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Truncated nonconvex state-dependent sweeping process: implicit and semi-implicit adapted Moreau's catching-up algorithms

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Abstract. In this paper, we deal with the existence of solutions for perturbed state-dependent Moreau's sweeping processes. Two ways are investigated to realize such a study, depending on the nature of the used scheme, namely implicit or semi-implicit. In both cases, our evolution problem is described in a general Hilbert space by a prox-regular moving set controlled through the truncated Hausdorff–Pompeiu distance. The normal cone involved is perturbed by a sum of a single-valued mapping and a multimapping.

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1. Introduction

Given a time interval $I := [0, T]$, a Hilbert space $\mathcal{H}$ and a closed-valued multimapping $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$, one can consider the problem of finding an absolutely continuous mapping $u(\cdot) : I \to \mathcal{H}$ satisfying

$$(\mathcal{P}) \begin{cases} -\dot{u}(t) \in N(C(t, u(t)); u(t)) + G(t, u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t, u(t)) \quad \text{for all } t \in I, \\ u(0) = u_0 \in C(0, u_0), \end{cases}$$

where $N(\cdot, \cdot)$ denotes a general concept of normal cone in $\mathcal{H}$ and $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a multimapping. The differential inclusion $(\mathcal{P})$ with $C(t, x) =: D(t)$ convex, $G \equiv 0$ and $N(\cdot, \cdot)$ the normal cone of convex analysis, that is,

$$(\mathcal{Q}) \begin{cases} -\dot{u}(t) \in N(D(t); u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in D(t) \quad \text{for all } t \in I, \\ u(0) = u_0 \in D(0), \end{cases}$$

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has been introduced in 1971 by Moreau in the famous "Travaux du Séminaire d'Analyse Convexe de Montpellier" [21] and called "sweeping process" ("processus de rafle" in French) due to its mechanical interpretation. Over the years, many variants of the so-called Moreau's sweeping process have been developed in the literature: stochastic [8], perturbed [2,20], nonconvex [15, 17,29], in Banach spaces framework [6]. To the best of our knowledge, the state-dependent sweeping process $(\mathcal{P})$ appeared for the first time in the thesis of Chraibi [11] for a convex moving set $C(t,x)$, in the particular case $\mathcal{H} = \mathbb{R}^3$ and $G \equiv 0$. The second work on that topic has been realized a decade later by Kunze and Monteiro Marques [18] in a general Hilbert setting. To construct a solution $u(\cdot)$ of $(\mathcal{P})$ with $G \equiv 0$, they used the implicit scheme

$$t_i^n := i\frac{T}{2^n}, \quad u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_{i+1}^n)}(u_i^n) \tag{1.1}$$

which is well defined thanks to the convexity of each $C(t,x)$ and an extension of Schauder's fixed point theorem (see Sect. 2). Their crucial assumptions are on one hand the Lipschitz behavior of the moving set $C(\cdot,\cdot)$, namely the existence of two reals $L_1, L_2 \geq 0$ such that

$$\operatorname{haus}(C(t,x), C(\tau,y)) \leq L_1\,|t-\tau| + L_2\,\|x-y\| \tag{1.2}$$

and, on the other hand, some compactness assumption on $C(\cdot,\cdot)$ through the Kuratowski or ball measure of noncompactness $\gamma(\cdot)$ of $\mathcal{H}$ (see Sect. 2); more precisely

$$\gamma(C(t,B)\cap \delta\mathbb{B}) < \gamma(B), \tag{1.3}$$

for each bounded subset B of $\mathcal{H}$ and for $\delta > 0$ large enough. It is worth pointing out that unlike $(\mathcal{Q})$, the existence of solutions for $(\mathcal{P})$ without any compactness-type assumption still remains an open question. In addition to their existence result, the authors of [18] showed that no solution of $(\mathcal{P})$ could be expected whenever $L_2 > 1$ in (1.2). We also refer to the works [9,10] for other developments based on fixed point theorems in the context of nonconvex prox-regular sets [12,26].
The evolution problem $(\mathcal{P})$ can also be handled without any fixed point arguments. This is the case for instance in [14] where Haddad developed the semi-implicit algorithm for a prox-regular moving set $C(\cdot,\cdot)$ and with a general multivalued perturbation $F(\cdot) := G(\cdot)$

$$t_i^n := i\frac{T}{2^n}, \quad u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n - (t_{i+1}^n - t_i^n) f_i^n), \tag{1.4}$$

with $f_i^n \in F(t_i^n, u_i^n)$, under the existence of a fixed strong compact set K such that $C(t,x) \subset K$ for every $(t,x) \in I \times \mathcal{H}$. Such a scheme has been also used in the much more general context of a subsmooth [3] moving set $C(\cdot,\cdot)$ [16]. Let us mention that, besides (1.1) and (1.4), the authors of [4] used another approach to get a solution of $(\mathcal{P})$ with $F \not\equiv 0$ based on a reduction technique to unconstrained differential inclusion.

The aim of the present paper is to develop existence results for the differential inclusion $(\mathcal{P})$ described by a prox-regular moving set of a general Hilbert space, with a perturbation $G := F + f$, where $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a multimapping and $f : I \times \mathcal{H} \to \mathcal{H}$ is a mapping. Doing so, we adapt the two major algorithms (1.1) and (1.4) and one of the main ideas of [5,22] in defining

$$u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, \omega_i^n)} \left(u_i^n - (t_{i+1}^n - t_i^n) f_i^n - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n) \mathrm{d}\tau \right),$$

with $f_i^n \in F(t_i^n, u_i^n)$ and where $\omega_i^n := u_i^n$ or $\omega_i^n := u_{i+1}^n$, depending on the scheme which is followed. As pointed out by Tolstonogov [28], the inequality (1.2) may fail for large classes of unbounded sets. This leads several researchers to weaken the inequality on the moving set in various forms (see, e.g., [1,23,24,27,28]). Following this way, we will replace in (1.2) the Hausdorff–Pompeiu distance $\operatorname{haus}(\cdot, \cdot)$ by the ρ-truncated one (resp., the ρ-excess if $C(\cdot, \cdot)$ is assumed to be bounded) that is $\operatorname{haus}_\rho(\cdot, \cdot)$ (resp., $\operatorname{exc}_\rho(\cdot, \cdot)$) for a suitable choice of $\rho > 0$. Concerning the perturbations, we will require that $f(\cdot, \cdot)$ is a Carathéodory mapping satisfying for some real $\alpha > 0$,

$$\|f(t, x)\| \leq \alpha(1 + \|x\|) \quad \text{for all } (t, x) \in I \times \mathcal{H} \tag{1.5}$$

and $F(\cdot, \cdot)$ is a scalarly upper semicontinuous multimapping with closed convex values such that for some real $\beta > 0$,

$$F(t, x) \subset \beta(1 + \|x\|)\mathbb{B} \quad \text{for all } (t, x) \in I \times \mathcal{H}. \tag{1.6}$$

The paper is organized as follows: Sect. 2 is devoted to recall fundamental background in variational analysis. Section 3 is concerned with specific results used in the proof of the existence theorems provided in Sects. 4 and 5.

2. Notation and preliminaries

In the whole paper, T is a positive real, I is the compact interval $[0, T]$, λ is the Lebesgue measure on I. The letter $\mathbb{R}_+ := [0, +\infty[$ (resp., $\mathbb{N}$) stands for the set of nonnegative reals (resp., set of integers starting from 1).

Throughout the paper, $\mathcal{H}$ is a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. The closed unit ball (resp., the open unit ball) of $\mathcal{H}$ is denoted by $\mathbb{B}$ (resp., $\mathbb{U}$) and the class of all bounded subsets of $\mathcal{H}$ is denoted by $\mathcal{B}$.

Let S be a nonempty subset of $\mathcal{H}$. The *support function of* S (resp., the *distance function from* S) is defined by

$$\sigma(v, S) := \sup_{x \in S} \langle v, x \rangle \quad \text{for all } v \in \mathcal{H}$$
$$(\text{resp., } d_S(x) =: d(x, S) := \inf_{y \in S} \|x - y\| \quad \text{for all } x \in \mathcal{H}).$$

As a classical application of the Hahn–Banach separation theorem, we know that for any two closed convex subsets S_1, S_2 of $\mathcal{H}$, one has

$$S_1 \subset S_2 \Leftrightarrow \sigma(\cdot, S_1) \leq \sigma(\cdot, S_2). \tag{2.1}$$

For any $x \in \mathcal{H}$, the (possibly empty) set of *nearest points* of x in S is defined as:

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

If $\mathrm{Proj}_S(x) = \{\overline{y}\}$ for some $\overline{y} \in S$, one says that $\mathrm{proj}_S(x)$ (or $P_S(x)$) is well defined and in such a case one sets $\mathrm{proj}_S(x) := \overline{y}$ (or $P_S(x) := \overline{y}$).

