,b}(t) - c\| + d_{C(t)}(c) \\ &\leq \sup_{t\in[T_0,T]} \big(\|U_{\lambda,b}(t)\| + \|c\| + \mathrm{haus}(C(T_0), C(t)) \big) \\ &\leq \sup_{t\in[T_0,T]} \|U_{\lambda,b}(t)\| + \|c\| + \mu(]T_0,T]) := M_2(\lambda). \qquad (4.14) \end{aligned}$$

The next lemma proves the desired convergence of $(V^k_{\lambda,b}(\cdot))_{k\geq 1}$ to $U_{\lambda,b}(\cdot)$.

Lemma 4.6. *The following hold.*
(a) One has the estimate

$$\sup_{t\in I} \left\| V^n_{\lambda,b}(t) - U_{\lambda,b}(t) \right\| \leq \sqrt{2(M_1 + M_2(\lambda))} \big(1 + \frac{T-T_0}{\lambda}\big) \sqrt{\varepsilon_n}.$$

In particular, for every real $\overline{\lambda} > 0$, the sequence $(V^k_{\overline{\lambda},b}(\cdot))_{k\geq 1}$ converges uniformly to $U_{\overline{\lambda},b}(\cdot)$ on I as $k \to +\infty$.
(b) One has the strong convergence

$$\lim_{k\to+\infty} \mathrm{proj}_{D_k(t)}(V^k_{\lambda,b}(t)) = \mathrm{proj}_{C(t)}(U_{\lambda,b}(t)) \quad \textit{for every}\, t \in I.$$

Proof. Fix any $t \in I$. Set $\phi_n(\tau) := \left\| V^n_{\lambda,b}(\tau) - U_{\lambda,b}(\tau) \right\|$ and $\psi_n(\tau) := \frac{1}{\lambda}\int_{T_0}^{\tau} e^{\frac{s-\tau}{\lambda}} \phi_n(s) ds$ for every $\tau \in I$. Set also $L := \sqrt{2(M_1 + M_2(\lambda))}$ and $K := L\sqrt{\varepsilon_n}$. Applying Lemma 2.5 and using (4.6), (4.13) and (4.14), we obtain

$$\begin{aligned}\left\| \operatorname{proj}_{D_n(t)}(V^n_{\lambda,b}(t)) - \operatorname{proj}_{C(t)}(U_{\lambda,b}(t)) \right\|^2 &\le \left\| V^n_{\lambda,b}(t) - U_{\lambda,b}(t) \right\|^2 \\ &\quad + 2h_n\big(d_{D_n(t)}(V^n_{\lambda,b}(t)) + d_{C(t)}(U_{\lambda,b}(t))\big) \\ &\le \phi_n(t)^2 + L^2\varepsilon_n. \end{aligned} \tag{4.15}$$

According to (4.9) and (4.10), we also have

$$\phi_n(t) \le \frac{1}{\lambda}\int_{T_0}^{t} e^{\frac{s-t}{\lambda}} \left\| \operatorname{proj}_{D_n(s)}(V^n_{\lambda,b}(s)) - \operatorname{proj}_{C(s)}(U_{\lambda,b}(s)) \right\| ds,$$

and then by virtue of (4.15)

$$\phi_n(t) \le \frac{1}{\lambda}\int_{T_0}^{t} e^{\frac{s-t}{\lambda}} \sqrt{\phi_n(s)^2 + L^2\varepsilon_n}\, ds.$$

It follows that

$$\phi_n(t) \le L\sqrt{\varepsilon_n}(1 - e^{\frac{T_0-t}{\lambda}}) + \psi_n(t) \le K + \psi_n(t), \tag{4.16}$$

which yields

$$\dot{\psi}_n(t) = \frac{1}{\lambda}(\phi_n(t) - \psi_n(t)) \le \frac{K}{\lambda}.$$

Since $t \in I$ is arbitrary in I, the latter inequality gives (thanks to $\psi_n(T_0) = 0$)

$$\psi_n(t) = \int_{T_0}^{t} \dot{\psi}_n(\tau)\, d\tau = \int_{T_0}^{t} \frac{1}{\lambda}(\phi_n(\tau) - \psi_n(\tau))\, d\tau \le \int_{T_0}^{T} \frac{K}{\lambda}\, d\tau = \frac{K}{\lambda}(T - T_0).$$

Using (4.16), this allows us to write

$$\phi_n(t) \le \frac{K}{\lambda}(T - T_0) + K = K(1 + \frac{T - T_0}{\lambda}),$$

and this is the inequality claimed by (a). The assertion (b) is a direct consequence of (a), the inequality (4.15) and $\varepsilon_n \downarrow 0$. □

Let us give an estimate on $d_{C(t)}(U_{\lambda,b}(t))$ which will be used in Lemma 4.8 and in Lemma 4.9.

Lemma 4.7. *For every $t \in I$, one has*

$$\begin{aligned}\left\| U_{\lambda,b}(t) - \operatorname{proj}_{C(t)}(U_{\lambda,b}(t)) \right\| &\le e^{-\frac{t-T_0}{\lambda}} \|b - c\| + \int_{]T_0,t]} e^{\frac{s-t}{\lambda}}\, d\mu(s) \\ &\le \|b - c\| + \mu(]T_0, t]). \end{aligned}$$

Proof. Fix any $t \in I = [T_0, T]$. Assume first that $t < T$. If $t = T_0$, the desired inequalities are obvious, so assume that $t > T_0$. Let $i \in \{0, \dots, p(n) - 1\}$ be such that $t \in [t_i^n, t_{i+1}^n[$. According to Lemma 4.6, we have

$$\lim_{k \to +\infty} \left\| V_{\lambda,b}^k(t) - \operatorname{proj}_{C(t)}(V_{\lambda,b}^k(t)) \right\| = \left\| U_{\lambda,b}(t) - \operatorname{proj}_{C(t)}(U_{\lambda,b}(t)) \right\|,$$

and

$$\lim_{k \to +\infty} \left\| V_{\lambda,b}^k(t) - \operatorname{proj}_{D_k(t)}(V_{\lambda,b}^k(t)) \right\| = \left\| U_{\lambda,b}(t) - \operatorname{proj}_{C(t)}(U_{\lambda,b}(t)) \right\|. \tag{4.17}$$

Now, let us define $\Pi_n :]T_0, t] \to \mathbb{R}$ by

$$\Pi_n(s) := \sum_{j=1}^{p(n)} e^{\frac{t_j^n - t}{\lambda}} \mathbf{1}_{]t_{j-1}^n, t_j^n]}(s) \quad \text{for all } s \in]T_0, t]$$

and observe that (since $t \in [t_i^n, t_{i+1}^n[$)

$$\int_{]T_0,t]} \Pi_n(s) d\mu(s) = \sum_{j=1}^{i} e^{\frac{t_j^n - t}{\lambda}} \mu(]t_{j-1}^n, t_j^n]) + e^{\frac{t_{i+1}^n - t}{\lambda}} \mu(]t_i^n, t]). \tag{4.18}$$

along with by (4.5)

$$e^{\frac{t_{i+1}^n - t}{\lambda}} \mu(]t_i^n, t]) \le e^{\frac{T-t}{\lambda}} \mu(]t_i^n, t]) \le e^{\frac{T-t}{\lambda}} \varepsilon_n. \tag{4.19}$$

On the other hand, note that for every $s \in]T_0, t]$

$$\lim_{k \to +\infty} \Pi_k(s) = e^{\frac{s-t}{\lambda}} \quad \text{and} \quad |\Pi_n(s)| \le e^{\frac{T-t}{\lambda}}.$$

Hence, we may write

$$\lim_{k \to +\infty} \int_{]T_0,t]} \Pi_k(s) d\mu(s) = \int_{]T_0,t]} e^{\frac{s-t}{\lambda}} d\mu(s). \tag{4.20}$$

Coming back to the inequality provided by Lemma 4.5 and using (4.17), (4.18), (4.19) and (4.20), we arrive to the first desired inequality on $[T_0, T[$. Now, assume that $t = T$. By virtue of what precedes, we have shown in particular

$$\left\| U_{\lambda,b}(\tau) - \operatorname{proj}_{C(\tau)}(U_{\lambda,b}(\tau)) \right\| \le e^{-\frac{\tau - T_0}{\lambda}} \|b - c\| + \int_{]T_0,T]} e^{\frac{s-\tau}{\lambda}} d\mu(s), \tag{4.21}$$

for all $\tau \in [T_0, T[$. From the equality $\mu(\{T\}) = 0$ and from the inequality valid for all $\tau \in [T_0, T[$,

$$\begin{aligned} \left| d_{C(\tau)}(U_{\lambda,b}(\tau)) - d_{C(T)}(U_{\lambda,b}(T)) \right| &\le \|U_{\lambda,b}(\tau) - U_{\lambda,b}(T)\| + \operatorname{haus}(C(\tau), C(T)) \\ &\le \|U_{\lambda,b}(\tau) - U_{\lambda,b}(T)\| + \mu(]\tau, T]) \end{aligned}$$

we see that $d_{C(t)}(U_{\lambda,b}(\tau)) \to d_{C(T)}(U_{\lambda,b}(T))$ as $\tau \uparrow T$. On the other hand, the Lebesgue dominated convergence theorem yields that $\int_{]T_0,T]} e^{\frac{s-\tau}{\lambda}} d\mu(s) \to$

$\int_{]T_0,T]} e^{\frac{s-T}{\lambda}}\, d\mu(s)$ as $\tau \uparrow T$. It remains to let $\tau \uparrow T$ in (4.21) to complete the proof of the first inequality. The second is a direct consequence of

$$\int_{]T_0,\tau]} e^{\frac{s-\tau}{\lambda}} d\mu(s) \le \int_{]T_0,\tau]} d\mu(s) = \mu(]T_0,\tau]) \quad \text{for all } \tau \in I.$$

□

Through the above lemma we can control uniformly (with respect to λ) the variation of $U_{\lambda,b}(\cdot)$ as follows.

Lemma 4.8. *The following estimate holds*

$$\int_{T_0}^{T} \left\| \dot{U}_{\lambda,b}(\tau) \right\| d\tau \le \|b-c\| + \mu(]T_0,T]).$$

More generally, for every $T_0 \le s \le t \le T$, one has

$$\int_{s}^{t} \left\| \dot{U}_{\lambda,b}(\tau) \right\| d\tau \le \left\| U_{\lambda,b}(s) - \mathrm{proj}_{C(s)}(U_{\lambda,b}(s)) \right\| + \mu(]s,t]). \tag{4.22}$$

Proof. Let us first write thanks to (4.8) and Lemma 4.7

$$\int_{T_0}^{T} \left\| \dot{U}_{\lambda,b}(t) \right\| dt \le \frac{1}{\lambda} \int_{T_0}^{T} (e^{-\frac{t-T_0}{\lambda}} \|b-c\| + \int_{]T_0,t]} e^{\frac{s-t}{\lambda}} d\mu(s))dt.$$

Some elementary computations give

$$\frac{1}{\lambda} \int_{T_0}^{T} e^{-\frac{t-T_0}{\lambda}} \|b-c\|\, dt \le \|b-c\|$$

and

$$\begin{aligned}
\frac{1}{\lambda} \int_{T_0}^{T} \int_{]T_0,t]} e^{\frac{s-t}{\lambda}} d\mu(s)dt &= \frac{1}{\lambda} \int_{T_0}^{T} \int_{]T_0,T]} \mathbf{1}_{]T_0,t]}(s) e^{\frac{s-t}{\lambda}} d\mu(s)dt \\
&= \frac{1}{\lambda} \int_{]T_0,T]} \int_{T_0}^{T} \mathbf{1}_{]T_0,t]}(s) e^{\frac{s-t}{\lambda}} dt d\mu(s) \\
&= \frac{1}{\lambda} \int_{]T_0,T]} \int_{s}^{T} e^{\frac{s-t}{\lambda}} dt d\mu(s) \\
&= \int_{]T_0,T]} (1 - e^{\frac{s-T}{\lambda}}) d\mu(s) \le \mu(]T_0,T]).
\end{aligned}$$

The first desired inequality then follows. Now, consider any $T_0 \le s \le t \le T$. The restriction of $U_{\lambda,b}(\cdot)$ to $[s,t]$ is the (unique) solution of the Cauchy problem

$$\begin{cases} \dot{\xi}(\tau) = -\frac{1}{2\lambda} \nabla d^2_{C(\tau)}(\xi(\tau)) & \text{a.e. } \tau \in [s,t] \\ \xi(s) = U_{\lambda,b}(s). \end{cases}$$

Hence, we can apply the above study to get

$$\int_{s}^{t} \left\| \dot{U}_{\lambda,b}(\tau) \right\| d\tau \le \left\| U_{\lambda,b}(s) - \mathrm{proj}_{C(s)}(U_{\lambda,b}(s)) \right\| + \mu(]s,t]).$$

The proof is then complete. ☐

From now on, we assume that $b = a \in C(T_0)$ (in particular, note that $c := \mathrm{proj}_{C(T_0)}(b) = a$) and we set $U_{\overline{\lambda}}(\cdot) := U_{\overline{\lambda},b}(\cdot)$ for every real $\overline{\lambda} > 0$.

Consider any sequence $(\lambda_n)_{n\geq 1}$ of positive reals with $\lambda_n \downarrow 0$. Lemma 4.8 entails in particular that $(U_{\lambda_n}(\cdot))_{n\geq 1}$ is uniformly bounded in norm and in variation. Indeed, keeping in mind that $U_{\lambda_n}(T_0) = a$ for every integer $n \geq 1$, we first observe that

$$\|U_{\lambda_n}(t)\| \leq \|a\| + \mu(]T_0, T]) =: M \quad \text{for all } n \in \mathbb{N}. \tag{4.23}$$

To see that the variation of $U_{\lambda_n}(\cdot)$ (with $n \in \mathbb{N}$) is uniformly bounded, it suffices to consider any subdivision $\sigma := (\tau_0, \ldots, \tau_p)$ of $I = [T_0, T]$ (where $p \geq 1$ is an integer) and to write

$$\begin{aligned}\sum_{j=0}^{p-1} \|U_{\lambda_n}(\tau_{j+1}) - U_{\lambda_n}(\tau_j)\| &\leq \sum_{j=0}^{p-1} \int_{\tau_j}^{\tau_{j+1}} \left\|\dot{U}_{\lambda_n}(t)\right\| dt \\ &\leq \int_{T_0}^{T} \left\|\dot{U}_{\lambda_n}(t)\right\| dt \leq \mu(]T_0, T]).\end{aligned}$$

Consequently, we have

$$\mathrm{var}(U_{\lambda_n}; [T_0, T]) \leq \mu(]T_0, T]) \quad \text{for all } n \in \mathbb{N}.$$

Applying Theorem 2.4, we may suppose without loss of generality that there is a mapping $U(\cdot) : I \to \mathcal{H}$ with bounded variation such that

$$U_{\lambda_n}(t) \overset{w}{\to} U(t) \quad \text{for all } t \in I. \tag{4.24}$$

The latter weak pointwise convergence combined with the weak lower semicontinuity of $\|\cdot\|$ entails through (4.23)

$$\|U(t)\| \leq M \quad \text{for all } t \in I. \tag{4.25}$$

We are going to highlight certain properties of $U(\cdot)$ and its right-continuous envelope $U^+(\cdot)$.