Let S' be another nonempty subset of $\mathcal{H}$ and let $\rho \in]0, +\infty]$ be an extended real. One defines the *ρ-pseudo excess of S over S'* (also called the *pseudo excess of the ρ-truncation of S over S'*) as the extended real

$$\mathrm{exc}_\rho(S, S') := \sup_{x \in S \cap \rho\mathbb{B}} d(x, S').$$

The *Hausdorff–Pompeiu ρ-pseudo distance between S and S'* is defined as:

$$\mathrm{haus}_\rho(S, S') := \max\{\mathrm{exc}_\rho(S, S'), \mathrm{exc}_\rho(S', S)\}.$$

If $\rho = +\infty$, we set by convention $\rho\mathbb{B} = \mathcal{H}$, so in this case the ρ-pseudo excess of S over S' (resp., the Hausdorff–Pompeiu ρ-pseudo distance between S and S') is the usual *excess of S over S'* (resp., the usual *Hausdorff–Pompeiu distance* between S and S'), i.e.,

$$\mathrm{exc}_\infty(S, S') = \sup_{x \in S} d(x, S') =: \mathrm{exc}(S, S')$$

(resp.,

$$\mathrm{haus}_\infty(S, S') = \max\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\} := \mathrm{haus}(S, S')).$$

For every real $\alpha > 0$ such that $\mathrm{exc}_\rho(S, S') < \alpha$, it is readily seen that

$$S \cap \rho\mathbb{B} \subset S' + \alpha\mathbb{B}. \tag{2.2}$$

Note also that

$$d(x', S') \leq \|x - x'\| + \mathrm{exc}_\rho(S, S') \quad \text{for all } x \in S \cap \rho\mathbb{B}, x' \in \mathcal{H},$$

or equivalently

$$d(x', S') \leq d(x', S \cap \rho\mathbb{B}) + \mathrm{exc}_\rho(S, S') \quad \text{for all } x' \in \mathcal{H}. \tag{2.3}$$

2.1. Proximal and Clarke normal cones

In this section, S is a subset of the real Hilbert space $\mathcal{H}$, U is a nonempty open subset of $\mathcal{H}$ and $f : U \to \mathbb{R} \cup \{+\infty\}$ is a function.

The *proximal normal cone* to S at $x \in \mathcal{H}$ is defined as the set

$$N^P(S; x) := \begin{cases} \{v \in \mathcal{H} : \exists r > 0, x \in \mathrm{Proj}_S(x + rv)\} & \text{if } x \in S, \\ \emptyset & \text{otherwise.} \end{cases}$$

For each $x \in S$, it is known that $N^P(S; x)$ is a convex cone (not necessarily closed) containing 0. From the definition, it is readily seen that for all $v \in \mathcal{H}$ with $\mathrm{Proj}_S(v) \neq \emptyset$,

$$v - w \in N^P(S; w) \quad \text{for all } w \in \mathrm{Proj}_S(v). \tag{2.4}$$

Through the concept of proximal normal cone, one defines the *proximal subdifferential* $\partial_P f(x)$ of f at $x \in U$ as the set

$$\partial_P f(x) := \left\{ v \in \mathcal{H} : (v, -1) \in N^P\big(\operatorname{epi} f; (x, f(x))\big) \right\},$$

where epi f is the *epigraph* of f, i.e.,

$$\operatorname{epi} f := \{(u, r) \in \mathcal{H} \times \mathbb{R} : u \in U, f(u) \leq r\}$$

and where $\mathcal{H} \times \mathbb{R}$ is endowed with the usual Hilbert product structure. In particular, note that $\partial_P f(x) = \emptyset$ if f is not finite at $x \in U$.

For each $x \in S$, the set of all vectors $h \in \mathcal{H}$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ of S with $x_n \to x$, for every sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals with $t_n \to 0$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $h_n \to h$ satisfying

$$x_n + t_n h_n \in S \quad \text{for all } n \in \mathbb{N}$$

is called the *Clarke tangent cone* to S at x and is denoted $T^C(S; x)$. It is known that this set is a closed convex cone containing 0. The *Clarke normal cone* of S at $x \in S$ is denoted by $N^C(S; x)$ and defined as the polar cone of $T^C(S; x)$, that is

$$N^C(S; x) := \left\{ v \in \mathcal{H} : \langle v, h\rangle \leq 0, \forall h \in T^C(S; x) \right\}.$$

It is usual to set for every $x \in \mathcal{H} \backslash S$, $T^C(S; x) := \emptyset =: N^C(S; x)$. Then, it can be checked in a direct way

$$N^P(S; x) \subset N^C(S; x) \quad \text{for all } x \in \mathcal{H}. \tag{2.5}$$

As for the proximal subdifferential, one defines the *Clarke subdifferential* $\partial_C f(x)$ of f at $x \in U$ by

$$\partial_C f(x) := \left\{ v \in \mathcal{H} : \ (v, -1) \in N^C\big(\operatorname{epi} f; (x, f(x))\big) \right\},$$

so $\partial_C f(x) = \emptyset$ whenever f is not finite at $x \in U$. From the inclusion (2.5), we observe that the proximal subdifferential is always included in the Clarke one, or in other words

$$\partial_P f(x) \subset \partial_C f(x) \quad \text{for all } x \in U.$$

If S is closed, the following relations between the proximal (resp., Clarke) subdifferential of the distance function of S and the proximal (resp., Clarke) normal cone to S hold true for all $x \in S$

$$\partial_P d_S(x) = N^P(S; x) \cap \mathbb{B} \quad \text{and} \quad \partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}. \tag{2.6}$$

If the function f is γ-Lipschitz near $x \in U$ for some real $\gamma \geq 0$, it is well known that $\partial_C f(x) \subset \gamma\mathbb{B}$ and

$$f^o(x; \cdot) = \sigma(\cdot; \partial_C f(x)),$$

where $f^o(x; \cdot)$ is the *Clarke directional derivative* of f at x, that is

$$f^o(x; h) := \limsup_{t\downarrow 0, x'\to x} t^{-1}(f(x' + th) - f(x')) \quad \text{for all } h \in \mathcal{H}.$$

Under such a Lipschitz assumption, $f^o(x;\cdot)$ is γ-Lipschitz on $\mathcal{H}$, convex and positively homogeneous. If U is convex, f Lipschitz continuous near x and convex on U, then

$$f^o(x;h) = f'(x;h) \quad \text{for all } h \in \mathcal{H}, \tag{2.7}$$

where $f'(x;\cdot)$ denotes the *standard directional derivative* of f at x, i.e.,

$$f'(x;h) := \lim_{t\downarrow 0} t^{-1}(f(x+th) - f(x)) \quad \text{for all } h \in \mathcal{H}.$$

2.2. Prox-regular sets in Hilbert setting

As mentioned above, we are interested in this paper in a variant of Moreau's sweeping process described by a prox-regular moving set. For the convenience of the reader, let us give some basic facts about prox-regularity. More details on this topic are available in the survey [12].

Definition 2.1 [26]. Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0,+\infty]$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P(S;x) \cap \mathbb{B}$ and for all $t \in]0,r[$, one has $x \in \mathrm{Proj}_S(x+tv)$.

The following theorem provides some useful characterizations and properties of uniform prox-regular sets (see, e.g., [12]). Before stating it, recall that for any extended real $r > 0$, the *r-open enlargement* of a subset S of $\mathcal{H}$ is defined as:

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}.$$

Theorem 2.2. *Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0,+\infty]$. Consider the following assertions.*

(a) *The set S is r-prox-regular.*

(b) *For all $x_1, x_2 \in S$, for all $v \in N^P(S;x_1)$, one has*

$$\langle v, x_2 - x_1\rangle \le \frac{1}{2r}\|v\|\,\|x_1 - x_2\|^2.$$

(c) *The mapping $\mathrm{proj}_S : U_r(S) \to S$ is well-defined and locally Lipschitz on $U_r(S)$.*

(d) *For all $u \in U_r(S)\setminus S$, one has (with $x = \mathrm{proj}_S(u)$)*

$$x = \mathrm{proj}_S\Big(x + t\frac{u-x}{\|u-x\|}\Big) \quad \textit{for all}\, t \in [0,r[.$$

(e) *For any $x \in S$, one has*

$$N^P(S;x) = N^C(S;x) \quad \textit{and} \quad \partial_P d_S(x) = \partial_C d_S(x).$$

Then, the assertions (a), (b), (c) *and* (d) *are pairwise equivalent and each one implies* (e).