Lemma 4.9. *Assume that the function* $\mathrm{var}(C; [T_0, \cdot])$ *is continuous at some point* $\overline{t} \in [T_0, T]$. *Then, one has* $U(\overline{t}) \in C(\overline{t}) \cap M\mathbb{B}$ *as well as the equalities*

$$\lim_{\lambda \downarrow 0} \left\|U_\lambda(\overline{t}) - \mathrm{proj}_{C(\overline{t})}(U_\lambda(\overline{t}))\right\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \left\|U_{\lambda_n}(\overline{t}) - U(\overline{t})\right\| = 0.$$

Proof. If $\overline{t} = T_0$, there is nothing to establish since $a \in C(T_0)$. Then, suppose that $\overline{t} > T_0$. Since $\overline{t}$ is a continuity point of $v(\cdot)$, it is clear that $\mu(\{\overline{t}\}) = 0$. This and the first inequality of Lemma 4.7 furnish for every real $\lambda > 0$

$$\left\|U_\lambda(\overline{t}) - \mathrm{proj}_{C(\overline{t})}(U_\lambda(\overline{t}))\right\| \leq \int_{]T_0, \overline{t}[} e^{\frac{s-\overline{t}}{\lambda}} d\mu(s).$$

Then, it suffices to apply Lebesgue dominated convergence theorem to justify the first claimed equality. Now let us write for any integer $n \geq 1$ and $h \in \mathcal{H}$

$$\begin{aligned}\left|\langle p_n(\bar{t}), h\rangle - \langle U(\bar{t}), h\rangle\right| &\leq \left|\langle p_n(\bar{t}) - U_{\lambda_n}(\bar{t}), h\rangle\right| + \left|\langle U_{\lambda_n}(\bar{t}) - U(\bar{t}), h\rangle\right| \\ &\leq \left\|p_n(\bar{t}) - U_{\lambda_n}(\bar{t})\right\| \|h\| + \left|\langle U_{\lambda_n}(\bar{t}) - U(\bar{t}), h\rangle\right|,\end{aligned}$$

where $p_n(\bar{t}) := \mathrm{proj}_{C(\bar{t})}(U_{\lambda_n}(\bar{t}))$. Hence, we have $p_n(\bar{t}) \stackrel{w}{\to} U(\bar{t})$. This convergence property, the inequality (4.25) and the fact that $(p_n(\bar{t}))_{n\geq 1}$ is a sequence of the (weakly) closed convex $C(\bar{t})$ give

$$U(\bar{t}) \in C(\bar{t}) \cap M\mathbb{B}.$$

Thanks to our ball-compactness assumption, we know that $C(\bar{t}) \cap M\mathbb{B}$ is (strongly) compact, so the weak convergence $p_n(\bar{t}) \stackrel{w}{\to} U(\bar{t})$ holds for the strong topology, i.e., $p_n(\bar{t}) \to U(\bar{t})$. It remains to combine this with the strong convergence $p_n(\bar{t}) - U_{\lambda_n}(\bar{t}) \to 0$ established above to complete the proof. □

Through a density argument, the latter lemma entails the following one.

Lemma 4.10. *For every $t \in I$, one has*

$$U^+(t) \in C(t).$$

Proof. By continuity of $v(\cdot) := \mathrm{var}(C; [T_0, \cdot])$ at T by assumption, Lemma 4.9 gives $U(T) \in C(T)$, or equivalently $U^+(T) \in C(T)$. Now fix any $t \in [T_0, T[$. By virtue of the fact that the function $v(\cdot)$ is continuous on a dense set of $[T_0, T]$, we can choose a sequence $(t_n)_{n\in\mathbb{N}}$ of $[T_0, T[$ such that $t_n \downarrow t$ with $v(\cdot)$ continuous at t_n for every $n \in \mathbb{N}$. Using Lemma 4.9, we know that $U(t_n) \in C(t_n)$ for every $n \in \mathbb{N}$, so

$$d_{C(t)}(U(t_n)) \leq \mathrm{haus}(C(t_n), C(t)) \leq \mu(]t, t_n]) \quad \text{for all } n \in \mathbb{N}.$$

Passing to the limit yields

$$d_{C(t)}(U^+(t)) = \lim_{n\to+\infty} d_{C(t)}(U(t_n)) \leq \lim_{n\to+\infty} \mu(]t, t_n]) = 0.$$

It remains to invoke the closedness property of $C(t)$ to obtain the desired inclusion. □

We control now the variation of the mapping $U^+(\cdot)$.

Lemma 4.11. *For every $T_0 \leq s \leq t \leq T$, the inequality*

$$\left\|U^+(t) - U^+(s)\right\| \leq \mu(]s, t])$$

holds, so the measures dU^+ as well as $|dU^+|$ are absolutely continuous with respect to μ.

Further, at every $\tau \in [T_0, T]$ where $\mathrm{var}(C; [T_0, \cdot])$ *is continuous one has*

$$U^+(\tau) = U(\tau).$$

Proof. Let any $T_0 \le s \le t \le T$. If $s = t$, the inequality is trivial, so assume that $s < t$. Choose two sequences $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ with

$$s \le s_k < t \le t_k$$

along with the convergences $t_k \to t$ and $s_k \to s$ and such that $v(\cdot)$ is continuous at each t_k and s_k for all $k \in \mathbb{N}$. Fix for a moment any $k \in \mathbb{N}$. By Lemma 4.8, we have for all $n \in \mathbb{N}$,

$$\|U_{\lambda_n}(t_k) - U_{\lambda_n}(s_k)\| \le \left\|U_{\lambda_n}(s_k) - \mathrm{proj}_{C(s_k)}(U_{\lambda_n}(s_k))\right\| + \mu(]s_k, t_k]).$$

Hence, letting $n \to +\infty$ (see Lemma 4.9)

$$\|U(t_k) - U(s_k)\| \le \mu(]s_k, t_k]).$$

Passing to the limit as $k \to +\infty$ gives the desired inequality, that is

$$\left\|U^+(t) - U^+(s)\right\| \le \mu(]s, t]). \tag{4.26}$$

Now, assume that $v(\cdot)$ is continuous at $\tau \in [T_0, T]$. We may assume that $\tau < T$. Let a sequence $(\tau_k)_{k\in\mathbb{N}}$ of $]\tau, T]$ with $\tau_k \downarrow \tau$ and such that $v(\cdot)$ is continuous at each τ_k for all $k \in \mathbb{N}$. Following the development above, we may write

$$\|U(\tau_k) - U(\tau)\| \le \mu(]\tau, \tau_k]) \quad \text{for all}\, k \in \mathbb{N}.$$

Letting $k \to +\infty$ yields $\|U^+(\tau) - U(\tau)\| \le 0$, i.e.,

$$U^+(\tau) = U(\tau).$$

Since $v(\cdot) := \mathrm{var}(C; [T_0, \cdot])$ is continuous at the left end-point T_0 by its right continuity assumption, we obtain in particular that $U^+(T_0) = U(T_0)$. This and (4.26) give that $\|dU^+([s, t])\| \le \mu([s, t])$ for all $s, t \in [T_0, T]$ with $s \le t$. It results that both measures dU^+ and $|dU^+|$ are absolutely continuous with respect to μ, and the proof is complete. □

As seen in the above proof of Lemma 4.11 we note that

$$U^+(T_0) = U(T_0) = a, \tag{4.27}$$

that is, $U(\cdot)$ is continuous at the left end-point T_0.

Lemma 4.12. *One has the following equalities*

$$U^-(t) = U(t) \quad \textit{and} \quad \lim_{n\to+\infty} \|U_{\lambda_n}(t) - U(t)\| = 0 \quad \textit{for all}\, t \in I.$$

Proof. Fix any $t \in I$. By virtue of (4.24), it suffices to show that

$$\lim_{n\to+\infty} \left\|U_{\lambda_n}(t) - U^-(t)\right\| = 0.$$

If $t = T_0$, there is nothing to prove since $U^-(T_0) = U(T_0)$ and

$$U_{\lambda_n}(T_0) = U(T_0) \quad \text{for all}\, n \ge 1.$$

So, assume that $t \in]T_0, T]$. Let any real $\varepsilon > 0$. Choose any $s \in]T_0, t[$ such that $v(\cdot)$ is continuous at s and satisfying

$$\mu(]s, t[) \le \frac{\varepsilon}{4}. \tag{4.28}$$

By virtue of Lemma 4.9, pick any integer $N \geq 1$ with

$$\|U_{\lambda_n}(s) - U(s)\| \leq \frac{\varepsilon}{4} \quad \text{and} \quad \left\|U_{\lambda_n}(s) - \mathrm{proj}_{C(s)}(U_{\lambda_n}(s))\right\| \leq \frac{\varepsilon}{4} \tag{4.29}$$

for all integer $n \geq N$. Let any integer $n \geq N$ and any real $t' \in [s, t[$. By Lemma 4.8, we see that

$$\begin{aligned}\left\|U_{\lambda_n}(t') - U_{\lambda_n}(s)\right\| &\leq \left\|U_{\lambda_n}(s) - \mathrm{proj}_{C(s)}(U_{\lambda_n}(s))\right\| + \mu(]s, t'])\\ &\leq \frac{\varepsilon}{4} + \mu(]s, t[) \leq \frac{\varepsilon}{2}.\end{aligned}$$

Combining the latter inequality with (4.29), Lemma 4.11 and (4.28)

$$\begin{aligned}\left\|U_{\lambda_n}(t') - U^+(t')\right\| &\leq \left\|U_{\lambda_n}(t') - U_{\lambda_n}(s)\right\|\\ &\quad + \left\|U_{\lambda_n}(s) - U^+(s)\right\| + \left\|U^+(s) - U^+(t')\right\|\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \mu(]s, t'])\\ &\leq \frac{3\varepsilon}{4} + \mu(]s, t[) \leq \varepsilon.\end{aligned}$$

Letting $t' \uparrow t$ yields

$$\left\|U_{\lambda_n}(t) - U^-(t)\right\| \leq \varepsilon.$$

Consequently, we have $\lim\limits_{n \to +\infty} U_{\lambda_n}(t) = U^-(t)$ and the proof is complete. $\square$

With the above result at hand and the inclusion $U^+(t) \in C(t)$ for every $t \in I$, we can derive that $U^+(t)$ is the metric projection of $U(t)$ onto $C(t)$ for every $t \in I$.

Lemma 4.13. *For every $t \in I$, one has*

$$U^+(t) = \mathrm{proj}_{C(t)}(U(t)).$$

Proof. Let $t \in I$. If $t = T$, there is nothing to prove according to the inclusion provided by Lemma 4.10 and the convention $U^+(T) := U(T)$. So, assume that $t < T$. Let $t' \in [T_0, T[$ with $t' > t$. From Lemma 4.8, we have for every integer $n \geq 1$,

$$\left\|U_{\lambda_n}(t') - U_{\lambda_n}(t)\right\| \leq d_{C(t)}(U_{\lambda_n}(t)) + \mu(]t, t']).$$

Taking the limit as $n \to +\infty$ (thanks to Lemma 4.12) gives

$$\left\|U(t') - U(t)\right\| \leq d_{C(t)}(U(t)) + \mu(]t, t'])$$

and letting $t' \downarrow t$ yields to

$$\left\|U^+(t) - U(t)\right\| \leq d_{C(t)}(U(t)).$$

Putting the latter inequality and Lemma 4.10 together ensures the equality

$$U^+(t) = \mathrm{proj}_{C(t)}(U(t)).$$

$\square$

In order to show that $U^+(\cdot)$ is the solution of the Moreau's sweeping process (4.3) with initial condition $U^+(T_0) = a$, the following lemma is needed.

Lemma 4.14. *Let $t_1, t_2 \in I$ with $t_1 < t_2$, $\phi : [t_1, t_2] \to \mathcal{H}$ be a right-continuous with bounded variation selection of $C(\cdot)$. Then, one has*

$$\int_{[t_1,t_2[} \left\langle \phi(\tau), \frac{dU}{|dU|}(\tau) \right\rangle |dU|\,(\tau) \geq \frac{1}{2}(\|U(t_2)\|^2 - \|U(t_1)\|^2).$$

Proof. Fix any integer $n \geq 1$ and set $p_n(\cdot) := \operatorname{proj}_{C(\cdot)}(U_{\lambda_n}(\cdot))$. Since $U_{\lambda_n}(\cdot)$ satisfies the Cauchy problem (4.8), we have

$$\dot{U}_{\lambda_n}(t) = \frac{1}{\lambda_n}(p_n(t) - U_{\lambda_n}(t)) \quad \text{a.e. } t \in I.$$

From (2.1) we see that for almost every $t \in I$

$$-\dot{U}_{\lambda_n}(t) = \frac{1}{\lambda_n}\Big(U_{\lambda_n}(t) - \operatorname{proj}_{C(t)}(U_{\lambda_n}(t))\Big) \in N\big(C(t); \operatorname{proj}_{C(t)}(U_{\lambda_n}(t))\big),$$

hence in particular (see (2.3))

$$\left\langle \dot{U}_{\lambda_n}(t), \phi(t) - \operatorname{proj}_{C(t)}(U_{\lambda_n}(t)) \right\rangle \geq 0 \quad \text{a.e. } t \in I.$$

We derive from this

$$\int_{[t_1,t_2[} \left\langle \dot{U}_{\lambda_n}(s), \phi(s) \right\rangle ds \geq \int_{[t_1,t_2[} \left\langle \dot{U}_{\lambda_n}(s), p_n(s) \right\rangle ds. \tag{4.30}$$

On the other hand, it is clear that for almost every $t \in I$,

$$\left\langle p_n(t), \dot{U}_{\lambda_n}(t) \right\rangle = \left\langle p_n(t) - U_{\lambda_n}(t), \frac{1}{\lambda_n}(p_n(t) - U_{\lambda_n}(t)) \right\rangle + \left\langle U_{\lambda_n}(t), \dot{U}_{\lambda_n}(t) \right\rangle,$$

so

$$\left\langle p_n(t), \dot{U}_{\lambda_n}(t) \right\rangle \geq \left\langle U_{\lambda_n}(t), \dot{U}_{\lambda_n}(t) \right\rangle.$$

Putting the latter inequality and (4.30) together, we arrive to

$$\int_{[t_1,t_2[} \left\langle \dot{U}_{\lambda_n}(s), \phi(s) \right\rangle ds \geq \int_{[t_1,t_2[} \left\langle U_{\lambda_n}(s), \dot{U}_{\lambda_n}(s) \right\rangle ds = \frac{1}{2}(\|U_{\lambda_n}(t_2)\|^2 - \|U_{\lambda_n}(t_1)\|^2).$$

Since U_{λ_n} is absolutely continuous, we know by (2.10) that

$$\int_{[t_1,t_2[} \left\langle \dot{U}_{\lambda_n}(s), \phi(s) \right\rangle ds = \int_{[t_1,t_2[} \left\langle \frac{dU_{\lambda_n}}{|dU_{\lambda_n}|}(s), \phi(s) \right\rangle |dU_{\lambda_n}|(s).$$

Further, Lemma 4.12 tells us that $U(\cdot)$ is left-continuous and $U_{\lambda_n}(t) \to U(t)$ for all $t \in I$. Then, we may apply Proposition 2.3 to obtain the desired inequality of the lemma. $\square$

We arrive now to the last one of the series of lemmas justifying Theorem 4.3.