According to (e) of Theorem 2.2, we put

$$N(S;x) := N^P(S;x) = N^C(S;x) \quad \text{for all } x \in S,$$

whenever S is a uniform prox-regular set of the real Hilbert space $\mathcal{H}$.

The following result is strongly inspired from [25, Theorem 2].

Proposition 2.3. *Let S_1, S_2 be r-prox-regular subsets of $\mathcal{H}$ with $r \in]0,+\infty[$, $\zeta \in]0,1[$, $\delta \in [0,+\infty[$, $x \in U_{r\zeta}(S_1) \cap U_{r\zeta}(S_2) \cap \delta\mathbb{B}$. If $\operatorname{haus}_{r\zeta+\delta}(S_1,S_2) \leq r$, then one has*

$$\left\|\operatorname{proj}_{S_1}(x) - \operatorname{proj}_{S_2}(x)\right\| \leq \left(\frac{2\zeta r}{1-\zeta}\operatorname{haus}_{r\zeta+\delta}(S_1,S_2)\right)^{1/2}.$$

Proof. Set $s := r\zeta + \delta$ and $h := \operatorname{haus}_s(S_1,S_2)$. Assume that $h \leq r$. For each $i \in \{1,2\}$, $\operatorname{Proj}_{S_i}(x)$ is reduced to a singleton $\{x_i\}$ (thanks to $x \in U_{r\zeta}(S_i)$ and the fact that S_i is r-prox-regular). Observe that

$$\|x_2\| \leq \|x_2 - x\| + \|x\| = d_{S_2}(x) + \|x\| < s,$$

i.e., $x_2 \in S_2 \cap s\mathbb{B}$. It follows that

$$d_{S_1}(x_2) \leq \sup_{x \in S_2 \cap s\mathbb{B}} d_{S_1}(x) = \operatorname{exc}_s(S_2,S_1) \leq h.$$

We claim that

$$2\langle x - x_1, x_2 - x_1\rangle \leq \zeta(\|x_1 - x_2\|^2 + 2rh).$$

We may assume that $x \neq x_1$, hence $x \notin S_1$. In particular, we have $x \in U_r(S_1)\backslash S_1$, so we can apply Theorem 2.2 to get

$$x_1 = \operatorname{proj}_{S_1}\left(x_1 + \frac{t(x-x_1)}{\|x-x_1\|}\right) \quad \text{for all } t \in [0,r[.$$

From the latter equality, we note that for all $z \in S_1$, for all $t \in [0,r[$,

$$\left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - x_2\right\| \geq \left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - z\right\| - \|x_2 - z\| \geq t - \|x_2 - z\|.$$

Passing to the supremum yields for all $t \in [0,r[$,

$$\left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - x_2\right\| \geq \sup_{z \in S_1}(t - \|x_2 - z\|) = t - d_{S_1}(x_2) \geq t - h.$$

Taking the limit as $t \uparrow r$ in both sides of the latter inequality, we arrive to

$$\left\|x_1 + \frac{r(x-x_1)}{\|x-x_1\|} - x_2\right\| \geq r - h.$$

We deduce from this (thanks to the inequality $r \geq h$)

$$\|x_1 - x_2\|^2 + \frac{2r}{\|x-x_1\|}\langle x - x_1, x_1 - x_2\rangle + r^2 \geq r^2 - 2rh,$$

or equivalently

$$2r\langle x - x_1, x_2 - x_1\rangle \leq \|x - x_1\|(\|x_1 - x_2\|^2 + 2rh).$$

Keeping in mind that $d_{S_1}(x) = \|x - x_1\| < r\zeta$, we obtain

$$2\langle x - x_1, x_2 - x_1\rangle \leq \zeta(\|x_1 - x_2\|^2 + 2rh),$$

which is the inequality claimed above. In the same way, we show

$$2\langle x - x_2, x_1 - x_2\rangle \leq \zeta(\|x_1 - x_2\|^2 + 2rh).$$

Adding the two latter inequalities, we have

$$\|x_1 - x_2\|^2 \leq \zeta(\|x_1 - x_2\|^2 + 2rh)$$

or equivalently

$$\|x_1 - x_2\|^2 \leq \frac{2r\zeta h}{1-\zeta}.$$

The proof is then complete. □

The case $r = +\infty$ in the latter proposition is due to Moreau (see, e.g., [19, Proposition 4.7]).

Proposition 2.4. *Let C, C' be two nonempty closed convex subsets of $\mathcal{H}$ and $x \in \mathcal{H}$. Then, one has*

$$\|\mathrm{proj}_C(x) - \mathrm{proj}_{C'}(x)\| \leq 2\big(d_C(x) + d_{C'}(x)\big)\mathrm{haus}(C, C').$$

Before stating the last result of this section, let us recall that a function $f : C \to \mathbb{R} \cup \{+\infty\}$ defined on a nonempty convex subset C of $\mathcal{H}$ is said to be σ-semiconvex (on C) for some $\sigma \in \mathbb{R}_+ := [0, +\infty[$ provided

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \frac{\sigma}{2}t(1-t)\|x - y\|^2,$$

for all $x_1, x_2 \in C$, for all $t \in]0, 1[$ or equivalently if $f + \frac{\sigma}{2}\|\cdot\|^2$ is convex on C.

Theorem 2.5. *Let S be an r-prox-regular subset of $\mathcal{H}$ for some $r \in]0, +\infty]$. Then, for any $s \in]0, r[$, for any nonempty convex set $C \subset U_s(S)$, the function d_S is $(r-s)^{-1}$-semiconvex on C.*

2.3. Kuratowski and ball measure of noncompactness

Our development below will require an extension of a classical result in functional analysis due to Schauder, which says that a continuous compact mapping $f : C \to C$ defined on C a nonempty closed bounded convex subset of a Banach space has a fixed point (see, e.g., [13]). Before stating it, we have to introduce the concept of measure of noncompactness. Recall that $\mathcal{B}$ denotes the class of all bounded subsets of $\mathcal{H}$.

Definition 2.6. One calls Kuratowski (resp., ball) measure of noncompactness on $\mathcal{H}$, the mapping $K : \mathcal{B} \to [0, +\infty[$ (resp., $B : \mathcal{B} \to [0, +\infty[$) defined by

$$K(\Omega) := \inf R(\Omega) \quad \text{for all } \Omega \in \mathcal{B},$$

(resp.,

$$B(\Omega) := \inf S(\Omega) \quad \text{for all } \Omega \in \mathcal{B}),$$

where for each $\Omega \in \mathcal{B}$, $R(\Omega)$ (resp., $S(\Omega)$) denotes the set of all reals $d > 0$ such that Ω admits a finite cover of subsets of $\mathcal{H}$ with diameter less or equal than d (resp., of balls of $\mathcal{H}$ with radius d).

Throughout the paper, the letter γ indifferently stands for the Kuratowski and ball measure of noncompactness. Before giving some first properties of $\gamma(\cdot)$, let us note the following obvious inequality:

$$\gamma(\alpha\mathbb{B}) \leq 2\alpha \quad \text{for all } \alpha \in \ ]0, +\infty[. \tag{2.8}$$

Proposition 2.7 [13, Proposition 7.2]. *The following hold.*

(a) *For all $\Omega \in \mathcal{B}$, $\gamma(\Omega) = 0$ if and only if $\mathrm{cl}_{\mathcal{H}}\Omega$ is compact.*
(b) *For all $\Omega_1, \Omega_2 \in \mathcal{B}$ with $\Omega_1 \subset \Omega_2$, one has $\gamma(\Omega_1) \le \gamma(\Omega_2)$.*
(c) *For all $\Omega_1, \Omega_2 \in \mathcal{B}$, one has $\gamma(\Omega_1 + \Omega_2) \le \gamma(\Omega_1) + \gamma(\Omega_2)$.*
(d) *For all $\Omega_1, \Omega_2 \in \mathcal{B}$, one has $\gamma(\Omega_1 \cup \Omega_2) \le \max\{\gamma(\Omega_1), \gamma(\Omega_2)\}$.*

The extension of Schauder's fixed point involves $\gamma(\cdot)$-condensing mappings.

Definition 2.8. Let Ω be a nonempty subset of $\mathcal{H}$, $f : \Omega \to \mathcal{H}$ be a continuous mapping on Ω. One says that f is $\gamma(\cdot)$-condensing whenever $\gamma(f(B)) < \gamma(B)$ for every bounded set $B \subset \Omega$ with $\gamma(B) > 0$.

Now, we are in position to state the following fixed point result.

Theorem 2.9 [13, Theorem 9.1] . *Let C be a nonempty closed bounded convex subset of $\mathcal{H}$, $f : C \to C$ be a $\gamma(\cdot)$-condensing mapping. Then, f has a fixed point.*

3. Preparatory results

This section is devoted to develop particular results which will be needed in the proof of the main results of the paper.

Before stating the first theorem, let us recall the concept of mapping with bounded variation. Let $u : I = [0, T] \to \mathcal{H}$ be a mapping. Any finite sequence $\sigma = (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that $0 = t_0 < \cdots < t_k = T$ is called a *subdivision* σ of I. One associates with such a subdivision σ, the real $S_\sigma := \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$. The *variation* of u on I is defined as the extended real

$$\mathrm{var}(u; I) := \sup_{\sigma \in \mathcal{S}} S_\sigma,$$

where $\mathcal{S}$ is the set of all subdivisions of I. The mapping u is said to be of *bounded variation on I* if $\mathrm{var}(u; I) < +\infty$.

The first result says that a sequence of uniformly bounded in norm and in variation mappings has a pointwise weakly convergent subsequence (see, e.g., [19, Theorem 2.1 p.10–11]).

Theorem 3.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of mappings from I to $\mathcal{H}$. Assume that:*

(a) *there exists a real $M > 0$ such that*

$$\|f_n(t)\| \le M \quad \text{for all } n \in \mathbb{N},\ t \in I;$$

(b) *there exists a real $L > 0$ such that*

$$\mathrm{var}(f_n; I) \le L \quad \text{for all } n \in \mathbb{N}.$$

Then, there exist a mapping $f : I \to \mathcal{H}$ with bounded variation on I and a subsequence $(f_{s(n)})_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that

$$f_{s(n)}(t) \xrightarrow{w} f(t) \quad \text{for all } t \in I,$$

where $\xrightarrow{w}$ denotes the weak convergence in $\mathcal{H}$.

Our aim is now to establish the following scalar upper semicontinuity property for prox-regular sets. It is a variant of a previous result due to Thibault and the author [23]. For the convenience of the reader, let us sketch a proof.

Proposition 3.2. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be an r-prox-regular valued multimapping for some $r \in]0, +\infty]$. Assume that there exist $\rho \in]0, +\infty]$, $L_1, L_2 \in [0, +\infty[$ such that for all $\tau, t \in I$ with $\tau \leq t$,*

$$\mathrm{exc}_\rho(C(\tau, x), C(t, y)) \leq L_1(t - \tau) + L_2 \|x - y\| . \tag{3.1}$$

Let $\overline{t} \in I$, $\overline{x} \in \mathcal{H}$ with $\overline{x} \in C(\overline{t}, \overline{x}) \cap \rho\mathbb{U}$, $(t_n)_{n\in\mathbb{N}}$ be a sequence of $[\overline{t}, T]$ with $t_n \to \overline{t}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}$ with $x_n \to \overline{x}$ and $x_n \in C(t_n, x_n)$ for all $n \in \mathbb{N}$. Then, one has

$$\limsup_{n\to+\infty} d^o_{C(t_n,x_n)}(x_n; h) \leq d^o_{C(\overline{t},\overline{x})}(\overline{x}; h) \quad \textit{for all}\, h \in \mathcal{H},$$

or equivalently

$$\limsup_{n\to+\infty} \sigma(h, \partial_C d_{C(t_n,x_n)}(x_n)) \leq \sigma(h, \partial_C d_{C(\overline{t},\overline{x})}(\overline{x})) \quad \textit{for all}\, h \in \mathcal{H}.$$

Before giving the proof, we need the following lemmas (see also [23]).

Lemma 3.3. *Let U be an open subset of $\mathcal{H}$, $x \in U$ and $g : U \to \mathbb{R}$ be a function. If there exists a real $\delta > 0$ with $B(x, \delta) \subset U$ and such that g is σ-semiconvex on $B(x, \delta)$ for a real $\sigma \geq 0$, then one has for all $h \in \mathbb{B}$,*

$$g^o(x; h) = \inf_{t\in]0,\delta[} t^{-1}\left[g(x + th) - g(x) + \frac{\sigma}{2}(\|x + th\|^2 - \|x\|^2)\right] - \sigma \langle x, h\rangle = g'(x; h).$$

Proof. Assume that there exists a real $\delta > 0$ such that $B(x, \delta) \subset U$ and g is σ-semiconvex on $B(x, \delta)$ for a real $\sigma \geq 0$. Fix any $h \in \mathbb{B}$. Set $f := g + \frac{\sigma}{2} \|\cdot\|^2$ which is convex on $B(x, \delta)$ according to the σ-semiconvexity on $B(x, \delta)$ of g. From (2.7), one observes that

$$\begin{aligned} f'(x; h) &= f^o(x; h) \\ &= \limsup_{t\downarrow 0, x'\to x} t^{-1}\left[(g(x' + th) - g(x')) + \frac{\sigma}{2}\|x' + th\|^2 - \frac{\sigma}{2}\|x'\|^2\right] \\ &= g^o(x; h) + D(\frac{\sigma}{2}\|\cdot\|^2)(x)(h). \end{aligned}$$

Since f is convex on $B(x, \delta)$ and $x + th \in B(x, \delta)$ for each $t \in]0, \delta[$, we have

$$f'(x; h) = \inf_{t\in]0,\delta[} t^{-1}(f(x + th) - f(x)).$$

It follows that

$$g^o(x; h) = -\sigma \langle x, h\rangle + \inf_{t\in]0,\delta[} t^{-1}(f(x + th) - f(x)),$$

so the first equality claimed is established. For the second, it remains to see that

$$\begin{aligned} g^o(x;h) &= f'(x;h) - D\left(\frac{\sigma}{2}\|\cdot\|^2\right)(x)(h) \\ &= f'(x;h) - \left(\frac{\sigma}{2}\|\cdot\|^2\right)'(x;h) \\ &= \left(f - \frac{\sigma}{2}\|\cdot\|^2\right)'(x;h) = g'(x;h). \end{aligned}$$

The proof is then complete. □

Lemma 3.4. *Let S be an r-prox-regular subset of $\mathcal{H}$ for some $r \in]0,+\infty]$. Then, for each $s \in]0,r[$, one has for all $(x,h) \in S \times \mathbb{B}$,*

$$\begin{aligned} (d_S)^o(x;h) &= \lim_{t\downarrow 0} t^{-1} d_S(x+th) \\ &= \inf_{t\in]0,s[} t^{-1}\left[d_S(x+th) + \frac{1}{2(r-s)}(\|x+th\|^2 - \|x\|^2)\right] - \frac{1}{r-s}\langle x,h\rangle . \end{aligned}$$

Proof. If $r = +\infty$, we know that S is convex as well as its associated distance function d_S. This justifies the equality claimed. Suppose now $r < +\infty$. Fix any $s \in]0,r[$. Let $(x,h) \in S \times \mathbb{B}$. It is easy to check that $B(x,s) \subset U_s(S)$, so we can apply Theorem 2.5 to get that d_S is $\frac{1}{r-s}$-semiconvex on $B(x,s)$. It remains to combine Lemma 3.3 with the equality $d_S(x) = 0$. □

Now, we are able to prove Proposition 3.2.