Lemma 4.15. *The mapping $U^+(\cdot) : I \to \mathcal{H}$ is the right-continuous solution with bounded variation of the Moreau sweeping process*

$$\begin{cases} -dU^+ \in N(C(t); U^+(t)) \\ U^+(T_0) = a. \end{cases}$$

Proof. The right-continuous mapping $U^+(\cdot)$ satisfies the equality $U^+(T_0) = a$ by (4.27). Further, it is of bounded variation on I with $U^+(t) \in C(t)$ for every $t \in I$ (see Lemma 4.11 and Lemma 4.10) and dU^+ is also absolutely continuous with respect to μ according to Lemma 4.11. It remains to show

$$-\frac{dU^+}{d\mu}(t) \in N(C(t); U^+(t)) \quad \mu\text{-a.e. } t \in I.$$

By (2.8) we observe that $dU^+ = dU$ since $U^+(T_0) = U(T_0)$ by (4.27). Put $A_1 := \{\tau \in I : \mu(\{\tau\}) > 0\}$. For any $t \in A_1$, we observe that

$$-\frac{dU^+}{d\mu}(t) = -\frac{U^+(t) - U^-(t)}{\mu(\{t\})} = \frac{U(t) - \mathrm{proj}_{C(t)}(U(t))}{\mu(\{t\})},$$

thanks to Moreau-Valadier's equality (2.9) and to Lemma 4.12 and to Lemma 4.13. Hence, it remains to apply (2.1) and Lemma 4.13 to obtain

$$-\frac{dU^+}{d\mu}(t) \in N(C(t); \mathrm{proj}_{C(t)}(U(t))) = N(C(t); U^+(t)).$$

Now denote by A_0 the set of $\tau \in [T_0, T[$ with $\mu(\{\tau\}) = 0$ such that $\frac{dU^+}{d\mu}(\tau)$ exists and (2.9) holds true. Lemma 4.11 ensures that the mapping $U^+(\cdot)$ is continuous at every $\tau \in A_0$ since $\mu(\{\tau\}) = 0$ for any $\tau \in A_0$. This and (2.7) give in particular for any $\tau \in A_0$ with $\tau \neq T_0$ that $U^+(\tau) = (U^+)^-(\tau) = U^-(\tau)$, so $U^+(\tau) = U(\tau)$ since U is left-continuous (see Lemma 4.12). It ensues that

$$U^+(\tau) = U(\tau) \quad \text{for all } \tau \in A_0, \tag{4.31}$$

since $U^+(T_0) = U(T_0)$ by (4.27) as already said above. Fix any $t \in A_0$. Let $(\eta_n)_{n\geq 1}$ be a sequence of $]0, T - t]$ with $\eta_n \downarrow 0$ such that $U(\cdot)$ is continuous at each $t + \eta_n$ with $n \geq 1$. Fix any $x \in C(t)$ and (see Corollary 4.2) pick a right-continuous with bounded variation selection $\phi(\cdot) : [t, T] \to \mathcal{H}$ of $C(\cdot)$ with $\phi(t) = x$. Let any integer $n \geq 1$. Applying Lemma 4.14 gives

$$\begin{aligned} \int_{[t,t+\eta_n[} \left\langle \phi(\tau), \frac{dU}{|dU|}(\tau) \right\rangle |dU|(\tau) &\geq \frac{1}{2}(\|U(t+\eta_n)\|^2 - \|U(t)\|^2) \\ &= \frac{1}{2}\langle U(t+\eta_n) + U(t), U(t+\eta_n) - U(t)\rangle. \end{aligned}$$

Since $U(\cdot)$ is left-continuous at t (by Lemma 4.12) and continuous at $t + \eta_n$, we have

$$dU^+([t, t+\eta_n[) = dU([t, t+\eta_n[) = U(t+\eta_n) - U(t) = dU^+([t, t+\eta_n]).$$

Hence writing $\phi(\cdot) = x - (x - \phi(\cdot))$, we obtain

$$\begin{aligned}\left\langle x, dU^+[t, t+\eta_n] \right\rangle \geq & \frac{1}{2} \left\langle U(t+\eta_n) + U(t), dU^+[t, t+\eta_n] \right\rangle \\ & + \int_{[t, t+\eta_n[} \left\langle x - \phi(\tau), \frac{dU^+}{|dU^+|}(\tau) \right\rangle \left| dU^+ \right| (\tau).\end{aligned} \tag{4.32}$$

If for infinitely many $n \in \mathbb{N}$ we have $\mu([t, t+\eta_n]) = 0$, Lemma 4.11 tells us that $dU^+([t, t+\eta_n]) = 0$ for such integers n, hence $\frac{dU^+}{d\mu}(t) = 0$ according to the convention $\frac{0}{0} = 0$ in (2.6), so $-\frac{dU^+}{d\mu}(t) \in N(C(t); U^+(t))$. Suppose that $\mu([t, t+\eta_n]) > 0$ for large n, say for $n \geq n_0$. By Lemma 4.11 and by the equality $\phi(t) = x$, for any $n \geq n_0$ with $\delta_n := \sup\limits_{t' \in [t, t+\eta_n]} \|\phi(t) - \phi(t')\|$ we have

$$\frac{1}{\mu([t, t+\eta_n])} \left| \int_{[t, t+\eta_n[} \left\langle x - \phi(\tau), \frac{dU^+}{|dU^+|}(\tau) \right\rangle \left| dU^+ \right| (\tau) \right| \leq \delta_n.$$

Clearly, by the right-continuity of ϕ at t we have $\delta_n \to 0$ as $n \to +\infty$. Dividing both members of (4.32) by $\mu([t, t+\eta_n])$ and making $n \to +\infty$ give $\left\langle x - \frac{1}{2}(U^+(t) + U(t)), \frac{dU^+}{d\mu}(t) \right\rangle \geq 0$, or equivalently by (4.31)

$$\left\langle x - U^+(t), \frac{dU^+}{d\mu}(t) \right\rangle \geq 0.$$

Since $x \in C(t)$ is arbitrary, this means (see (2.3)) that

$$-\frac{dU^+}{d\mu}(t) \in N(C(t); U^+(t)).$$

Finally, noting that $T \in A_1$ when $\mu(\{T\}) > 0$, we see that $\mu(I \setminus (A_1 \cup A_0)) = 0$ if either $\mu(\{T\}) > 0$ or $\mu(\{T\}) = 0$. This finishes the proof of the lemma. $\square$

The proof of Theorem 4.3 is then achieved. $\square$

Remark 4.16. It is clear (see Lemma 4.9) that we can replace the ball-compactness assumption on the moving set $C(\cdot)$ by the compactness of each $C(t) \cap M\mathbb{B}$ (with $t \in I$) where $M := \|a\| + \mu(]T_0, T])$. $\square$

Remark 4.17. Besides the pointwise convergence of $(U_\lambda(\cdot))_{\lambda > 0}$ as $\lambda \downarrow 0$ to $U^-(\cdot)$, it is known that the family of projections $\left(\mathrm{proj}_{C(\cdot)}(U_\lambda(\cdot))\right)_{\lambda > 0}$ converges uniformly to $U^+(\cdot)$ as $\lambda \downarrow 0$. It should be noted that $(U_\lambda(\cdot))_{\lambda > 0}$ also converges in the sense of *filled-in graphs* to $U^+(\cdot)$ as $\lambda \downarrow 0$. The definition of such a graph convergence and the above mentioned results can be found in the paper [30] or in the monograph [31, Theorem 5.1] by M.D.P. Monteiro Marques. $\square$

5. Regularization under prox-regularity and Lipschitz continuity

The regularization of nonconvex sweeping process began with Thibault's paper [48] under the prox-regularity of the sets $C(t)$ and the Lipschitz continuity of the multimapping $C : I \rightrightarrows \mathcal{H}$. The main theorem in [48] can be stated as follows. Recall that for any $a \in \mathcal{H}$, one sets $B(a, r) := \mathcal{H} =: B[a, r]$ whenever $r = +\infty$.

Theorem 5.1 (Thibault, [48])**.** *Let $C : [T_0, T] = I \rightrightarrows \mathcal{H}$ be a multimapping whose values are r-prox-regular for some extended real $r \in]0, +\infty]$ and let $a \in C(T_0)$. Assume that this multimapping is Lipschitz continuous in the sense that there exists a real $\kappa \geq 0$ such that*

$$\operatorname{haus}(C(s), C(t)) \leq \kappa |s - t| \quad \textit{for all } s, t \in I. \tag{5.1}$$

Let θ be a positive real number such that $\theta < r/(3\kappa)$.

Then, for any $\lambda \in]0, \kappa^{-1} r[$, the (classical) differential equation over $[T_0, T_0 + \theta] \times B(a, \frac{r}{3})$

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda} \nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(T_0) = a \end{cases}$$

is well defined and has a unique solution $u_\lambda(\cdot)$ on $[T_0, T_0 + \theta]$, and the family $(u_\lambda(\cdot))_{0 < \lambda < \kappa^{-1} r}$ converges uniformly on $[T_0, T_0 + \theta]$ as $\lambda \downarrow 0$ to a solution of the differential inclusion sweeping process

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \textit{a.e. } t \in I \\ u(t) \in C(t) \quad \textit{for all } t \in I \\ u(T_0) = a \in C(T_0). \end{cases}$$

M. Sene and L. Thibault considered later in [46] the situation when an external force is present through a mapping f depending both on time and on state. They showed under the prox-regularity of the sets $C(t)$ and the Lipschitz continuity of the multimapping $C(\cdot)$ that a regularization process can also be provided for the dynamical system

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) \quad \text{a.e. } t \in I \\ u(t) \in C(t) \quad \text{for all } t \in I \\ u(T_0) = a \in C(T_0), \end{cases} \tag{5.2}$$

whenever the mapping $f(t, \cdot)$ is Lipschitz continuous and bounded. Their extension of Theorem 5.1 to such a situation is the following.

Theorem 5.2 (Sene-Thibault, [46])**.** *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$ which is Lipschitz continuous in the sense that there exists a real $\kappa \geq 0$ such that*

$$\operatorname{haus}(C(s), C(t)) \leq \kappa |s - t| \quad \textit{for all } t \in I. \tag{5.3}$$

Let $a \in C(T_0)$ and let $f : I \times B(a, r/3) \to \mathcal{H}$ be a mapping which is Bochner measurable with respect to $t \in I$ and such that:

(i) *there exists a real* $\beta > 0$ *such that*

$$\|f(t,x)\| \le \beta \quad \textit{for all}\, t \in I \textit{ and}\, x \in B(a, r/3);$$

(ii) *there exists* $k \in \mathbb{R}_+$ *such that for all* $t \in I$ *and for all* $x, y \in B(a, r/3)$,

$$||f(t,x) - f(t,y)|| \le k||x - y||.$$

Let θ *be a positive real number such that* $\theta < \frac{r}{3(2\beta+\kappa)}$ *and let the extended real* $\lambda_r := r/(\beta + \kappa)$.

Under the above assumptions, for any $\lambda \in]0, \lambda_r[$, *the differential equation over* $[T_0, T_0 + \theta] \times B(a, r/3)$

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) - f(t, u_\lambda(t)) \\ u_\lambda(T_0) = a \end{cases}$$

is well defined and has a unique solution u_λ *on* $[T_0, T_0 + \theta]$, *and the family* $(u_\lambda)_{0<\lambda<\lambda_r}$ *converges uniformly on* $[T_0, T_0 + \theta]$ *as* $\lambda \downarrow 0$ *to a solution of the dynamical differential inclusion*

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) \quad \textit{a.e.}\, t \in I \\ u(t) \in C(t) \quad \textit{for all}\, t \in I \\ u(T_0) = a \in C(T_0), \end{cases}$$

Further, this solution stays in $B(a, r/3)$ *and the solution inside this ball is unique.*

If the mapping f *is defined on* $I \times \mathcal{H}$ *and satisfies the assumptions (i) and (ii) for all* $t \in I$ *and* $x, y \in \mathcal{H}$, *then dividing* $I = [T_0, T]$ *into a finite number of intervals with length less than or equal to* θ *yields the existence of a unique solution* $u(\cdot)$ *of the above differential inclusion over* I. *Further, one has*

$$\|\dot{u}(t)\| \le 2\beta + \kappa \quad \textit{a.e.}\, t \in I.$$

6. Regularization under prox-regularity and bounded truncated variation

As mentioned by A.A. Tolstonogov [50], there are some practical situations where an unbounded moving set does not fulfill the control (5.1)-(5.3). Roughly speaking, a possible and efficient way to relax such an assumption consists in replacing the classical Hausdorff-Pompeiu distance $\mathrm{haus}(\cdot,\cdot)$ by the truncated one $\widehat{\mathrm{haus}}_\rho(\cdot,\cdot)$. It is worth pointing out that only very few studies in that direction have been achieved. We refer to [50, 18, 2] for the convex setting and to [49] for the prox-regular one. Except [50] each of these papers makes a great use of the famous Moreau's catching-up algorithm.

The aim of this section is to show how the family of solutions of suitable regularizations of the sweeping differential inclusion of (5.2) converges to the solution of this differential inclusion when $C(t)$ is a prox-regular moving set of the general Hilbert space $\mathcal{H}$ with the bounded truncated variation assumption

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) := \sup_{x \in \rho\mathbb{B}} |d(x, C(s)) - d(x, C(t))| \le |v(t) - v(s)|,$$

where $v(\cdot)$ is a Lipschitz continuous mapping. The main difference with the previous section is that here the truncated Hausdorff distance $\widehat{\mathrm{haus}}_\rho(C(s), C(t))$ is involved instead of the usual Hausdorff-Pompeiu distance $\mathrm{haus}(C(s), C(t))$. The convergence under the significantly weakened truncated Lipschitz variation assumption is the challenge of the section.

Let us start with the following crucial lemma which is in the line of Lemma 2.5. It ensures that for a given prox-regular moving set $C(t) \subset \mathcal{H}$ the mapping $t \mapsto \mathrm{proj}_{C(t)}(x)$ is continuous at $\bar{t} \in I$ for x sufficiently close to $C(t)$ with t near $\bar{t}$. It is an adaptation of the proof of [5, Theorem 2] (see also [44, Theorem 2]).

Lemma 6.1. *Let S_1, S_2 be r-prox-regular subsets of $\mathcal{H}$ with $r \in]0, +\infty[$, $\gamma \in]0, 1[$, $\rho \in]0, +\infty[$, $x \in \rho\mathbb{B} \cap U_{r\gamma}(S_1) \cap U_{r\gamma}(S_2)$. If $\mathrm{haus}_{r\gamma+\rho}(S_1, S_2) \le r$, then one has*

$$\left\| \mathrm{proj}_{S_1}(x) - \mathrm{proj}_{S_2}(x) \right\| \le \left(\frac{2\gamma r}{1-\gamma} \mathrm{haus}_{r\gamma+\rho}(S_1, S_2) \right)^{1/2}.$$

Proof. Set $s := r\gamma + \rho$ and $h := \mathrm{haus}_s(S_1, S_2)$. Assume that $h \le r$. For each $i \in \{1, 2\}$, $\mathrm{Proj}_{S_i}(x)$ is reduced to a singleton $\{x_i\}$ (thanks to $x \in U_{r\gamma}(S_i)$ and the fact that S_i is r-prox-regular). Observe that

$$\|x_2\| \le \|x_2 - x\| + \|x\| = d_{S_2}(x) + \|x\| < r\gamma + \rho = s,$$

hence $x_2 \in S_2 \cap s\mathbb{B}$. It follows that

$$d_{S_1}(x_2) \le \sup_{x \in S_2 \cap s\mathbb{B}} d_{S_1}(x) \le h.$$

We claim that

$$2 \langle x - x_1, x_2 - x_1 \rangle \le \gamma(\|x_1 - x_2\|^2 + 2rh).$$

We may assume that $x \neq x_1$, hence $x \notin S_1$. In particular, we have $x \in U_r(S_1) \setminus S_1$, so we can apply Theorem 2.8(e) to get

$$x_1 = \mathrm{proj}_{S_1}\Big(x_1 + \frac{t(x - x_1)}{\|x - x_1\|}\Big) \quad \text{for all } t \in [0, r[.$$

Note that for all $z \in S_1$, for all $t \in [0, r[$,

$$\left\| x_1 + \frac{t(x - x_1)}{\|x - x_1\|} - x_2 \right\| \ge \left\| x_1 + \frac{t(x - x_1)}{\|x - x_1\|} - z \right\| - \|x_2 - z\| \ge t - \|x_2 - z\|.$$

Passing to the supremum yields for all $t \in [0, r[$,

$$\left\| x_1 + \frac{t(x - x_1)}{\|x - x_1\|} - x_2 \right\| \ge \sup_{z \in S_1} (t - \|x_2 - z\|) = t - d_{S_1}(x_2) \ge t - h.$$

Taking the limit as $t \uparrow r$ in both sides of the latter inequality, we get

$$\left\| x_1 + \frac{r(x - x_1)}{\|x - x_1\|} - x_2 \right\| \geq r - h.$$

We deduce from this (thanks to the inequality $r \geq h$)

$$\|x_1 - x_2\|^2 + \frac{2r}{\|x - x_1\|} \langle x - x_1, x_1 - x_2 \rangle + r^2 \geq r^2 - 2rh,$$

or equivalently

$$2r \langle x - x_1, x_2 - x_1 \rangle \leq \|x - x_1\| \left(\|x_1 - x_2\|^2 + 2rh \right).$$

Keeping in mind that $d_{S_1}(x) = \|x - x_1\| < r\gamma$, we obtain

$$2 \langle x - x_1, x_2 - x_1 \rangle \leq \gamma(\|x_1 - x_2\|^2 + 2rh),$$

which is the inequality claimed above. In the same way, we show

$$2 \langle x - x_2, x_1 - x_2 \rangle \leq \gamma(\|x_1 - x_2\|^2 + 2rh).$$

Adding the two latter inequalities, we have

$$\|x_1 - x_2\|^2 \leq \gamma(\|x_1 - x_2\|^2 + 2rh)$$

or equivalently

$$\|x_1 - x_2\|^2 \leq \frac{2r\gamma h}{1 - \gamma}.$$

The proof is then complete. □

As already said, the prox-regular moving set $C(\cdot)$ will be assumed to have a Lipschitz variation. The first result below provides the convergence to a local solution of (5.2). It is in the line of Theorem 5.1 and of Theorem 5.2.