Proof. (of Proposition 3.2) Fix any $h \in \mathbb{B}$. Let $s \in]0,r[$. Since $C(\bar{t},\overline{x})$ is r-prox-regular, the mapping $\mathrm{proj}_{C(\bar{t},\overline{x})} : U_r(C(\bar{t},\overline{x})) \to \mathcal{H}$ is well-defined and norm-to-norm continuous. In particular, we have

$$\lim_{x\to\overline{x}} \mathrm{proj}_{C(\bar{t},\overline{x})}(x) = \mathrm{proj}_{C(\bar{t},\overline{x})}(\overline{x}) = \overline{x} \in \rho\mathbb{U},$$

so we can find a real $\alpha \in]0,s[$ such that for all $x \in B(\overline{x},\alpha)$,

$$\mathrm{proj}_{C(\bar{t},\overline{x})}(x) \in \rho\mathbb{U} \subset \rho\mathbb{B}.$$

From the latter inclusion and the inequalities valid for all $x \in B(\overline{x},\alpha)$, we see that

$$\begin{aligned} \|x - \mathrm{proj}_{C(\bar{t},\overline{x})}(x)\| &= d_{C(\bar{t},\overline{x})}(x) \\ &\le d_{C(\bar{t},\overline{x})\cap\rho\mathbb{B}}(x) \\ &\le \|x - \mathrm{proj}_{C(\bar{t},\overline{x})}(x)\|, \end{aligned}$$

which entails

$$d_{C(\bar{t},\overline{x})}(x) = d_{C(\bar{t},\overline{x})\cap\rho\mathbb{B}}(x) \quad \text{for all } x \in B(\overline{x},\alpha). \tag{3.2}$$

Applying Lemma 3.4 gives for every $n \in \mathbb{N}$, $\tau \in]0,\alpha[$, we have

$$\begin{aligned} d^o_{C(t_n,x_n)}(x_n;h) \le \tau^{-1} d_{C(t_n,x_n)}(x_n+\tau h) &- \frac{1}{r-s}\langle x_n,h\rangle \\ &+ \frac{\tau^{-1}}{2(r-s)}(\|x_n+\tau h\|^2 - \|x_n\|^2). \end{aligned} \tag{3.3}$$

Furthermore, for all $n \in \mathbb{N}$, for all $\tau \in]0, \alpha[$, we see that

$$\begin{aligned}
&\tau^{-1} d_{C(t_n,x_n)}(x_n + \tau h) \\
&\quad \le \tau^{-1}\big(d_{C(t_n,x_n)}(\overline{x} + \tau h) + \|x_n - \overline{x}\|\big) \\
&\quad \le \tau^{-1}\big(d_{C(\bar{t},\overline{x})\cap\rho\mathbb{B}}(\overline{x} + \tau h) + \mathrm{exc}_\rho(C(\bar{t},\overline{x}), C(t_n,x_n)) + \|x_n - \overline{x}\|\big) \\
&\quad \le \tau^{-1}\big(d_{C(\bar{t},\overline{x})\cap\rho\mathbb{B}}(\overline{x} + \tau h) + L_1(t_n - \bar{t}) + L_2\|x_n - \overline{x}\| + \|x_n - \overline{x}\|\big) \\
&\quad = \tau^{-1}\big(d_{C(\bar{t},\overline{x})}(\overline{x} + \tau h) + L_1(t_n - \bar{t}) + (1 + L_2)\|x_n - \overline{x}\|\big)
\end{aligned}$$

where the second inequality is due to (2.3), the third to (3.1) and where the equality is a direct consequence of (3.2). This entails that for all $\tau \in]0, \alpha[$,

$$\limsup_{n\to+\infty} \tau^{-1} d_{C(t_n,x_n)}(x_n + \tau h) \le \tau^{-1} d_{C(\bar{t},\overline{x})}(\overline{x} + \tau h). \tag{3.4}$$

On the other hand for all $\tau \in]0, \alpha[$,

$$\begin{aligned}
&\limsup_{n\to+\infty}\left[-\frac{1}{r-s}\langle x_n, h\rangle + \frac{1}{2(r-s)\tau}(\|x_n + \tau h\|^2 - \|x_n\|^2)\right] \\
&\quad = -\frac{1}{r-s}\langle \overline{x}, h\rangle + \frac{1}{2(r-s)\tau}(\|\overline{x} + \tau h\|^2 - \|\overline{x}\|^2) \\
&\quad = \frac{\tau}{2(r-s)}\|h\|^2.
\end{aligned} \tag{3.5}$$

Putting (3.3), (3.4) and (3.5) together, we see that for all $\tau \in]0, \alpha[$,

$$\limsup_{n\to+\infty} d^o_{C(t_n,x_n)}(x_n; h) \le \tau^{-1} d_{C(\bar{t},\overline{x})}(\overline{x} + \tau h) + \frac{\tau}{2(r-s)}\|h\|^2.$$

On the other hand, since $\overline{x} \in C(\bar{t}, \overline{x})$, we have

$$\limsup_{\tau\downarrow 0} \tau^{-1} d_{C(\bar{t},\overline{x})}(\overline{x} + \tau h) \le d^o_{C(\bar{t},\overline{x})}(\overline{x}; h),$$

hence

$$\limsup_{n\to+\infty} d^o_{C(t_n,x_n)}(x_n; h) \le d^o_{C(\bar{t},\overline{x})}(x; h).$$

The latter inequality being true for any $h \in \mathbb{B}$, the positive homogeneity of the Clarke directional derivative guarantees that it holds for all $h \in \mathcal{H}$. $\square$

4. Existence result for unbounded moving set through implicit algorithm

As mentioned above, in [18] Kunze and Monteiro Marques established in any Hilbert space the existence of solutions for the following state-dependent sweeping process

$$(\mathcal{SDSP})\begin{cases} -\dot{u}(t) \in N(C(t, u(t)); u(t)) + F(t, u(t)) + f(t, u(t)) & \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t, u(t)) \quad \text{for all } t \in I, \\ u(0) = u_0, \end{cases}$$

with $F, f \equiv 0$ and $C(\cdot,\cdot)$ closed convex valued, under the Lipschitz behavior of $C(\cdot,\cdot)$

$$\operatorname{haus}(C(t,x), C(\tau,y)) \le L_1 |t-\tau| + L_2 \|x-y\|, \tag{4.1}$$

where $L_1 \ge 0$ and $0 \le L_2 < 1$ and under the compactness assumption (1.3). Their technique involves the following implicit algorithm:

$$t_i^n := i\frac{T}{2^n}, \quad u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_{i+1}^n)}(u_i^n), \tag{4.2}$$

which is strongly based on the fixed point result stated in Theorem 2.9.

Our aim in the present section is to develop such an implicit approach for an r-prox-regular moving set $C(\cdot,\cdot)$ (possibly unbounded) with a perturbation described through a sum of a Carathéodory mapping $f(\cdot,\cdot) : I \times \mathcal{H} \to \mathcal{H}$ and a scalarly upper semicontinuous multimapping $F(\cdot,\cdot) : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ with closed convex values satisfying respectively (1.5) and (1.6). It is worth mentioning that the unperturbed (i.e., $F, f \equiv 0$) prox-regular case has already been considered in [9] with the help of (4.2) in a general separable Hilbert space. Besides the perturbation $f + F$ and the prox-regularity of $C(\cdot,\cdot)$, we also weaken the control on the moving set in (4.1) by replacing the Pompeiu–Hausdorff distance $\operatorname{haus}(\cdot,\cdot)$ by a truncated Hausdorff distance $\operatorname{haus}_\rho(\cdot,\cdot)$ i.e.,

$$\operatorname{haus}_\rho(C(t,x), C(\tau,y)) \le L_1 |t-\tau| + L_2 \|x-y\|,$$

for some suitable extended real $\rho > 0$ depending on the data $r, \|u_0\|, T, \alpha$ and β.

To construct our version of the implicit algorithm (4.2), namely

$$u_0^n := u_0, \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_{i+1}^n)}\Big(u_i^n - (t_{i+1}^n - t_i^n)\zeta(t_i^n, u_i^n) - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)\mathrm{d}\tau\Big)$$

and

$$\zeta(t_i^n, u_i^n) \in F(t_i^n, u_i^n),$$

the following result will be needed. With $F, f \equiv 0$, it can be seen as an extension of [18, Lemma 2.1] to the prox-regular setting.

Proposition 4.1. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping, $s, \theta \in I$, $t \in [s,T]$, $u \in \mathcal{H}$ with $u \in C(s,u)$, $z \in F(\theta,u)$.*

Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping and $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping such that:

(i) *the mapping $f(\cdot,u)$ is measurable and there exists a real $\alpha \ge 0$ such that*

$$\|f(\tau,u)\| \le \alpha(1+\|u\|) \quad \textit{for all}\, \tau \in [s,t];$$

(ii) *there exists a real $\beta \ge 0$ such that*

$$F(\theta,u) \subset \beta(1+\|u\|)\mathbb{B}.$$

With $\vartheta := (\alpha+\beta)(1+\|u\|)$ assume that:

(iii) *there exist two reals* $L_1 \geq 0$, $0 \leq L_2 < 1$ *and two extended reals* $r > \frac{(L_1+(1+L_2)\vartheta)(t-s)}{1-L_2}$, $\rho \geq \vartheta(t-s) + \|u\| + r$ *such that for every* $\tau, \tau' \in I$, $x, y \in \mathcal{H}$, $C(\tau, x)$ *is* r*-prox-regular and*

$$\mathrm{haus}_\rho(C(\tau, x), C(\tau', y)) \leq L_1 |\tau - \tau'| + L_2 \|x - y\|;$$

(iv) *there exists a real* $\delta \geq \frac{L_1+2\vartheta}{1-L_2}(t-s) + \|u\|$ *such that for every bounded subset* B *of* $\mathcal{H}$ *with* $\gamma(B) > 0$,

$$\gamma(C(t, B) \cap \delta\mathbb{B}) < \gamma(B).$$

Then, there exists $v \in \mathcal{H}$ *such that*

$$\|u - v\| \leq \frac{L_1 + 2\vartheta}{1 - L_2}(t - s) \quad \textit{and} \quad v = \mathrm{proj}_{C(t,v)}\left(u - (t-s)z - \int_s^t f(\tau, u)d\tau\right).$$

Proof. We may assume that $t \neq s$. Set

$$u' := u - (t-s)z - \int_s^t f(\tau, u)\mathrm{d}\tau \quad \text{and} \quad c := \frac{\big(L_1 + (1+L_2)\vartheta\big)(t-s)}{1 - L_2}.$$

From (i) and (ii), we observe that

$$\|u - u'\| \leq (t-s)\|z\| + \int_s^t \|f(\tau, u)\| d\tau \leq \vartheta(t-s). \tag{4.3}$$

Thanks to (iii), pick any real $r' \in]c, r[$. Assume for a moment that $c = 0$. Then, observe that $\vartheta = 0$ and $L_1 = 0$. Using the latter inequality and assumption (iii), we get

$$u' = u \quad \text{and} \quad \mathrm{haus}_\rho(C(s, u), C(t, u)) = 0.$$

It follows (keeping in mind that $u \in C(s, u) \cap \rho\mathbb{B}$ and that $C(\cdot, \cdot)$ is closed-valued since it takes prox-regular values) $u \in C(t, u)$. Consequently, we have

$$\|u - u'\| \leq \frac{L_1 + 2\vartheta}{1 - L_2}(t - s) \quad \text{and} \quad u = \mathrm{proj}_{C(t,u)}(u').$$

So, we may assume that $c > 0$. By virtue of the inclusion $u \in C(s, u) \cap \rho\mathbb{B}$, (4.3) and the inequality $c < r'$, we see that for all $w \in B[u', c]$,

$$\begin{aligned}
d_{C(t,w)}(u') &\leq d_{C(t,w)}(u) + \|u - u'\| \\
&\leq \mathrm{exc}_\rho(C(s, u), C(t, w)) + \vartheta(t - s) \\
&\leq (L_1 + \vartheta)(t - s) + L_2 \|u - w\| \\
&\leq (L_1 + \vartheta)(t - s) + L_2(\|u - u'\| + \|u' - w\|) \\
&\leq \big(L_1 + (1 + L_2)\vartheta\big)(t - s) + L_2 c \\
&= \frac{\big(L_1 + (1 + L_2)\vartheta\big)(t - s)}{1 - L_2}(1 - L_2 + L_2) \\
&= c < r'.
\end{aligned} \tag{4.4}$$

According to the r-prox-regularity of $C(t, w)$ for each $w \in B[u', c]$, the latter inequality allows us to consider the mapping $g : B[u', c] \to B[u', c]$ defined by

$$g(w) := \mathrm{proj}_{C(t,w)}(u') \quad \text{for all } w \in B[u', c].$$

Now, set $\kappa := \|u\| + \vartheta(t-s)$ and note that (see (4.3)) $\|u'\| \le \kappa$. Hence, (by virtue of (4.4)) we have

$$u' = u-(t-s)z-\int_s^t f(\tau,u)\mathrm{d}\tau \in U_{r'}(C(t,w))\cap\kappa\mathbb{B} \quad \text{for all} w \in B[u',c]. \quad (4.5)$$

Now, we claim that g is continuous on Ω. Fix any $\overline{v} \in \Omega := B[u',c]$ and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of Ω converging to $\overline{v}$. Let us distinguish two cases.

Case 1: $r = +\infty$. From (iii), we see that $\rho = +\infty$, so $\mathrm{haus}_\rho(\cdot,\cdot)$ coincides with $\mathrm{haus}(\cdot,\cdot)$. Combining Proposition 2.4, (iii) and (4.5), we obtain $g(v_n) \to g(\overline{v})$.

Case 2: $r < +\infty$. From (4.5) again, we have for all $n \in \mathbb{N}$,

$$u' = u-(t-s)z-\int_s^t f(\tau,u)\mathrm{d}\tau \in U_{r'}(C(t,v_n)) \cap U_{r'}(C(t,\overline{v})) \cap \kappa\mathbb{B}.$$

By assumption (keeping in mind that $\rho \ge r' + \kappa$) we have

$$\mathrm{haus}_{r'+\kappa}(C(t,v_n), C(t,\overline{v})) \to 0.$$

Thus, for $n \in \mathbb{N}$ large enough

$$\mathrm{haus}_{r'+\kappa}(C(t,v_n), C(t,\overline{v})) \le r.$$

This allows us to apply Proposition 2.3 to obtain that $g(v_n) \to g(\overline{v})$.
In any case, g is continuous at $\overline{v}$. Consequently, g is continuous on Ω. Now, fix any (bounded) subset B of Ω with $\gamma(B) > 0$. From the very definition of g, we have $g(B) \subset C(t,B)$. Thanks to (4.4), note that for each $b \in B$, $\|g(b) - u'\| = d_{C(t,b)}(u') \le c$ which entails the inclusion

$$g(B) \subset (c+\kappa)\mathbb{B} \subset \delta\mathbb{B}.$$

Combining Proposition 2.7 with assumption (iv), we arrive to

$$\gamma(g(B)) \le \gamma(C(t,B) \cap \delta\mathbb{B}) < \gamma(B).$$

Applying Theorem 2.9, we get a fixed point $v \in B[u',c]$ for g, that is,

$$\|u'-v\| \le \frac{(L_1 + (1+L_2)\vartheta)(t-s)}{1-L_2} \quad \text{and} \quad v = \mathrm{proj}_{C(t,v)}(u').$$

In particular, we obtain through (4.3),

$$\|u-v\| \le \left(\frac{L_1+(1+L_2)\vartheta}{1-L_2} + \vartheta\right)(t-s) = \frac{L_1+2\vartheta}{1-L_2}(t-s).$$

The proof is complete. □

Remark 4.2. According to Proposition 2.7, it is clear that (iv) above is satisfied if $C(t,B)$ is relatively ball-compact (i.e., the intersection of $C(t,B)$ with any closed ball of $\mathcal{H}$ is relatively compact) which is always the case if $\dim\mathcal{H} < +\infty$. □

Now, we are able to prove one of the main results of the paper which ensures the existence of solutions for the differential inclusion ($\mathcal{SDSP}$). Before stating it, let us recall that a multimapping $F : \mathcal{T} \rightrightarrows \mathcal{H}$ from a real Hausdorff topological space $\mathcal{T}$ to the Hilbert space $\mathcal{H}$ is said to be *scalarly* *upper semicontinuous* provided that for any $\zeta \in \mathcal{H}$, the function $\sigma(\zeta, F(\cdot)) : \mathcal{T} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is upper semicontinuous.

Theorem 4.3. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$, $u_0 \in \mathcal{H}$ with $u_0 \in C(0, u_0)$, $\rho_0 \in]\|u_0\|, +\infty[$. Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping and $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping such that:*

(i) *$f(\cdot, x)$(resp., $f(t, \cdot)$) is measurable for each $x \in \bigcup_{(t,a)\in I\times\mathcal{H}} C(t, a)$ (resp., continuous for each $t \in I$) and there exists a real $\alpha \geq 0$ such that*

$$\|f(t, x)\| \leq \alpha(1 + \|x\|) \quad \textit{for all}\, t \in I, x \in \bigcup_{(t,a)\in I\times\mathcal{H}} C(t, a);$$

(ii) *$F(\cdot, \cdot)$ is nonempty closed convex valued and scalarly upper-semicontinuous and there exists a real $\beta \geq 0$ such that*

$$F(t, x) \subset \beta(1 + \|x\|)\mathbb{B} \quad \textit{for all}\, t \in I, x \in \bigcup_{(t,a)\in I\times\mathcal{H}} C(t, a).$$

Assume that:

(iii) *there exist a real $L_1 \geq 0$, $L_2 \in [0, 1[$ with $\frac{1-L_2}{2(\alpha+\beta)} > T$ and an extended real $\rho \geq \frac{\rho_0+T(1-L_2)^{-1}(L_1+2(\alpha+\beta))}{1-2T(\alpha+\beta)(1-L_2)^{-1}} + r$ such that for every $\tau, t \in I$, $x, y \in \mathcal{H}$,*

$$\mathrm{haus}_\rho(C(t, x), C(\tau, y)) \leq L_1 |t - \tau| + L_2 \|x - y\| ;$$