Theorem 6.2. *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$ and $a \in C(T_0)$, and let $f : I \times B(a, \frac{r}{3}) \to \mathcal{H}$ be a mapping. Assume that:*

(i) *there exists a real $\beta > 0$ such that for all $t \in I$ and $x \in B(a, \frac{r}{3})$,*

$$\|f(t, x)\| \leq \beta;$$

(ii) *the mapping $f(\cdot, x)$ is Bochner measurable for each $x \in B(a, \frac{r}{3})$ and there exists $k \in \mathbb{R}_+$ such that for all $t \in I$ and for all $x_1, x_2 \in B(a, \frac{r}{3})$,*

$$\|f(t, x_1) - f(t, x_2)\| \leq k \|x_1 - x_2\|;$$

(iii) *there exist a function $v : I \to \mathbb{R}$ which is κ-Lipschitz continuous for some real $\kappa \geq 0$ on I and an extended real $\rho \geq \|a\| + r$ such that for all $s, t \in I$ with $s \leq t$,*

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) := \sup_{x \in \rho\mathbb{B}} |d(x, C(s)) - d(x, C(t))| \leq |v(t) - v(s)|. \tag{6.1}$$

Let θ be any positive real with $\theta \le T - T_0$ and satisfying $\theta < \frac{r}{3(2\beta+\kappa)}$. Then, for each real $\lambda > 0$, there exists one and only one mapping $u_\lambda(\cdot) : [T_0, T_0+\theta] \to B(a, \frac{r}{3})$ solution of the regularized differential equation

$$\begin{cases} -\dot{u}_\lambda(t) = \frac{1}{2\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) + f(t, u_\lambda(t)) & a.e.\ t \in [T_0, T_0+\theta], \\ u_\lambda(T_0) = a. \end{cases}$$

This family $(u_\lambda(\cdot))_{\lambda>0}$ converges uniformly on $[T_0, T_0+\theta]$ when $\lambda \downarrow 0$ to a $(2\beta+\kappa)$-Lipschitz continuous mapping $u : [T_0, T_0+\theta] \to B(a, \frac{r}{3})$ solution of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) & a.e.\ t \in [T_0, T_0+\theta], \\ u(t) \in C(t) & \text{for all}\, t \in [T_0, T_0+\theta], \\ u(T_0) = a. \end{cases}$$

Furthermore, one has the error estimation for every real $\lambda > 0$,

$$\sup_{t\in[T_0,T_0+\theta]} \|u_\lambda(t) - u(t)\|^2 \le 2\lambda(\beta+\kappa)^2 \int_{T_0}^{T_0+\theta} \exp\big(K(T_0+\theta-s)\big)ds,$$

where $K := 2[9(\frac{\beta+\kappa}{r}) + k]$, and

$$\|\dot{u}(t) + f(t, u(t))\| \le \beta + \kappa \quad a.e.\ t \in [T_0, T_0+\theta].$$

Proof. Fix any positive real θ with $\theta \le T - T_0$ and $\theta < \frac{r}{3(2\beta+\kappa)}$. Observe that for every Lebesgue measurable set $A \subset [T_0, T]$ with $\mathcal{L}(A) \le \theta$ (where we recall that $\mathcal{L}$ stands for the Lebesgue measure on I) we have

$$\int_A |\dot{v}(s)|\, d(s) \le \kappa\mathcal{L}(A) \le \kappa\theta < \frac{r}{3}. \tag{6.2}$$

Thanks to the equality $d(a, C(T_0)) = 0$, the inequality $\rho \ge \|a\|$, (6.1) and (6.2), observe that for all $x \in B[a, \frac{r}{3}]$, for all $t \in [T_0, T_0+\theta]$,

$$\begin{aligned} d(x, C(t)) &\le d(x, C(t)) - d(a, C(t)) + d(a, C(t)) - d(a, C(T_0)) \\ &\le \|x-a\| + |v(t) - v(T_0)| \\ &\le \frac{r}{3} + \left|\int_{T_0}^{t} \dot{v}(s)ds\right| \\ &\le \frac{r}{3} + \int_{T_0}^{T_0+\theta} |\dot{v}(s)|\, ds < \frac{r}{3} + \frac{r}{3} = \frac{2r}{3}, \end{aligned}$$

which yields

$$x \in U_{\frac{2}{3}r}(C(t)) \quad \text{for all}\, (t,x) \in [T_0, T_0+\theta] \times B[a, \frac{r}{3}]. \tag{6.3}$$

The latter inclusion along with the r-prox-regularity of each $C(t)$ with $t \in [T_0, T_0+\theta]$ allows us (thanks to Theorem 2.8) to consider the mapping $h : [T_0, T_0+\theta] \times B[a, \frac{r}{3}] \to \mathcal{H}$ defined by

$$h(t,x) := x - \mathrm{proj}_{C(t)}(x) \quad \text{for all}\, (t,x) \in [T_0, T_0+\theta] \times B[a, \frac{r}{3}].$$

Using Theorem 2.8 again, we have

$$h(t,x) = \nabla(\frac{1}{2}d^2_{C(t)})(x) \quad \text{for all } (t,x) \in [T_0, T_0+\theta] \times B[a, \frac{r}{3}]$$

and

$$\left\| \mathrm{proj}_{C(t)}(x_1) - \mathrm{proj}_{C(t)}(x_2) \right\| \leq \frac{1}{1-\frac{2}{3}} \|x_1 - x_2\| = 3 \|x_1 - x_2\|,$$

for all $t \in [T_0, T_0+\theta]$, for all $x_1, x_2 \in U_{\frac{2}{3}r}(C(t))$. In particular, this says that

$$h(t,\cdot) \text{ is } 3 - \text{Lipschitz on } B[a, \frac{r}{3}] \text{ for each } t \in [T_0, T_0+\theta]. \tag{6.4}$$

We continue the proof with a series of lemmas.

Lemma 6.3. *For each $x \in B[a, \frac{r}{3}]$, the mapping $(f+h)(\cdot, x)$ is Bochner integrable on $[T_0, T_0+\theta]$.*

Proof. Let $x \in B[a, \frac{r}{3}]$, $\bar{t} \in [T_0, T_0+\theta]$. Fix any sequence $(t_n)_{n\in\mathbb{N}}$ of $[T_0, T_0+\theta]$ with $t_n \to \bar{t}$. Let us distinguish two cases.
Case 1: $r = +\infty$. In such a case, $\rho = +\infty$ and $\widehat{\mathrm{haus}}_\rho(\cdot,\cdot) = \mathrm{haus}(\cdot,\cdot)$. Then, as a direct consequence of Lemma 2.5 and the convergence $\mathrm{haus}(C(t_n), C(\bar{t})) \to 0$, we get the continuity of $\mathrm{proj}_{C(\cdot)}(x)$ at $\bar{t}$.
Case 2: $r < +\infty$. Thanks to the inclusion (6.3), we have (with $\gamma := \frac{2}{3}$)

$$x \in \bigcap_{t\in[T_0,T_0+\theta]} U_{r\gamma}(C(t)).$$

Note that for $n \in \mathbb{N}$ sufficiently large, say $n \geq N$,

$$\widehat{\mathrm{haus}}_{\|a\|+r}(C(t_n), C(\bar{t})) \leq r,$$

by virtue of (2.11) and of the convergence $\widehat{\mathrm{haus}}_{\|a\|+r}(C(t_n), C(\bar{t})) \to 0$. Since $\|x\| < \|a\| + r \leq \rho$, we can apply Lemma 6.1 to get for every integer $n \geq N$

$$\begin{aligned}
\left\| \mathrm{proj}_{C(t_n)}(x) - \mathrm{proj}_{C(\bar{t})}(x) \right\| &\leq \Big(\frac{2\gamma r}{1-\gamma} \mathrm{haus}_{\|a\|+r}\big(C(t_n), C(\bar{t})\big)\Big)^{\frac{1}{2}} \\
&\leq \Big(\frac{2\gamma r}{1-\gamma} \widehat{\mathrm{haus}}_{\|a\|+r}\big(C(t_n), C(\bar{t})\big)\Big)^{\frac{1}{2}} \\
&\leq \Big(\frac{2\gamma r}{1-\gamma} \widehat{\mathrm{haus}}_{\rho}\big(C(t_n), C(\bar{t})\big)\Big)^{\frac{1}{2}} \\
&\leq \Big(\frac{2\gamma r}{1-\gamma} \left|v(t_n) - v(\bar{t})\right| \Big)^{\frac{1}{2}}
\end{aligned}$$

and this obviously entails that $\mathrm{proj}_{C(\cdot)}(x)$ is continuous at $\bar{t}$.
In both cases, $h(\cdot, x)$ is continuous at $\bar{t}$. The Bochner integrability of $(f + h)(\cdot, x)$ then follows. □

For each real $\lambda > 0$, let us consider the following differential equation over $[T_0, T_0 + \theta[\times B(a, \frac{r}{3})$

$$(E_\lambda) : \begin{cases} \dot{u}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u(t)) - f(t, u(t)) & \text{a.e. } t \in [T_0, T_0 + \theta[\\ u(T_0) = a. \end{cases}$$

The development above guarantees that for each real $\lambda > 0$, this differential equation has a unique solution $u_\lambda(\cdot)$ defined on its maximal interval of existence $[T_0, T_\lambda[\subset [T_0, T_0 + \theta[$.

For each real $\lambda > 0$, let us set

$$g_\lambda(t) := d(u_\lambda(t), C(t)) \quad \text{and} \quad z_\lambda(t) := -f(t, u_\lambda(t)) \quad \text{for all } t \in [T_0, T_\lambda[.$$

The property of this function $g_\lambda(\cdot)$ in Lemma 6.6 will use the following lemma.

Lemma 6.4. *Let S be a nonempty closed set of $\mathcal{H}$ and $z : I \to \mathcal{H}$ be a mapping. Let the function $\delta : I \to \mathbb{R}$ be defined by $\delta(t) := (1/2)d^2_S(z(t))$ for all $t \in I = [T_0, T]$. Then, at every $t \in [T_0, T[$ where z is derivable on the right, the right derivative $\dot{\delta}(t^+)$ exists and*

$$\dot{\delta}(t^+) = -\sigma\big(-\dot{z}(t^+); \partial(\frac{1}{2}d^2_S)(z(t))\big) = d_S(z(t))\sigma\big(-\dot{z}(t^+); \partial d_S(z(t))\big).$$

Proof. We already recalled in Proposition 2.1 that the function $\psi := -(1/2)d^2_S(\cdot)$ is tangentially regular on the whole space $\mathcal{H}$. Let $t \in I$ as in the statement (if any). Then writing for $s > 0$ small enough by the local Lipschitz property of $d^2_S(\cdot)$

$$s^{-1}[(-\delta)(t + s) - (-\delta)(t)] = s^{-1}[\psi(z(t) + s\dot{z}(t^+)) - \psi(z(t))] + \varepsilon(s)$$

(where $\varepsilon(s) \to 0$ as $s \downarrow 0$), we see that

$$-\dot{\delta}(t^+) = \psi'(z(t); \dot{z}(t^+)) = \psi^o(z(t); \dot{z}(t^+)) = \sigma\big(\dot{z}(t^+); \partial\psi(z(t))\big).$$

Since $\partial\psi(z(t)) = \partial((-1/2)d^2_S)(z(t)) = -(1/2)\partial(d^2_S)(z(t)) = -d_S(z(t))\partial d_S(z(t))$, we deduce that $-\dot{\delta}(t^+) = d_S(z(t))\sigma\big(-\dot{z}(t^+); \partial d_S(z(t))\big)$, which justifies the lemma. $\square$

The next lemma is stated in a form which is also useful for the next section.

Lemma 6.5. *Let $S(t)$ be a nonempty closed set of $\mathcal{H}$ for each $t \in I = [T_0, T]$ and let $\rho_0 \in]0, +\infty]$ be such that for any $s, t \in I$,*

$$\widehat{\text{haus}}_{\rho_0}(S(s), S(t)) = \sup_{x \in \rho_0\mathbb{B}} |d(x, S(s)) - d(x, S(t))| \le |v_0(s) - v_0(t)| \tag{6.5}$$

where $v_0 : I \to \mathbb{R}$ is some absolutely continuous function. Let J be a subinterval of I with nonempty interior and $z : J \to \rho_0\mathbb{B}$ be a locally absolutely continuous mapping. Let also $g(\cdot) : J \to \mathbb{R}_+$ be defined by

$$g(t) := d(z(t), S(t)) \quad \textit{for all}\, t \in J.$$

Then, g is locally absolutely continuous on J and at each $t \in \operatorname{int} J$ where g, z and v_0 are derivable (set whose complement in J is of null Lebesgue-measure) one has

$$\dot{g}(t)g(t) \le -\sigma\big(-\dot{z}(t), \partial(\frac{1}{2}d^2_{S(t)})(z(t))\big) + g(t)\,|\dot{v}_0(t)|\,.$$

Proof. First, observe that for all $t, s \in J$,

$$\begin{aligned} |g(t) - g(s)| &= |d(z(t), S(t)) - d(z(s), S(s))| \\ &\le |d(z(t), S(t)) - d(z(s), S(t))| + |d(z(s), S(t)) - d(z(s), S(s))| \\ &\le \|z(t) - z(s)\| + |v_0(t) - v_0(s)|\,, \end{aligned}$$

where the latter inequality is due to the inclusion $z(J) \subset \rho_0\mathbb{B}$ and (6.5). The function g is then locally absolutely continuous on J. Let us define the function $\varphi : I \times \mathcal{H} \to \mathbb{R}$ by

$$\varphi(t, x) := \frac{1}{2}d^2_{S(t)}(x) \quad \text{for all } (t, x) \in I \times \mathcal{H}.$$

Let τ_0, τ_1 be reals with $\tau_0 < \tau_1$ such that $\operatorname{int} J =]\tau_0, \tau_1[$. Fix any $\bar{t} \in \operatorname{int} J$ such that g, v and z are derivable at $\bar{t}$. For all $s \in]0, \tau_1 - \bar{t}[$ writing

$$\begin{aligned} &\frac{1}{2s}[g(\bar{t}+s)^2 - g(\bar{t})^2] \\ =&\frac{1}{s}[\varphi(\bar{t}+s, z(\bar{t}+s)) - \varphi(\bar{t}, z(\bar{t}))] \\ =&\frac{1}{s}[\varphi(\bar{t}+s, z(\bar{t}+s)) - \varphi(\bar{t}, z(\bar{t}+s))] + \frac{1}{s}[\varphi(\bar{t}, z(\bar{t}+s)) - \varphi(\bar{t}, z(\bar{t}))] \\ =&\frac{1}{2s}[d_{S(\bar{t}+s)}(z(\bar{t}+s)) - d_{S(\bar{t})}(z(\bar{t}+s))][d_{S(\bar{t}+s)}(z(\bar{t}+s)) + d_{S(\bar{t})}(z(\bar{t}+s))] \\ &+ \frac{1}{s}[\varphi(\bar{t}, z(\bar{t}+s)) - \varphi(\bar{t}, z(\bar{t}))] \end{aligned}$$

we note that

$$\begin{aligned} \frac{1}{2s}[g(\bar{t}+s)^2 - g(\bar{t})^2] \le \frac{1}{2s}\left|v_0(\bar{t}+s) - v_0(\bar{t})\right| &[d_{S(\bar{t}+s)}(z(\bar{t}+s)) + d_{S(\bar{t})}(z(\bar{t}+s))] \\ &+ \frac{1}{s}[\varphi(\bar{t}, z(\bar{t}+s)) - \varphi(\bar{t}, z(\bar{t}))]. \end{aligned}$$

Passing to the limit as $s \downarrow 0$ and using Lemma 6.4, we obtain

$$\dot{g}(\bar{t})g(\bar{t}) \le \left|\dot{v}_0(\bar{t})\right| g(\bar{t}) - \sigma\big(-\dot{z}(t), \partial(\frac{1}{2}d^2_{S(t)})(z(t))\big).$$

as desired. □

Applying the above lemma with $\rho_0 := \rho$, $S(\cdot) := C(\cdot)$ and noting (thanks to the fact that $C(\cdot)$ takes r-prox-regular values) for every $t \in I$ and $x \in \mathcal{H}$ with $d(x, C(t)) < r$ that $\partial(1/2)d^2_{C(t)}(x) = x - \operatorname{proj}_{C(t)}(x)$, we directly derive the following lemma.