(iv) *there exists a real $\delta > \frac{\|u_0\|+T(1-L_2)^{-1}(L_1+2(\alpha+\beta))}{1-2T(\alpha+\beta)(1-L_2)^{-1}} =: l$ such that for every bounded subset B of $\mathcal{H}$ with $\gamma(B) > 0$,*

$$\gamma(C(t, B) \cap \delta\mathbb{B}) < \gamma(B).$$

Then, there exists a Lipschitz continuous mapping $u : I \to \mathcal{H}$ satisfying the state-dependent sweeping process ($\mathcal{SDSP}$) along with

$$\|\dot{u}(t)\| \leq \frac{L_1 + 2(\alpha + \beta)(1 + l)}{1 - L_2} \quad \lambda\textit{-a.e. } t \in I.$$

Proof. For each $n \in \mathbb{N}$, let us consider the partition of I defined by

$$t_i^n := i\frac{T}{2^n} \quad \text{for all}\, i \in \{0, \ldots, 2^n\}$$

and let us set

$$\vartheta := (\alpha + \beta)(1 + l) \quad \text{and} \quad \eta := \frac{L_1 + 2\vartheta}{1 - L_2}.$$

Using the inequalities $r > 0$, $\rho_0 > \|u_0\|$, $\rho > l + r$ and $\delta > l$, we can choose $N \in \mathbb{N}$ such that for all integer $n \geq N$,

$$r > \eta\frac{T}{2^n}, \quad \rho \geq \vartheta\frac{T}{2^n} + l + r \quad \text{and} \quad \delta \geq \eta\frac{T}{2^n} + l. \tag{4.6}$$

For each $(t, x) \in I \times \mathcal{H}$, choose (keeping in mind that F takes nonempty values) $\zeta(t, x) \in F(t, x)$. Set $u_0^n := u_0$ for each integer $n \geq N$. Fix for a moment any integer $n \geq N$. By induction, let us construct a finite sequence $(u_i^n)_{1\leq i\leq 2^n}$ of $\mathcal{H}$ such that for all $i \in \{1, \ldots, 2^n\}$,

$$u_i^n := \mathrm{proj}_{C(t_i^n, u_i^n)} \left(u_{i-1}^n - (t_i^n - t_{i-1}^n)\zeta(t_{i-1}^n, u_{i-1}^n) - \int_{t_{i-1}^n}^{t_i^n} f(\tau, u_{i-1}^n)\mathrm{d}\tau \right), \tag{4.7}$$

$$\left\| u_i^n - u_{i-1}^n \right\| \leq \frac{L_1 + 2(\alpha+\beta)(1 + \left\| u_{i-1}^n \right\|)}{1 - L_2}(t_i^n - t_{i-1}^n) \tag{4.8}$$

and

$$\left\| u_{i-1}^n \right\| < l. \tag{4.9}$$

Observe first that $\|u_0^n\| < l$. Further, from (4.6), it is clear that

$$r > \frac{L_1 + (1 + L_2)(\alpha+\beta)(1 + \|u_0^n\|)}{1 - L_2}(t_1^n - t_0^n),$$
$$\rho \geq (\alpha+\beta)(1 + \|u_0^n\|)(t_1^n - t_0^n) + \|u_0^n\| + r$$

and

$$\delta \geq \frac{L_1 + 2(\alpha+\beta)(1 + \|u_0^n\|)}{1 - L_2}(t_1^n - t_0^n) + \|u_0^n\|.$$

Hence, we can apply Proposition 4.1 to get $u_1^n \in \mathcal{H}$ satisfying

$$u_1^n := \mathrm{proj}_{C(t_1^n, u_1^n)} \left(u_0^n - (t_1^n - t_0^n)\zeta(t_0^n, u_0^n) - \int_{t_0^n}^{t_1^n} f(\tau, u_0^n)\mathrm{d}\tau \right)$$

as well as

$$\|u_1^n - u_0^n\| \leq \frac{L_1 + 2(\alpha+\beta)(1 + \|u_0^n\|)}{1 - L_2}(t_1^n - t_0^n).$$

Fix any $k \in \{1, \ldots, 2^n - 1\}$. If $k = 1$, there is nothing to prove, so assume that $k > 1$. Suppose that all the steps of the induction from 1 to $k - 1$ have been realized. Fix any $q \in \{1, \ldots, k - 1\}$. Thanks to the inequality

$$\left\| u_q^n - u_{q-1}^n \right\| \leq \frac{L_1 + 2(\alpha+\beta)(1 + \left\| u_{q-1}^n \right\|)}{1 - L_2}(t_q^n - t_{q-1}^n)$$

we have

$$\left\| u_q^n \right\| \leq \left\| u_{q-1}^n \right\| + \frac{T}{2^n}(1 - L_2)^{-1}\big(L_1 + 2(\alpha+\beta)(1 + \left\| u_{q-1}^n \right\|)\big).$$

It follows that

$$\begin{aligned} \left\| u_q^n \right\| &\leq \|u_0^n\| + \frac{T}{2^n}(1 - L_2)^{-1} \sum_{j=0}^{q-1} \big(L_1 + 2(\alpha+\beta)(1 + \left\| u_j^n \right\|)\big) \\ &\leq \|u_0^n\| + T(1 - L_2)^{-1} \left(L_1 + 2(\alpha+\beta) \left(1 + \max_{0\leq j\leq q} \left\| u_j^n \right\| \right) \right) \end{aligned}$$

which entails

$$\max_{0\le j\le q}\left\|u_j^n\right\| \le \|u_0^n\| + T(1-L_2)^{-1}\left(L_1 + 2(\alpha+\beta)\left(1+\max_{0\le j\le q}\left\|u_j^n\right\|\right)\right),$$

or equivalently

$$(1-2(\alpha+\beta)T(1-L_2)^{-1})\max_{0\le j\le q}\left\|u_j^n\right\| \le \|u_0^n\| + T(1-L_2)^{-1}(L_1+2(\alpha+\beta)).$$

Coming back to the definition of l, we see that $\max_{0\le j\le q}\left\|u_j^n\right\| \le l$. By (4.6), we have

$$\begin{aligned} r &> \frac{L_1 + (1+L_2)(\alpha+\beta)(1+\left\|u_{k-1}^n\right\|)}{1-L_2}(t_k^n - t_{k-1}^n),\\ \rho &\ge (\alpha+\beta)(1+\left\|u_{k-1}^n\right\|)(t_k^n - t_{k-1}^n) + \left\|u_{k-1}^n\right\| + r \end{aligned}$$

and

$$\delta \ge \frac{L_1 + 2(\alpha+\beta)(1+\left\|u_{k-1}^n\right\|)}{1-L_2}(t_k^n - t_{k-1}^n) + \left\|u_{k-1}^n\right\|.$$

This allows us to apply again Proposition 4.1 to obtain $u_k^n \in \mathcal{H}$ satisfying

$$u_k^n := \mathrm{proj}_{C(t_k^n, u_k^n)}\left(u_{k-1}^n - (t_k^n - t_{k-1}^n)\zeta(t_{k-1}^n, u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(\tau, u_{k-1}^n)\mathrm{d}\tau\right)$$

along with

$$\left\|u_k^n - u_{k-1}^n\right\| \le \frac{\big(L_1 + 2(\alpha+\beta)(1+\left\|u_{k-1}^n\right\|)\big)(t_k^n - t_{k-1}^n)}{1-L_2}.$$

This completes the induction. The inequalities (4.8) and (4.9) furnish

$$\begin{aligned} \|u_{2^n}^n\| &\le \left\|u_{2^n-1}^n\right\| + \frac{L_1 + 2(\alpha+\beta)(1+\left\|u_{2^n-1}^n\right\|)}{1-L_2}\frac{T}{2^n}\\ &< l + \frac{L_1 + 2(\alpha+\beta)(1+l)}{1-L_2}\frac{T}{2^n}, \end{aligned}$$

which gives an integer $N' \ge N$ such that for all integer $n \ge N'$, $\|u_{2^n}^n\| \le l$. Thus, for all integer $n \ge N'$, we get

$$\|u_i^n\| \le l \quad \text{for all } i \in \{0,\dots,2^n\}. \tag{4.10}$$

We deduce for each $i \in \{0,\dots,2^n\}$,

$$\begin{aligned} z_i^n := \zeta(t_i^n, u_i^n) &\in F(t_i^n, u_i^n)\\ &\subset \beta(1+\|u_i^n\|)\mathbb{B}\\ &\subset \beta(1+l)\mathbb{B}. \end{aligned} \tag{4.11}$$