Lemma 6.6. *Let J be a subinterval of $I = [T_0, T]$ with nonempty interior and $z : J \to \rho\mathbb{B}$ be a locally absolutely continuous mapping. Let also $g(\cdot) : J \to \mathbb{R}_+$ be defined by*

$$g(t) := d(z(t), C(t)) \quad \textit{for all}\, t \in J.$$

Then, $g(\cdot)$ is locally absolutely continuous on J. If in addition, $g(t) < r$ for all $t \in J$, then one has

$$\dot{g}(t)g(t) \le \left\langle \dot{z}(t), z(t) - \mathrm{proj}_{C(t)}(z(t)) \right\rangle + g(t)\,|\dot{v}(t)| \quad \textit{a.e.}\ t \in J.$$

With this lemma at hand, we can prove the following estimates for $g_\lambda(\cdot)$ and $\dot{g}_\lambda(\cdot)$.

Lemma 6.7. *For each real $\lambda > 0$, the function $g_\lambda : [T_0, T_\lambda[\to \mathbb{R}$ is locally absolutely continuous on $[T_0, T_\lambda[$ and*

$$\dot{g}_\lambda(t) \le \beta + |\dot{v}(t)| - \frac{1}{\lambda} g_\lambda(t) \le \beta + \kappa - \frac{1}{\lambda} g_\lambda(t) \quad \textit{a.e.}\ t \in [T_0, T_\lambda[.$$

Further, one has

$$g_\lambda(t) \le e^{-\frac{t}{\lambda}} \int_{T_0}^{t} (\beta + |\dot{v}(s)|) e^{\frac{s}{\lambda}}\, ds \le \lambda(\beta + \kappa) \quad \textit{for all}\, t \in [T_0, T_\lambda[.$$

Proof. Fix any real $\lambda > 0$. Since $u_\lambda(t) \in B(a, \frac{r}{3})$ for each $t \in [T_0, T_\lambda[$, we have (thanks to the choice of ρ and (6.3)) that

$$u_\lambda(t) \in \rho\mathbb{B} \quad \text{and} \quad d(u_\lambda(t), C(t)) < \frac{2r}{3} \quad \text{for all}\, t \in [T_0, T_\lambda[.$$

Applying Lemma 6.6, we get

$$\dot{g}_\lambda(t) g_\lambda(t) \le \left\langle \dot{u}_\lambda(t), u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t)) \right\rangle + g_\lambda(t)\,|\dot{v}(t)| \quad \text{a.e.}\ t \in [T_0, T_\lambda[.$$

On the other hand, from the definition of $u_\lambda(\cdot)$, we note that

$$\dot{u}_\lambda(t) = -\frac{1}{\lambda}[u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t))] + z_\lambda(t) \quad \text{a.e.}\ t \in [T_0, T_\lambda[,$$

so the latter inequality gives

$$\begin{aligned} \dot{g}_\lambda(t) g_\lambda(t) \le g_\lambda(t)\,|\dot{v}(t)| &- \frac{1}{\lambda} \left\langle u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t)), u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t)) \right\rangle \\ &+ \left\langle z_\lambda(t), u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t)) \right\rangle. \end{aligned}$$

Thanks to the equality valid for all $t \in [T_0, T_\lambda[$,

$$g_\lambda(t) = d(u_\lambda(t), C(t)) = \left\| u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t)) \right\|,$$

we can write

$$\dot{g}_\lambda(t) g_\lambda(t) \le g_\lambda(t)\,|\dot{v}(t)| - \frac{1}{\lambda} g_\lambda^2(t) + \|z_\lambda(t)\|\, g_\lambda(t) \quad \text{a.e.}\ t \in [T_0, T_\lambda[.$$

Fix any $t_0 \in]T_0, T_\lambda[$ where $\dot{g}_\lambda(t_0)$, $\dot{v}(t_0)$ and $\dot{u}_\lambda(t_0)$ exist and where the latter inequality holds. If $g_\lambda(t_0) > 0$, we have

$$\dot{g}_\lambda(t_0) \le |\dot{v}(t_0)| - \frac{1}{\lambda} g_\lambda(t_0) + \|z_\lambda(t_0)\|. \tag{6.6}$$

Assume that $g_\lambda(t_0) = 0$. We claim that $\dot{g}_\lambda(t_0) = 0$. Since $g_\lambda \ge 0$ and $\lim_{s\to 0} \frac{1}{s} g_\lambda(t_0 + s)$ exists, we have

$$\lim_{s\downarrow 0} \frac{1}{s} g_\lambda(t_0 + s) \ge 0 \quad \text{and} \quad \lim_{s\uparrow 0} \frac{1}{s} g_\lambda(t_0 + s) \le 0 \tag{6.7}$$

and this entails that $\dot{g}_\lambda(t_0) = 0$, so (6.6) still holds true. Then, we have established that for almost every $t \in [T_0, T_\lambda[$,

$$\begin{aligned} \dot{g}_\lambda(t) &\le |\dot{v}(t)| - \frac{1}{\lambda} g_\lambda(t) + \|z_\lambda(t)\| \\ &\le |\dot{v}(t)| - \frac{1}{\lambda} g_\lambda(t) + \beta \\ &\le \kappa - \frac{1}{\lambda} g_\lambda(t) + \beta. \end{aligned} \tag{6.8}$$

Applying the Gronwall lemma below with (6.8) and the equality $g_\lambda(T_0) = 0$, we get

$$g_\lambda(t) \le e^{-\frac{t}{\lambda}} \int_{T_0}^t (\beta + |\dot{v}(s)|) e^{\frac{s}{\lambda}} ds \quad \text{for all } t \in [T_0, T_\lambda[.$$

It remains to invoke the κ-Lipschitz property of $v(\cdot)$ to get

$$g_\lambda(t) \le e^{-\frac{t}{\lambda}} \int_{T_0}^t (\beta + |\dot{v}(s)|) e^{\frac{s}{\lambda}} ds \le \lambda(\beta + \kappa).$$

The proof is complete. □

Lemma 6.8 (Gronwall)**.** *Let $\varphi : I = [T_0, T] \to \mathbb{R}$ be an absolutely continuous function on I, $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ be Lebesgue integrable functions on I. If for almost every $t \in I$,*

$$\dot{\varphi}(t) \le b(t) + a(t)\varphi(t),$$

then for all $t \in I$,

$$\varphi(t) \le \varphi(T_0) \exp\left(\int_{T_0}^t a(s) ds \right) + \int_{T_0}^t b(\tau) \exp\left(\int_\tau^t a(s) ds \right) d\tau.$$

The next lemma provides a uniform boundedness of the family of derivatives $(\dot{u}(\cdot))_{\lambda > 0}$.

Lemma 6.9. *For each real $\lambda > 0$ and for almost every $t \in [T_0, T_\lambda[$, one has*

$$\|\dot{u}_\lambda(t) - z_\lambda(t)\| \le \beta + \kappa,$$

in particular

$$\|\dot{u}_\lambda(t)\| \le 2\beta + \kappa. \tag{6.9}$$

Proof. Fix any real $\lambda > 0$. By definition of $u_\lambda(\cdot)$, we have

$$\dot{u}_\lambda(t) - z_\lambda(t) = -\frac{1}{\lambda}[u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t))] \quad \text{a.e. } t \in [T_0, T_\lambda[,$$

hence

$$\|\dot{u}_\lambda(t) - z_\lambda(t)\| = \frac{1}{\lambda} g_\lambda(t) \quad \text{a.e. } t \in [T_0, T_\lambda[.$$

According to Lemma 6.7, we deduce

$$\|\dot{u}_\lambda(t) - z_\lambda(t)\| \leq \beta + \kappa \quad \text{a.e. } t \in [T_0, T_\lambda[.$$

The proof is complete since $\|z_\lambda(t)\| \leq \beta$ for all $t \in I$ (see assumption (i)). □

Fix any real $\lambda > 0$. The inequality (6.9) says that $u_\lambda(\cdot)$ is $(2\beta + \kappa)$-Lipschitz continuous on $[T_0, T_\lambda[$. Since $T_\lambda \in \mathbb{R}$, the limit $\lim\limits_{t \uparrow T_\lambda} u_\lambda(t)$ exists in $\mathcal{H}$ and the extended mapping still denoted $u_\lambda(\cdot)$, defined at T_λ by $u_\lambda(T_\lambda) = \lim\limits_{t \uparrow T_\lambda} u_\lambda(t)$, is Lipschitz continuous on $[T_0, T_\lambda]$. Keeping in mind that $\theta < \frac{r}{3(2\beta+\kappa)}$, we see that

$$T_\lambda - T_0 \leq \theta < \frac{r}{3(2\beta + \kappa)}.$$

Then, since $u_\lambda(T_\lambda) = \lim\limits_{t \uparrow T_\lambda} u_\lambda(t)$, it ensues that

$$\|u_\lambda(T_\lambda) - a\| = \|u_\lambda(T_\lambda) - u_\lambda(T_0)\| \leq (2\beta + \kappa)(T_\lambda - T_0) < \frac{r}{3}. \tag{6.10}$$

Thus, the Lipschitz mapping $u_\lambda(\cdot) : [T_0, T_\lambda] \to B(a, \frac{r}{3})$ (extended at T_λ) satisfies

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda} \nabla d^2_{C(t)}(u_\lambda(t)) - f(t, u_\lambda(t)) \quad \text{a.e. } t \in [T_0, T_\lambda], \\ u_\lambda(T_0) = a. \end{cases}$$

Moreover, note that $T_\lambda = T_0 + \theta$. Indeed, if $T_\lambda < T_0 + \theta$, (6.10) allows us to extend $u_\lambda(\cdot)$ on the right of T_λ in a solution of (E_λ) with the range of the extension of $u_\lambda(\cdot)$ included in $B(a, \frac{r}{3})$ and this cannot hold true according to the maximality of $[T_0, T_\lambda[$.

It is then established that for any real $\lambda > 0$, there is one and only one Lipschitz continuous mapping $u_\lambda : [T_0, T_0 + \theta] \to B(a, \frac{r}{3})$ satisfying

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda} \nabla d^2_{C(t)}(u_\lambda(t)) - f(t, u_\lambda(t)) \quad \text{a.e. } t \in [T_0, T_0 + \theta], \\ u_\lambda(T_0) = a. \end{cases}$$

Our aim is now to establish that $(u_\lambda(\cdot))_{\lambda > 0}$ satisfies the Cauchy criterion as $\lambda \downarrow 0$.

Lemma 6.10. *For all $\lambda_1, \lambda_2 \in]0, +\infty[$, for all $t \in [T_0, T_0 + \theta]$, one has*

$$\|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|^2 \leq 2(\lambda_1 + \lambda_2)(\beta + \kappa)^2 \int_{T_0}^t \exp\big(2[9(\frac{\beta + \kappa}{r}) + k](t - s)\big) ds.$$

Proof. Let $\lambda_1, \lambda_2 \in]0, +\infty[$. Let A be a Lebesgue negligible subset of $[T_0, T_0+\theta[$ such that for each $t \in [T_0, T_0+\theta[\setminus A$ and each $i \in \{1,2\}$ (see the definition of $u_{\lambda_i}(\cdot)$ and Lemma 6.9)

$$\dot{u}_{\lambda_i}(t) = -\frac{1}{2\lambda_i}\nabla d^2_{C(t)}(u_{\lambda_i}(t)) + z_{\lambda_i}(t)$$

and

$$\|\dot{u}_{\lambda_i}(t) - z_{\lambda_i}(t)\| \leq \beta + \kappa. \tag{6.11}$$

Fix any $\zeta \in \{\lambda_1, \lambda_2\}$. From (2.1) note that

$$z_\zeta(t) - \dot{u}_\zeta(t) = \frac{1}{\zeta}\big(u_\zeta(t) - \mathrm{proj}_{C(t)}(u_\zeta(t))\big) \in N^P\big(C(t); \mathrm{proj}_{C(t)}(u_\zeta(t))\big).$$

Combining the latter inclusion and (6.11) with Theorem 2.8(c), we obtain

$$\Big\langle -\dot{u}_{\lambda_1}(t) + z_{\lambda_1}(t) + \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t), \mathrm{proj}_{C(t)}(u_{\lambda_1}(t)) - \mathrm{proj}_{C(t)}(u_{\lambda_2}(t))\Big\rangle$$
$$\geq -\frac{\beta+\kappa}{r}\Big\|\mathrm{proj}_{C(t)}(u_{\lambda_1}(t)) - \mathrm{proj}_{C(t)}(u_{\lambda_2}(t))\Big\|^2.$$

Using the equality

$$\mathrm{proj}_{C(t)}(u_\zeta(t)) = \zeta(\dot{u}_\zeta(t) - z_\zeta(t)) + u_\zeta(t),$$

the fact that $\mathrm{proj}_{C(t)}(\cdot)$ is 3-Lipschitz continuous on $B(a, \frac{r}{3})$ and the inclusion $u_\zeta(t) \in B(a, \frac{r}{3})$, we get

$$\langle -\dot{u}_{\lambda_1}(t) + z_{\lambda_1}(t) + \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t), \lambda_1(\dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t)) - \lambda_2(\dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t))\rangle$$
$$+ \langle -\dot{u}_{\lambda_1}(t) + z_{\lambda_1}(t) + \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t), u_{\lambda_1}(t) - u_{\lambda_2}(t)\rangle$$
$$\geq -9\frac{\beta+\kappa}{r}\|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|^2.$$

An elementary computation gives

$$\begin{aligned}
&-\lambda_1\|\dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t)\|^2 - \lambda_2\|\dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t)\|^2 \\
&+ (\lambda_1+\lambda_2)\langle \dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t), \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t)\rangle \\
&+ \langle z_{\lambda_1}(t) - z_{\lambda_2}(t), u_{\lambda_1}(t) - u_{\lambda_2}(t)\rangle - \langle \dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t), u_{\lambda_1}(t) - u_{\lambda_2}(t)\rangle \\
\geq& -9\frac{\beta+\kappa}{r}\|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|^2.
\end{aligned}$$

With $\psi(\cdot) := \|u_{\lambda_1}(\cdot) - u_{\lambda_2}(\cdot)\|^2$ the latter inequality can be rewritten as

$$\begin{aligned}
\frac{1}{2}\dot{\psi}(t) \leq 9\frac{\beta+\kappa}{r}\psi(t) &- \lambda_1\|\dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t)\|^2 - \lambda_2\|\dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t)\|^2 \\
&+ (\lambda_1+\lambda_2)\langle \dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t), \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t)\rangle \\
&+ \langle z_{\lambda_1}(t) - z_{\lambda_2}(t), u_{\lambda_1}(t) - u_{\lambda_2}(t)\rangle
\end{aligned}$$

and this obviously entails

$$\begin{aligned}
\frac{1}{2}\dot{\psi}(t) \leq 9\frac{\beta+\kappa}{r}\psi(t) &+ (\lambda_1+\lambda_2)\langle \dot{u}_{\lambda_1}(t) - z_{\lambda_1}(t), \dot{u}_{\lambda_2}(t) - z_{\lambda_2}(t)\rangle \\
&+ \langle z_{\lambda_1}(t) - z_{\lambda_2}(t), u_{\lambda_1}(t) - u_{\lambda_2}(t)\rangle.
\end{aligned} \tag{6.12}$$

On the other hand, since $f(t,\cdot)$ is k-Lipschitz continuous on $B(a,\frac{r}{3})$ and $u_\zeta(t)\in B(a,\frac{r}{3})$, we have

$$\begin{aligned}\|z_{\lambda_1}(t)-z_{\lambda_2}(t)\| &= \|f(t,u_{\lambda_1}(t))-f(t,u_{\lambda_2}(t))\| \\ &\le k\,\|u_{\lambda_1}(t)-u_{\lambda_2}(t)\|.\end{aligned} \tag{6.13}$$

Coming back to (6.12) and using (6.11) and (6.13), we then see that

$$\frac{1}{2}\dot{\psi}(t)\le [9\frac{\beta+\kappa}{r}+k]\psi(t)+(\lambda_1+\lambda_2)(\beta+\kappa)^2.$$

It remains to combine Lemma 6.8 with the latter inequality and $\psi(T_0)=0$ to complete the proof. □

Lemma 6.11. *The family $(u_\lambda(\cdot))_{\lambda>0}$ converges uniformly on $[T_0,T_0+\theta]$ to a mapping $u:[T_0,T_0+\theta]\to B(a,\frac{r}{3})$ satisfying the differential inclusion*

$$\begin{cases}-\dot{u}(t)\in N(C(t);u(t))+f(t,u(t)) \quad a.e.\ t\in[T_0,T_0+\theta],\\ u(t)\in C(t) \quad \text{for all}\, t\in[T_0,T_0+\theta],\\ u(T_0)=a.\end{cases}$$

Furthermore, one has for every real $\lambda>0$,

$$\sup_{t\in[T_0,T_0+\theta]}\|u_\lambda(t)-u(t)\|^2\le 2\lambda(\beta+\kappa)^2\int_{T_0}^{T_0+\theta}\exp\big(K(T_0+\theta-s)\big)ds,$$

where $K:=2[9(\frac{\beta+\kappa}{r})+k]$, and for almost every $t\in[T_0,T_0+\theta]$

$$\|\dot{u}(t)+f(t,u(t))\|\le\beta+\kappa \quad \text{and} \quad \|\dot{u}(t)\|\le 2\beta+\kappa.$$

Proof. According to Lemma 6.10, the familly $(u_\lambda(\cdot))_{\lambda>0}$ converges uniformly as $\lambda\downarrow 0$ to a continuous mapping $u:[T_0,T_0+\theta]\to\mathcal{H}$ and this mapping $u(\cdot)$ satisfies the first estimate in the statement. Fix for a moment any real $\lambda>0$. From Lemma 6.7, we get

$$d(u_\lambda(t),C(t))=:g_\lambda(t)\le\lambda(\beta+\kappa) \quad \text{for all}\, t\in[T_0,T_0+\theta[. \tag{6.14}$$

Writing for every $t\in[T_0,T_0+\theta[$

$$\begin{aligned}|g_\lambda(t)-g_\lambda(T_0+\theta)| &= \big|d_{C(t)}(u_\lambda(t))-d_{C(T_0+\theta)}(u_\lambda(T_0+\theta))\big| \\ &\le \|u_\lambda(t)-u_\lambda(T_0+\theta)\|+\widehat{\mathrm{haus}}_\rho(C(t),C(T_0+\theta))\end{aligned}$$

we see (through (6.1) and the Lipschitz property of $u_\lambda(\cdot)$) that $\lim_{t\uparrow T_0+\theta}g_\lambda(t)=g_\lambda(T_0+\theta)$. Coming back to (6.14) and letting $t\uparrow T_0+\theta$, we arrive to $g_\lambda(T_0+\theta)\le\lambda(\beta+\kappa)$. Hence, we have

$$d(u_\lambda(t),C(t))=g_\lambda(t)\le\lambda(\beta+\kappa) \quad \text{for all}\, t\in[T_0,T_0+\theta].$$

Passing to the limit as $\lambda\downarrow 0$ in the latter inequality gives (thanks to the closedness property of all sets $C(t)$)

$$u(t)\in C(t) \quad \text{for all}\, t\in[T_0,T_0+\theta]. \tag{6.15}$$

By Lemma 6.9 we also know that for each real $\lambda > 0$,

$$\|\dot{u}_\lambda(t)\| \le 2\beta + \kappa \quad \text{a.e. } t \in [T_0, T_0 + \theta],$$

so we can find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive reals with $\lambda_n \downarrow 0$ such that $(\dot{u}_{\lambda_n}(\cdot))_{n\in\mathbb{N}}$ converges weakly in $L^2([T_0, T_0+\theta], \mathcal{H})$ to some mapping $h(\cdot) \in L^2([T_0, T_0+\theta], \mathcal{H})$. For any $t \in [T_0, T_0+\theta]$, any $z \in \mathcal{H}$ and any $n \in \mathbb{N}$, we can write

$$\left\langle z, \int_{T_0}^t \dot{u}_{\lambda_n}(s) ds \right\rangle = \int_{T_0}^T \left\langle z\mathbf{1}_{[T_0,t]}(s), \dot{u}_{\lambda_n}(s)\right\rangle ds.$$

This allows us to see that for all $t \in [T_0, T_0+\theta]$,

$$\int_{T_0}^t \dot{u}_{\lambda_n}(s) ds \to \int_{T_0}^t h(s) ds \quad \text{weakly in } \mathcal{H}.$$

For each $t \in [T_0, T_0+\theta]$, the strong convergence of $(u_{\lambda_n}(t))_{n\in\mathbb{N}}$ to $u(t)$ in $\mathcal{H}$ and the equality $u_{\lambda_n}(t) = a + \int_{T_0}^t \dot{u}_{\lambda_n}(s) ds$ valid for all $n \in \mathbb{N}$ entail

$$u(t) = a + \int_{T_0}^t h(s) ds. \tag{6.16}$$

We deduce that $u(\cdot)$ is absolutely continuous on $[T_0, T_0+\theta]$ with $\dot{u}(\cdot) = h(\cdot)$ almost everywhere on $[T_0, T_0+\theta]$ and then

$$\dot{u}_{\lambda_n}(\cdot) \to \dot{u}(\cdot) \quad \text{weakly in } L^2([T_0, T_0+\theta], \mathcal{H}).$$

Set $z(\cdot) := -f(\cdot, u(\cdot))$ and keep in mind that $z_\lambda(\cdot) = -f(\cdot, u_\lambda(\cdot))$ for every real $\lambda > 0$. Note that

$$\dot{u}_{\lambda_n}(\cdot) - z_{\lambda_n}(\cdot) \to \dot{u}(\cdot) - z(\cdot) \quad \text{weakly in } L^2([T_0, T_0+\theta], \mathcal{H}), \tag{6.17}$$

and hence thanks to Lemma 6.9 and Mazur's lemma it is easily seen that

$$\|\dot{u}(t) - z(t)\| \le \beta + \kappa \quad \text{and} \quad \|\dot{u}(t)\| \le 2\beta + \kappa \quad \text{a.e. } t \in [T_0, T_0+\theta].$$

The latter inequality, (6.16) and $\dot{u}(\cdot) = h(\cdot)$ give

$$\|u(t) - a\| \le (t - T_0)(2\beta + \kappa) < \frac{r}{3} \quad \text{for all } t \in [T_0, T_0+\theta], \tag{6.18}$$

since $\theta < \frac{r}{3(2\beta+\kappa)}$. Consequently, we get the inclusion $u([T_0, T_0+\theta]) \subset B(a, \frac{r}{3})$. Applying Mazur's lemma with (6.17), there are for each $n \in \mathbb{N}$ some integer $r(n) > n$ and a familly $(\alpha_{k,n})_{n\le k\le r(n)}$ of $[0,1]$ with $\sum_{k=n}^{r(n)} \alpha_{k,n} = 1$ such that $\sum_{k=n}^{r(n)} \alpha_{k,n}(z_{\lambda_k} - \dot{u}_{\lambda_k})$ converges strongly to $z(\cdot) - \dot{u}(\cdot)$ in $L^2([T_0, T_0+\theta], \mathcal{H})$. Now, consider any Lebesgue neligible set $N \subset [T_0, T_0+\theta]$ such that for every $n \in \mathbb{N}$ and every $t \in [T_0, T_0+\theta]$, $\dot{u}(t)$ and $\dot{u}_{\lambda_n}(t)$ exist. Without loss of generality, we may also suppose that for every $t \in [T_0, T_0+\theta] \setminus N$,

$$\lim_{n\to+\infty} \sum_{k=n}^{r(n)} \alpha_{k,n}(z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t)) = z(t) - \dot{u}(t). \tag{6.19}$$

along with (see Lemma 6.9)

$$\|\dot{u}_{\lambda_n}(t) - z_{\lambda_n}(t)\| \le \beta + \kappa \quad \text{and} \quad \|\dot{u}_{\lambda_n}(t)\| \le 2\beta + \kappa, \tag{6.20}$$

for all $n \in \mathbb{N}$ and for all $t \in [T_0, T_0 + \theta] \setminus N$. Fix for a moment any $t \in [T_0, T_0 + \theta] \setminus N$ and any $n \in \mathbb{N}$. From (6.20), it is readily seen that

$$\begin{aligned} &\left| \sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle \right| \\ \le &(\beta + \kappa) \sum_{k=n}^{r(n)} \alpha_{k,n} \left\| u(t) - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\|. \end{aligned} \tag{6.21}$$

On the other hand, by (6.4) and (6.15) we see that

$$\lim_{k \to +\infty} \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) - u(t) = 0. \tag{6.22}$$

Coming back to (6.21), it is easily seen that the latter convergence entails

$$\lim_{n \to +\infty} \sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle = 0. \tag{6.23}$$

For every $n \in \mathbb{N}$ and every $x' \in \mathcal{H}$, combining the equality

$$\begin{aligned} &\sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), x' - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle \\ = &\left\langle \sum_{k=n}^{r(n)} \alpha_{k,n} (z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t)), x' - u(t) \right\rangle \\ &+ \sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle \end{aligned}$$

with (6.19) and (6.23), we get with $\xi(t, x') := \langle z(t) - \dot{u}(t), x' - u(t) \rangle$

$$\lim_{n \to +\infty} \sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), x' - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle = \xi(t, x'). \tag{6.24}$$

For each real $\lambda > 0$, the equality

$$z_\lambda(t) - \dot{u}_\lambda(t) = \frac{1}{2\lambda} \nabla d^2_{C(t)}(u_\lambda(t)) = \frac{1}{\lambda} [u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))]$$

along with (2.1), guarantees that

$$z_\lambda(t) - \dot{u}_\lambda(t) \in N\big(C(t); \operatorname{proj}_{C(t)}(u_\lambda(t))\big).$$

Thanks to the r-prox-regularity of $C(t)$ and (6.20), the latter inclusion entails for all $x' \in C(t)$ (see Theorem 2.6(b)),

$$\sum_{k=n}^{r(n)} \alpha_{k,n} \left\langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), x' - \mathrm{proj}_{C(t)}(u_{\lambda_k}(t)) \right\rangle$$
$$\leq \frac{\beta+\kappa}{2r} \sum_{k=n}^{r(n)} \alpha_{k,n} \left\| x' - \mathrm{proj}_{C(t)}(u_{\lambda_k}(t)) \right\|^2 .$$

Using (6.22) and (6.24), it follows that for all $x' \in C(t)$,

$$\xi(t, x') = \langle z(t) - \dot{u}(t), x' - u(t) \rangle \leq \frac{\beta+\kappa}{2r} \|x' - u(t)\|^2 .$$

As a consequence, we obtain

$$-\dot{u}(t) + z(t) \in N^P(C(t); u(t)) = N(C(t); u(t)),$$

which finishes the proof of the lemma. $\square$

The proof of Theorem 6.2 is then complete. $\square$

The existence result over $I = [T_0, T]$ can be deduced from a suitable finite partition of the interval $[T_0, T]$.

Theorem 6.12. *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$, $a \in C(T_0)$, $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Assume that:*

(i) *there exists a real $\beta > 0$ such that for all $t \in I$ and $x \in \mathcal{H}$,*
$$\|f(t, x)\| \leq \beta;$$

(ii) *the mapping $f(\cdot, x)$ is Bochner measurable for each $x \in \mathcal{H}$ and there exists $k \in \mathbb{R}_+$ such that for all $t \in I$ and for all $x_1, x_2 \in \mathcal{H}$,*
$$\|f(t, x_1) - f(t, x_2)\| \leq k \|x_1 - x_2\| ;$$

(iii) *there exist a function $v : I \to \mathbb{R}$ which is κ-Lipschitz continuous for some real $\kappa \geq 0$ on $[T_0, T]$, a real $\theta \in]0, \min\left\{T - T_0, \frac{r}{3(2\beta+\kappa)}\right\}[$ and an integer $p \geq 2$ with $p(T_0 + \theta) \geq T$ such that*
$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) := \sup_{x \in \rho\mathbb{B}} |d(x, C(s)) - d(x, C(t))| \leq |v(t) - v(s)|$$
for some extended real $\rho \geq \|a\| + \frac{p+2}{3} r$ and for all $s, t \in I$.

Then, there exists a $(2\beta + \kappa)$-Lipschitz continuous mapping $u : I \to B(a, \frac{pr}{3})$ solution of the differential inclusion

$$(6.25) \qquad \begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) & a.e.\ t \in I, \\ u(t) \in C(t) & for\ all\ t \in I, \\ u(T_0) = a. \end{cases}$$

Proof. For each $i \in \{1, \cdots, p\}$, put $T_i := T_0 + i\theta$. Set $a_0 := a$. According to Theorem 6.2, there is a $(2\beta + \kappa)$-Lipschitz mapping $u_0 : [T_0, T_1] \to B(a_0, \frac{r}{3})$ satisfying

$$\begin{cases} -\dot{u}_0(t) \in N(C(t); u_0(t)) + f_0(t, u_0(t)) & \text{a.e. } t \in [T_0, T_1], \\ u_0(t) \in C(t) & \text{for all } t \in [T_0, T_1], \\ u_0(T_0) = a_0, \end{cases}$$

where the mapping f_0 is the restriction of f to $[T_0, T_1] \times B(a_0, \frac{r}{3})$. Since $a_1 := u_0(T_1) \in C(T_1)$ and

$$\|a_1\| + r \le \frac{r}{3} + \|a_0\| + r = \frac{4r}{3} + \|a_0\| \le \rho,$$

we can apply Theorem 6.2 to get a $(2\beta+\kappa)$-Lipschitz mapping $u_1 : [T_1, T_2] \to B(a_1, \frac{r}{3})$ satisfying

$$\begin{cases} -\dot{u}_1(t) \in N(C(t); u_1(t)) + f_1(t, u_1(t)) & \text{a.e. } t \in [T_1, T_2], \\ u_1(t) \in C(t) & \text{for all } t \in [T_1, T_2], \\ u_1(T_1) = a_1, \end{cases}$$

where the mapping f_1 is the restriction of f to $[T_1, T_2] \times B(a_1, \frac{r}{3})$. We may proceed in this way up to the last closed interval $[T_{p-1}, T_p]$. Defining the mapping $u(\cdot) : [T_0, T_p] \supset [T_0, T] \to \bigcup_{i=0}^{p-1} B(a_i, \frac{r}{3}) \subset B(a, \frac{pr}{3})$ by putting $u(t) := u_i(t)$ for any $t \in [T_i, T_{i+1}]$ for all $i \in \{0, \ldots, n-1\}$, it is readily seen that $u(\cdot)$ provides a $(2\beta + \kappa)$-Lipschitz mapping satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + f(t, u(t)) & \text{a.e. } t \in I, \\ u(t) \in C(t) & \text{for all } t \in I, \\ u(T_0) = a. \end{cases}$$

The proof is then complete. $\square$

Corollary 6.13. *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$, $a \in C(T_0)$, $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Assume that (i) and (ii) of Theorem 6.12 hold. Assume also that there exist a function $v : I \to \mathbb{R}$ which is κ-Lipschitz continuous for some real $\kappa \ge 0$ on I such that*

$$\operatorname{haus}(C(s), C(t)) \le |v(t) - v(s)| \quad \textit{for all}\, s, t \in I.$$

Then, there exists a $(2\beta + \kappa)$-Lipschitz continuous mapping $u : I \to \mathcal{H}$ solution of (6.25).

Proof. Fixing any $\theta \in]0, \min\left\{T - T_0, \frac{r}{3(2\beta+\kappa)}\right\}$ and choosing an integer $p \ge 2$ with $p(T_0 + \theta) \ge T$, remembering that $\operatorname{haus}(\cdot, \cdot) = \widehat{\operatorname{haus}}_\infty(\cdot, \cdot)$, it suffices to apply Theorem 6.12 with $\rho := +\infty$. $\square$

The last result of this section is devoted to the reduction of the absolutely continuous case to the Lipschitzian one for $f \equiv 0$. We adapt a method due to J.J. Moreau ([36]) (see also, [48]).

Proposition 6.14. *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping, $a \in C(T_0)$. Assume that there exist $\rho, r \in]0, +\infty]$ with $\|a\| + r \leq \rho$ satisfying:*

(i) *for every $t \in I$, the set $C(t)$ is r-prox-regular;*
(ii) *there exists a function $v : I \to \mathbb{R}$ absolutely continuous on I such that for all $s, t \in I$ with $s \leq t$,*

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) \leq |v(t) - v(s)|.$$

If $\int_{T_0}^{T} |\dot{v}(s)|\, ds > 0$, then for each positive real $\theta \leq \int_{T_0}^{T} |\dot{v}(s)|\, ds$ satisfying $\theta < \frac{r}{3}$, there exist a real $\eta > 0$ (in fact, any real $\eta > 0$ with $\int_{T_0}^{\eta} |\dot{v}(s)|\, ds = \theta$ is appropriate) and an absolutely continuous mapping $u : [T_0, \eta] \to B(a, \frac{r}{3})$ solution of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & a.e.\ t \in [T_0, \eta], \\ u(t) \in C(t) & for\ all\, t \in [T_0, \eta], \\ u(0) = a. \end{cases} \tag{6.26}$$

Proof. Let $\omega(\cdot) : I \to \mathbb{R}$ be the function defined by

$$\omega(t) := \int_{T_0}^{t} |\dot{v}(s)|\, ds \quad \text{for all}\, t \in I.$$

Assume that $\omega(T) > 0$. Fix any positive real θ with $\theta \leq \omega(T) - \omega(T_0) = \omega(T)$ and $\theta < \frac{r}{3}$. Observe that for all $s, t \in I$ with $s \leq t$,

$$\widehat{\mathrm{haus}}_\rho(C(s) \cap \rho\mathbb{B}, C(t) \cap \rho\mathbb{B}) \leq \omega(t) - \omega(s).$$

Further, since $C(\cdot)$ is in particular closed-valued we have for all $\tau_1, \tau_2 \in I$ with $\omega(\tau_1) = \omega(\tau_2)$,

$$C(\tau_1) \cap \rho\mathbb{B} = C(\tau_2) \cap \rho\mathbb{B}. \tag{6.27}$$

Fix a selection τ of $\omega^{-1} : [0, \omega(T)] \rightrightarrows [T_0, T]$, that is, a mapping $\tau : [0, \omega(T)] \to [T_0, T]$ with $\tau(s) \in \omega^{-1}(s)$ for all $s \in [0, \omega(T)]$. This function τ is nondecreasing. Indeed, suppose the contrary, that is, there are $s_1 < s_2$ in $[0, \omega(T)]$ with $\tau(s_1) > \tau(s_2)$. By the nondecreasing property of ω one would have $\omega(\tau(s_1)) \geq \omega(\tau(s_2))$, or equivalently $s_1 \geq s_2$, which would contradict the inequality $s_1 < s_2$.

Now, consider the multimapping $D : [0, \omega(T)] \rightrightarrows \mathcal{H}$ defined by

$$D(s) := C(\tau(s)) \quad \text{for all } s \in [0, \omega(T)].$$

Note by (6.27) that

$$D(\omega(t)) \cap \rho\mathbb{B} = C\big(\tau(\omega(t))\big) \cap \rho\mathbb{B} = C(t) \cap \rho\mathbb{B} \quad \text{for all } t \in I. \tag{6.28}$$

For any $x \in \rho\mathbb{B}$ and $s_1, s_2 \in [0, \omega(T)]$ with $s_1 \le s_2$, we have

$$\begin{aligned}|d(x, D(s_1)) - d(x, D(s_2))| &= |d(x, C(\tau(s_1)) - d(x, C(\tau(s_2))| \\ &\le \widehat{\mathrm{haus}}_\rho(C(\tau(s_1)), C(\tau(s_2))) \\ &\le |\omega(\tau(s_2)) - \omega(\tau(s_1))| = |s_2 - s_1|\,.\end{aligned}$$

This says that for all $s_1, s_2 \in [0, \omega(T)]$,

$$\widehat{\mathrm{haus}}_\rho(D(s_1), D(s_2)) \le |s_2 - s_1|\,.$$

Since $\rho \ge \|a\| + r$, we can apply Theorem 6.2 to get a 1-Lipschitz continuous mapping $z : [0, \theta] \to B(a, \frac{r}{3})$ satisfying

$$\begin{cases} -\dot{z}(s) \in N(D(s); z(s)) & \text{a.e. } s \in [0, \theta], \\ z(s) \in D(s) & \text{for all } t \in [0, \theta], \\ z(0) = a. \end{cases}$$

Fix any real $\eta > 0$ with $\int_{T_0}^{\eta} |\dot{v}(s)|\, ds = \theta$. Let us define the mapping $u : [T_0, \eta] \to B(a, \frac{r}{3})$ by

$$u(t) := z(\omega(t)) \quad \text{for all } t \in [T_0, \eta].$$

The mapping u is absolutely continuous on $[T_0, \eta]$ with

$$\dot{u}(t) = |\dot{v}(t)|\, \dot{z}(\omega(t)) \quad \text{a.e. } t \in [T_0, \eta]. \tag{6.29}$$

Obviously, we have $u(T_0) = z(0) = a$ and thanks to the definition of $D(\cdot)$ and to (6.28)

$$u(t) \in D(\omega(t)) \cap \rho\mathbb{B} = C(t) \cap \rho\mathbb{B} \subset C(t) \quad \text{for all } t \in [T_0, T].$$

Let A be a Lebesgue negligible subset of $[T_0, \eta]$ such that for all $t \in [T_0, \eta] \setminus A$, $\dot{u}(t), \dot{v}(t), \dot{z}(\omega(t))$ exist and

$$-\dot{z}(\omega(t)) \in N(D(\omega(t)); z(\omega(t))).$$

Then fix any $t \in [T_0, T] \setminus A$. From (6.29) and the fact that $N(\cdot; \cdot)$ is a cone, we have

$$-\dot{u}(t) \in N(D(\omega(t)); u(t)).$$

On the other hand, by the inequality

$$\|u(t)\| \le \|a\| + \frac{r}{3} < \rho,$$

we know that $\rho\mathbb{B}$ is a neighborhood of $u(t)$, and hence it results (see (2.2)) that

$$-\dot{u}(t) \in N(C(t); u(t)).$$

The proof is then complete. □

Remark 6.15. It is readily seen that the constant mapping $u : I \to \mathcal{H}$ with $u(t) := a$ for all $t \in I$ satisfies (6.26) whenever $v(t) = 0$ for almost every $t \in I$. □

7. The case of sweeping process under compactness and α-far property

Let us consider the sweeping process

$$\dot{u}(t) \in -N(C(t); u(t)) \quad \text{with initial condition } u(T_0) = u_0 \in C(T_0), \tag{7.1}$$

where $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ is here a mutimapping with nonempty closed (not necessarily prox-regular) values. Assume that $C(\cdot)$ is κ-Lipschitz, that is,

$$\mathrm{haus}(C(s), C(t)) \le \kappa |s - t| \quad \text{for all } s, t \in I.$$

In this situation, the function $t \mapsto (1/2) d^2_{C(t)}(x)$ may fail to be differentiable for some $x \in \mathcal{H}$, but nevertheless we can consider for each real $\lambda > 0$ the differential inclusion

$$(\mathrm{DI}_\lambda) \begin{cases} \dot{u}_\lambda(t) \in -\frac{1}{2\lambda} \partial(d^2_{C(t)})(u_{\lambda(t)}) & \text{a.e. } t \in I, \\ u_\lambda(T_0) = u_0. \end{cases}$$

This differential inclusion is with convex weakly compact second member and without any constraint (unlike the case of (7.1)). When it is well-behaved with respect to existence of solutions, although there is no hope of uniqueness in general, (DI_λ) can be seen as a semi-regularization of (7.1). Although the multimapping $x \mapsto -(1/2)\partial(d^2_{C(t)})(x)$ is weakly compact convex valued and upper semicontinuous from $(\mathcal{H}, \|\cdot\|)$ into $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$ for every $t \in I$, it is not upper semicontinuous from $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$ into $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$. So, we may not apply existence results of absolutely continuous solutions for differential inclusions $\dot{z}(t) \in F(t, z(t))$, where $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a multimapping with nonempty weakly compact convex values such that $x \mapsto F(t, x)$ is upper semicontinuous from $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$ into $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$. This yields, as done recently and very efficiently by A. Jourani and E. Vilches [28], to evoke and recall a result of D. Bothe [8]. In fact, we merely recall a partial form of the result in [8]. It involves the concept of measure of noncompactness (see, e.g., [21] for the definition and basic properties).

Theorem 7.1 (Bothe, [8]). *Let $F : I = [T_0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed convex values. Assume:*

(i) *for each $x \in \mathcal{H}$ the multimapping $F(\cdot, x)$ admits a Bochner measurable selection;*

(ii) *for each $t \in I$ the multimapping $F(t, \cdot)$ is upper semicontinuous from $(\mathcal{H}, \|\cdot\|)$ into $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$;*

(iii) *there exists $\beta(\cdot) \in L^1(I)$ such that for every $t \in I$ and every $x \in \mathcal{H}$,*

$$\sup\{\|w\| : w \in F(t, x)\} \le \beta(t)(1 + \|x\|);$$

(iv) *there exists $k(\cdot) \in L^1(I)$ such that for every $t \in I$ and every bounded set B in $\mathcal{H}$*

$$\gamma\big(F(t, B)\big) \le k(t)\gamma(B),$$

where $\gamma(\cdot)$ is the Hausdorff measure of non-compactness.

Then, the differential inclusion

$$\dot{\zeta}(t) \in F(t, \zeta(t)) \quad \textit{with initial condition}\, \zeta(T_0) = \zeta_0 \in \mathcal{H}$$

admits at least one absolutely continuous solution.

Concerning the differential inclusion (DI_λ), taking Proposition 2.2 into account the condition (iv) in Theorem 7.1 leads to assume hereafter that all the nonempty sets $C(t)$ are strongly ball-compact. Under such an assumption, with $F(t,x) := -\partial\big((1/2)d^2_{C(t)}\big)(x)$ for every $(t,x) \in I \times \mathcal{H}$, we have for every bounded set $B \subset \mathcal{H}$ and every $t \in I$, by Proposition 2.2 that

$$F(t, B) \subset B - \overline{\mathrm{co}}\big(C(t) \cap r\mathbb{B}\big),$$

where $r := \sup_{b \in B} \big(\|b\| + d_S(b)\big) < +\infty$, hence $\gamma\big(F(t,B)\big) \le \gamma(B)$. From now on, unless otherwise stated the real Hilbert space $\mathcal{H}$ is assumed to be separable. With $\varphi_t(x) := 1/2 d^2_{C(t)}(x)$ for every $(t,x) \in I \times \mathcal{H}$, the equalities valid for all $n \in \mathbb{N}$, all $h \in \mathcal{H}$ and all $(t,x) \in I \times \mathcal{H}$,

$$\sigma(h; \partial\varphi_t(x)) = \varphi_t^o(x;h) = \inf_{n \in \mathbb{N}} \sup_{(s,z) \in Z_n} \frac{1}{s}[\varphi(t, z+sh) - \varphi(t,z)],$$

where for each $n \in \mathbb{N}$, $Z_n := \big\{(s,z) : s \in]0, \frac{1}{n}[\cap \mathbb{Q}, z \in B(x, \frac{1}{n}) \cap D\big\}$ and D is some countable dense subset of $\mathcal{H}$, ensure (see, e.g., [15, Theorem 3.37]) for each $x \in \mathcal{H}$ the measurability of the multimapping $t \mapsto \partial(\frac{1}{2}d^2_{C(t)})(x)$ as well as the existence of a Bochner measurable selection of this multimapping. Further, for each $t \in I$, the multimapping $x \mapsto \partial(1/2d^2_{C(t)})(x)$ is upper semicontinuous from $(\mathcal{H}, \|\cdot\|)$ into $(\mathcal{H}, w(\mathcal{H}, \mathcal{H}))$ (see Subsection 2.1). Then, by Theorem 7.1 we can fix, throughout the rest of this section, for each real $\lambda > 0$ an absolutely continuous solution $u_\lambda(\cdot)$ of the differential inclusion (DI_λ). Consider any real $\lambda > 0$. By Lemma 6.5 the function $g_\lambda(\cdot) : I \to \mathbb{R}$, defined by $g_\lambda(t) := d(u_\lambda(t), C(t))$ for all $t \in I$ is absolutely continuous and for almost every $t \in I$,

$$\dot{g}_\lambda(t) g_\lambda(t) \le \kappa\, g_\lambda(t) - g_\lambda(t)\sigma\big(-\dot{u}_\lambda(t), \partial d_{C(t)}(u_\lambda(t))\big).$$

Since $u_\lambda(\cdot)$ is an absolutely continuous solution of (DI_λ), for almost every $t \in I$, we can choose $\zeta_\lambda(t) \in \partial d_{C(t)}(u_\lambda(t))$ such that $-\dot{u}_\lambda(t) = \lambda^{-1} g_\lambda(t)\zeta_\lambda(t)$. It ensues for almost every $t \in I$ that

$$\sigma\big(-\dot{u}_\lambda(t), \partial d_{C(t)}(u_\lambda(t))\big) = \frac{1}{\lambda} g_\lambda(t)\sigma\big(\zeta_\lambda(t), \partial d_{C(t)}(u_\lambda(t))\big) \ge \frac{1}{\lambda} g_\lambda(t)\|\zeta(t)\|^2,$$

hence

$$\dot{g}_\lambda(t) g_\lambda(t) \le \kappa\, g_\lambda(t) - \frac{1}{\lambda}(g_\lambda(t))^2 \|\zeta_\lambda(t)\|^2.$$

The reasoning in (6.7) guarantees that for almost every $t \in I$

$$\dot{g}_\lambda(t) \le \kappa - \frac{1}{\lambda} g_\lambda(t)\|\zeta_\lambda(t)\|^2.$$

A right application of Gronwall Lemma like in the previous section requires that $\|\zeta(\cdot)\|$ be bounded from below by some positive real on the set $\{t \in$

$I : g_\lambda(t) = 0\}$. This leads (taking into account the situation of r-prox-regularity) to assume that there exist an extended real $r \in]0, +\infty]$ and a real $\alpha > 0$ such that

$$(\mathrm{Hyp}_{\alpha,r}) \quad d(0, \partial d_{C(t)}(x)) \geq \alpha \quad \text{for all } t \in I,\ x \in \mathrm{Tube}_r(C(t)),$$

where $\mathrm{Tube}_r(C(t)) := \{x \in \mathcal{H} : 0 < d_{C(t)}(x) < r\}$. This means for each $t \in I$ that *the Clarke subdifferential of* $d_{C(t)}$ *at each* $x \in \mathrm{Tube}_r(C(t))$ *is kept* α*-far away from zero*. Given a real $\alpha > 0$, closed sets for which there is some $r \in]0, +\infty]$ such that the property holds true are called α*-far* in ([25]) (see also [24, p. 551] for another related concept of *convex-like set*). Of course, according to Theorem 2.8(g) and to (2.4) any r-prox-regular set S of the Hilbert space $\mathcal{H}$ is 1-far relative to $\mathrm{Tube}_r(S)$, that is, $(\mathrm{Hyp}_{\alpha,r})$ holds with $\alpha = 1$. A. Jourani and E. Vilches [27, Proposition 3.9] proved the nice result that for any uniformly subsmooth set S of the Hilbert space $\mathcal{H}$ (see [4] for definition) and any $\alpha \in]0, 1[$ there exists some $r \in]0, +\infty]$ such that S satisfies $(\mathrm{Hyp}_{\alpha,r})$.

Throughout the remaining of the section we assume that the hypothesis $(\mathrm{Hyp}_{\alpha,r})$ is satisfied for any set $C(t)$ with $t \in I$ and *we follow for a very large part A. Jourani and E. Vilches* [28]. Fix for a moment $\lambda > 0$. Then by what precedes, on any interval $[T_0, \tau]$ (with $T_0 < \tau \leq T$) where $d_{C(t)}(u_\lambda(t)) < r$ (such intervals exist by continuity of $t \mapsto d_{C(t)}(u_\lambda(t))$ and by the equality $d_{C(T_0)}(u_\lambda(T_0)) = 0$) one has for almost every $t \in [T_0, \tau]$ that

$$\dot{g}_\lambda(t) \leq \kappa - \frac{\alpha^2}{\lambda} g_\lambda(t),$$

hence by Gronwall lemma (see the previous section) $g_\lambda(t) \leq \alpha^{-2}\kappa\lambda$ for all $t \in [T_0, \tau]$. Denote by T_λ the supremum of $\tau \in]T_0, T]$ such $d_{C(t)}(u_\lambda(t)) < r$ for all $t \in [T_0, \tau]$ we have

$$d_{C(T_\lambda)}(u_\lambda(T_\lambda)) = g_\lambda(T_\lambda) \leq \alpha^{-2}\kappa\lambda,$$

so for any positive $\lambda < \alpha^2 r/\kappa$ we see that $d_{C(T_\lambda)}(u_\lambda(T_\lambda)) < r$, hence $T_\lambda = T$ since otherwise we would get the contradiction that the property holds with some $\tau \in]T_\lambda, T]$.

For each $0 < \lambda < \alpha^2 r/\kappa$ we note for almost every $t \in I := [T_0, T]$ that $\|\dot{u}_\lambda(t)\| \leq g_\lambda(t)/\lambda$ by (DI_λ), thus $\|\dot{u}_\lambda(t)\| \leq \alpha^{-2}\kappa$ since $g_\lambda(t) \leq \alpha^{-2}\kappa\lambda$ by what precedes. Fix any sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $]0, \alpha^2 r/\kappa[$ tending to 0 and put $y_n(\cdot) := u_{\lambda_n}(\cdot)$ for all $n \in \mathbb{N}$. Since $\|\dot{y}_n(t)\| \leq \alpha^{-2}\kappa$ for almost every $t \in I$, from the sequence $(\dot{y}_n)_{n\in\mathbb{N}}$ we can extract a subsequence that we do not relabel which converges weakly in $L^2(I, \mathcal{H})$ to some $w(\cdot)$. For each $t \in I$ and each $n \in \mathbb{N}$ writing $y_n(t) = u_0 + \int_{T_0}^t \dot{y}_n(s)\,ds$ allows us to see that $y_n(t) \to u(t) := u_0 + \int_{T_0}^t w(s)\,ds$ weakly in $\mathcal{H}$ for every $t \in I$. By definition the mapping $u(\cdot)$ is absolutely continuous on I with $\dot{u}(\cdot) = w(\cdot)$ almost everywhere. Fix any $t \in I$ and put $\mathfrak{r}(t) := 1 + \sup\limits_{n\in\mathbb{N}} \big(\|y_n(t)\| + d_{C(t)}(y_n(t))\big) < +\infty$. Since $d(y_n(t), C(t) \cap \mathfrak{r}(t)\mathbb{B}) = d(y_n(t), C(t)) \to 0$ as $n \to \infty$, there is

$z_n(t) \in C(t) \cap \mathfrak{r}(t)\mathbb{B}$ such that $\|y_n(t) - z_n(t)\| \to 0$, so $z_n(t) \to u(t)$ weakly in $\mathcal{H}$. It ensues that $\|z_n(t) - u(t)\| \to 0$ as $n \to \infty$ since the strong and weak topologies on the strong compact set $C(t) \cap \mathfrak{r}(t)\mathbb{B}$ coincide, and this implies that $\|y_n(t) - u(t)\| \to 0$ as $n \to +\infty$. Remembering that $d(y_n(t), C(t)) \to 0$ as $n \to \infty$, we also get that $u(t) \in C(t)$ by strong closedness of $C(t)$.

Now consider the multimapping $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ with nonempty weakly compact convex values defined by $F(t,x) := \mathrm{co}\big(\{0\} \cup \frac{\kappa}{\alpha^2}\partial d_{C(t)}(x)\big)$ for all $(t,x) \in I \times \mathcal{H}$. Note that for each $t \in I$ the multimapping $F(t,\cdot)$ is scalarly upper semicontinuous and that for almost every $t \in I$ one has $-\dot{y}_n(t) \in F(t, y_n(t))$ for all $n \in \mathbb{N}$ since $-\dot{y}_n(t) \in \frac{1}{\lambda_n} g_{\lambda_n}(t)\partial d_{C(t)}(y_n(t))$ with $0 \le g_{\lambda_n}(t) \le \frac{\kappa\lambda_n}{\alpha^2}$. Since in addition $\dot{y}_n \to \dot{u}$ weakly in $L^2(I, \mathcal{H})$ as $n \to \infty$ and $\|y_n(t) - u(t)\| \to 0$ for every $t \in I$, Lemma 7.2 below ensures that for almost every $t \in I$ we have $\dot{u}(t) \in F(t, u(t))$. On the other hand, for each $t \in I$ the inclusion $u(t) \in C(t)$ assures us that $0 \in \frac{\kappa}{\alpha^2}\partial d_{C(t)}(u(t)) \subset N(C(t); u(t))$. Therefore, we get that $-\dot{u}(t) \in N(C(t); u(t))$ for almost every $t \in I$.

Lemma 7.2. *Assume that $\mathcal{H}$ is a general real Hilbert space (not necessarily separable). Consider any real $p \in [1, +\infty[$ and any sequence $(y_n(\cdot))_{n\in\mathbb{N}}$ converging weakly to $y(\cdot)$ in $L^p(I, \mathcal{H})$. Let $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed convex values such that $F(t,\cdot)$ is scalarly upper semicontinuous in $(\mathcal{H}, \|\cdot\|)$ for every $t \in I$. Let $(x_n(\cdot))_{n\in\mathbb{N}}$ be a sequence of mappings from I into $\mathcal{H}$ and let $x(\cdot) : I \to \mathcal{H}$ be a mapping such that for almost every $t \in I$, one has $\|x_n(t) - x(t)\| \to 0$ as $n \to \infty$. Assume for each $n \in \mathbb{N}$ that $y_n(t) \in F(t, x_n(t))$ for almost every $t \in I$. Then, one has*

$$y(t) \in F(t, x(t)) \quad a.e.\ t \in I.$$

Proof. By Mazur Lemma there is a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $L^p(I, \mathcal{H})$ converging strongly to y therein, with $\zeta_n \in \mathrm{co}\{y_k : k \ge n\}$. Extracting a subsequence if necessary, we may suppose for almost every $t \in I$ that $\|\zeta_n(t) - y(t)\| \to 0$ as $n \to \infty$. Let $L \subset I$ be a Lebesgue negligible set and such that for each $t \in I \setminus L$ one has $\|\zeta_n(t) - y(t)\| \to 0$ and $\|x_n(t) - x(t)\| \to 0$ as $n \to \infty$ along with $y_n(t) \in F(t, x_n(t))$ for all $n \in \mathbb{N}$. Then for each $t \in I \setminus L$, each $z \in \mathcal{H}$ and each $n \in \mathbb{N}$ we can write

$$\langle z, \zeta_n(t)\rangle \le \sup_{k\ge n}\langle z, y_k(t)\rangle \le \sup_{k\ge n}\sigma\big(z, F(t, x_k(t))\big).$$

Letting $n \to +\infty$, we get for each $t \in I \setminus L$ and each $z \in \mathcal{H}$

$$\langle z, y(t)\rangle \le \lim_{n\to+\infty}\sup_{k\ge n}\sigma\big(z, F(t, x_k(t))\big) \le \sigma\big(z, F(t, x(t))\big).$$

This being true for all $z \in \mathcal{H}$ we conclude that $y(t) \in F(t, x(t))$ for all $t \in I \setminus L$. $\square$

We can then state the following partial form of the result of A. Jourani and E. Vilches [28]. We remember the convention $1/\kappa = +\infty$ if $\kappa = 0$. We also recall that for any $S \subset \mathcal{H}$ and any extended real $r > 0$, $\mathrm{Tube}_r(S) := \{x \in \mathcal{H} : 0 < d_S(x) < r\}$.

Theorem 7.3 (Jourani-Vilches, [28]). *Assume that the real Hilbert space $\mathcal{H}$ is separable. Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed values for which there is a real $\kappa \geq 0$ such that*

$$\text{haus}(C(s), C(t)) \leq \kappa |s - t| \quad \textit{for all } s, t \in I, \tag{7.2}$$

and let $u_0 \in C(T_0)$. Assume that the following conditions (i) and (ii) hold:

(i) *there exist $r \in]0, +\infty]$ and a real $\alpha > 0$ such that for every $x \in \text{Tube}_r(C(t))$ one has*

$$d(0, \partial d_{C(t)}) \geq \alpha,$$

that is, the zero in $\mathcal{H}$ is kept α-far away from the C-subdifferential of $d_{C(t)}$ at every point in $\text{Tube}_r(C(t))$;

(ii) *for each $t \in I$ and each real $\mathfrak{r} > 0$ the set $C(t) \cap \mathfrak{r}\mathbb{B}$ is strongly compact in the space $\mathcal{H}$.*

Then the following hold.

(a) For each positive real $\lambda < \alpha^2 r/\kappa$, the differential inclusion (DI_λ) admits at least one absolutely continuous solution.

(b) For each sequence $(\lambda'_n)_{n\in\mathbb{N}}$ in $]0, +\infty[$, there exist a subsequence $(\lambda_n)_{n\in\mathbb{N}}$ and an absolutely continuous solution $u_{\lambda_n}(\cdot)$ of the differential inclusion (DI_{λ_n}) for each $n \in \mathbb{N}$ such that $(u_{\lambda_n}(\cdot))_{n\in\mathbb{N}}$ converges pointwise on I to an absolutely continuous solution $u(\cdot)$ of the sweeping process

$$\dot{u}(t) \in -N(C(t); u(t)) \quad \textit{with initial condition } u(T_0) = u_0,$$

and the derivative $\dot{u}(\cdot)$ of this solution satisfies $\|\dot{u}(t)\| \leq \alpha^{-2}\kappa$ for almost every $t \in I$.

We must point out that the case of state dependence, that is, the sweeping process $\dot{u}(t) \in -N\big(C(t, u(t)); u(t)\big)$, is also considered in [28] under similar hypotheses. E. Vilches [52] also studied the regularization process in [28] with an additional perturbed Lipschitz mapping.

Remark 7.4. It is worth mentioning that by (7.2) the condition in the assumption (ii) of Theorem 7.3 is equivalent to require that for each real $\mathfrak{r} > 0$ the set $C(I) \cap \mathfrak{r}\mathbb{B}$ is relatively compact (in fact compact), where $I := [T_0, T]$. More generally, suppose that $C(t, x)$ depends both on t and on another variable x in a normed space X, that is, $C : I \times X \rightrightarrows \mathcal{H}$ is a multimapping, and suppose that there are a function $L : X \to \mathbb{R}$ on X and a continuous function $v : I \to \mathbb{R}$ such that

$$\text{haus}\big(C(s, x), C(t, y)\big) \leq |L(x) - L(y)| + |v(s) - v(t)|,$$

for all $x, y \in \mathcal{H}$, all $s, t \in I$. Then the condition $C(t, A) \cap \mathfrak{r}\mathbb{B}$ is relatively compact for every $t \in I$, every real $\mathfrak{r} > 0$ and every bounded set A in X, is equivalent to the condition that $C(I \times A) \cap \mathfrak{r}\mathbb{B}$ is relatively compact for every real $\mathfrak{r} > 0$ and every bounded set A in X. It is sufficient to show the implication $\Rightarrow$. Let $\mathfrak{r} > 0$ and $A \subset X$ bounded. Take any sequence $(y_n)_{n\in\mathbb{N}}$ in $C(I \times A) \cap \mathfrak{r}\mathbb{B}$. For each $n \in \mathbb{N}$ there are $t_n \in I$ and $x_n \in A$ such that $y_n \in C(t_n, x_n) \cap \mathfrak{r}\mathbb{B}$. Extracting a subsequence if necessary we may suppose that $(t_n)_{n\in\mathbb{N}}$ converges to some $t_0 \in I$. Since $\mathrm{haus}\big(C(t_n, x_n), C(t_0, x_n)\big) \le |v(t_n) - v(t_0)|$, for each $n \in \mathbb{N}$ we can choose some $z_n \in C(t_0, x_n)$ such that $\|y_n - z_n\| \le |v(t_n) - v(t_0)| + 1/n$. From this and the boundedness of $(y_n)_{n\in\mathbb{N}}$ we see that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded, so there is some real $\mathfrak{r}_0$ such that we have $y_n \in C(t_0, A) \cap \mathfrak{r}_0\mathbb{B} + \varepsilon_n\mathbb{B}$, where $\varepsilon_n := (1/n) + |v(t_n) - v(t_0)|$, which furnishes some $b_n \in \mathbb{B}$ such that $y_n + \varepsilon_n b_n \in C(t_0, A) \cap \mathfrak{r}_0\mathbb{B}$. By relative compactness of $C(t_0, A) \cap \mathfrak{r}_0\mathbb{B}$ and convergence of $(\varepsilon_n b_n)_{n\in\mathbb{N}}$ (in fact to 0), it ensues that the sequence $(y_n)_{n\in\mathbb{N}}$ admits a convergent subsequence. This confirms the relative compactness of $C(I \times A) \cap \mathfrak{r}\mathbb{B}$. $\square$

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