For each integer $n \ge N'$, let us define $u_n : I \to \mathcal{H}$ by $u_n(T) = u_{2^n}^n$ and

$$u_n(t) := u_i^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0,\dots,2^n-1\}.$$

To apply Theorem 3.1, note that from (4.10)

$$\|u_n(t)\| \le l < \rho \quad \text{for all } n \ge N',\ t \in I. \tag{4.12}$$

and from (4.8)

$$\operatorname{var}(u_n; I) = \sum_{i=1}^{2^n} \left\|u_i^n - u_{i-1}^n\right\| \le \eta T \quad \text{for all } n \ge N'. \tag{4.13}$$

Combining (4.12) and (4.13) and applying Theorem 3.1, we may assume without loss of generality the existence of a mapping $u(\cdot) : I \to \mathcal{H}$ with bounded variation on I such that

$$u_n(t) \xrightarrow{w} u(t) \quad \text{for all } t \in I. \tag{4.14}$$

Observe that for each integer $n \ge N'$, for any $p, q \in \{0, \dots, 2^n\}$ with $p < q$,

$$\left\|u_p^n - u_q^n\right\| \le \sum_{k=0}^{q-p-1} \left\|u_{p+k+1}^n - u_{p+k}^n\right\| \le \eta \sum_{k=0}^{q-p+1} (t_{p+k+1}^n - t_{p+k}^n) \le \eta(t_q^n - t_p^n),$$

which yields for each integer $n \ge N'$,

$$\|u_n(t) - u_n(s)\| \le \eta \left(|t-s| + \frac{1}{2^n}\right) \quad \text{for all } t, s \in I.$$

Thanks to the weak lower semicontinuity of $\|\cdot\|$, the latter inequality entails along with the weak convergence in (4.14)

$$\|u(t) - u(s)\| \le \liminf_{n \to +\infty} \|u_n(t) - u_n(s)\| \le \eta\,|t-s| \quad \text{for all } t, s \in I.$$

Hence, $u(\cdot)$ is Lipschitz continuous on I and

$$\|\dot u(t)\| \le \eta \quad \lambda\text{-a.e. } t \in I. \tag{4.15}$$

Now, we show that $u(\cdot)$ is a solution of $(\mathcal{SDSP})$. Concerning the initial condition (i.e., the value of $u(0)$), since $u_n(0) = u_0^n = u_0$ for each integer $n \ge N'$, the weak convergence provided by (4.14) ensures that

$$u(0) = u_0. \tag{4.16}$$

We claim that $u(t) \in C(t, u(t))$ for all $t \in I$. For each integer $n \ge N'$, define the mapping $\delta_n : I \to I$ by $\delta_n(T) := T$ and

$$\delta_n(t) := t_i^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0, \dots, 2^n - 1\}.$$

By virtue of (4.12) and (4.7), we see that

$$u_n(t) \in C(\delta_n(t), u_n(t)) \cap l\mathbb{B} \quad \text{for all } n \ge N', t \in I. \tag{4.17}$$

Fix for a moment any integer $n \ge N'$ and $t \in [0, T] = I$. From the inequality

$$\operatorname{exc}_l[C(\delta_n(t), u_n(t)), C(t, u_n(t))] \le L_1\,|\delta_n(t) - t| < \frac{L_1 T}{2^n},$$

it follows that (see (2.2))

$$C(\delta_n(t), u_n(t)) \cap l\mathbb{B} \subset C(t, u_n(t)) + \frac{L_1 T}{2^n}\mathbb{B}$$

or equivalently

$$C(\delta_n(t), u_n(t)) \cap l\mathbb{B} \subset \left[C(t, u_n(t)) \cap \left(l + \frac{L_1 T}{2^n}\right)\mathbb{B}\right] + \frac{L_1 T}{2^n}\mathbb{B}.$$

According to the third inequality of (4.6) and the latter inclusion, we have

$$u_n(t) \in C(\delta_n(t), u_n(t)) \cap l\mathbb{B} \subset [C(t, u_n(t)) \cap \delta\mathbb{B}] + \frac{L_1 T}{2^n}\mathbb{B}. \tag{4.18}$$

By contradiction assume that $U(t) := \{u_k(t) : k \geq N'\}$ is not relatively compact or equivalently (see Proposition 2.7) $\gamma(U(t)) > 0$. Using assumption (iv), we get

$$\gamma(U(t)) > \gamma[C(t, U(t)) \cap \delta\mathbb{B}]$$

and this obviously gives a real $\Delta > 0$ such that

$$\gamma(U(t)) - \gamma[C(t, U(t)) \cap \delta\mathbb{B}] \geq 2\Delta > 0. \tag{4.19}$$

Choose any integer $n_0 \geq N'$ such that $\frac{L_1 T}{2^{n-1}} \leq \Delta$ for all integer $n \geq n_0$. Using Proposition 2.7, the inclusions provided by (4.18), the inequality (2.8) and the choice of n_0, we can write

$$\begin{aligned}
\gamma(U(t)) = \gamma(\{u_k(t) : k \geq n_0\}) &\leq \gamma\left(C(t, U(t)) \cap \delta\mathbb{B} + \frac{L_1 T}{2^{n_0}}\mathbb{B}\right) \\
&\leq \gamma(C(t, U(t)) \cap \delta\mathbb{B}) + \gamma\left(\frac{L_1 T}{2^{n_0}}\mathbb{B}\right) \\
&\leq \gamma(C(t, U(t)) \cap \delta\mathbb{B}) + \frac{L_1 T}{2^{n_0 - 1}} \\
&\leq \gamma(C(t, U(t)) \cap \delta\mathbb{B}) + \Delta. \qquad (4.20)
\end{aligned}$$

It remains to combine (4.19) and (4.20) to get

$$\gamma(U(t)) \leq \gamma(U(t)) - 2\Delta + \Delta = \gamma(U(t)) - \Delta,$$

which is the desired contradiction. Hence, for all $t \in I$, $U(t)$ is relatively compact in $\mathcal{H}$ and this yields through (4.14)

$$u_n(t) \to u(t) \quad \text{for all}\, t \in I. \tag{4.21}$$

Fix for a moment any $t \in I$ and $n \geq N'$. Using assumption (iii) again, we have

$$\begin{aligned}
\operatorname{exc}_l[C(\delta_n(t), u_n(t)), C(t, u(t))] &\leq L_1(t - \delta_n(t)) + L_2 \|u_n(t) - u(t)\| \\
&< \frac{L_1 T}{2^n} + L_2 \|u_n(t) - u(t)\|
\end{aligned}$$

and this ensures by (2.2) and (4.17)

$$u_n(t) \in C(\delta_n(t), u_n(t)) \cap l\mathbb{B} \subset C(t, u(t)) + \left(L_2 \|u_n(t) - u(t)\| + \frac{L_1 T}{2^n}\right)\mathbb{B}.$$

Consequently, we obtain

$$d_{C(t,u(t))}(u_n(t)) \to 0 \quad \text{for all}\, t \in I$$

and thanks to the closedness of $C(\cdot,\cdot)$, we arrive to

$$u(t) \in C(t, u(t)) \quad \text{for all } t \in I.$$

Now, for each integer $n \geq N'$, let us define the mapping $v_n : I \to \mathcal{H}$ by

$$v_n(t) := u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}\left(u_{i+1}^n - u_i^n + (t_{i+1}^n - t_i^n)z_i^n + \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)\mathrm{d}\tau\right)$$
$$- (t - t_i^n)z_i^n - \int_{t_i^n}^{t} f(\tau, u_i^n)\mathrm{d}\tau,$$

for all $t \in I$ with $i \in \{0, \ldots, 2^n - 1\}$ such that $t \in [t_i^n, t_{i+1}^n]$. Combining the definition of $v_n(\cdot), u_n(\cdot)$ (with $n \geq N'$), (4.8), (4.10), (4.11) and the assumption (i), we get for all $t \in [t_i^n, t_{i+1}^n[$ with $i \in \{0, \ldots, 2^n - 1\}$

$$\|v_n(t) - u_n(t)\| \leq \left\|u_{i+1}^n - u_i^n\right\| + 2(t_{i+1}^n - t_i^n)\left\|z_i^n\right\| + 2\int_{t_i^n}^{t_{i+1}^n} \|f(\tau, u_i^n)\| \,\mathrm{d}\tau$$
$$\leq \frac{T}{2^n}\big(\eta + 2(\alpha+\beta)(1+l)\big) = \frac{T}{2^n}(\eta + 2\vartheta).$$

In particular, we have $v_n(t) - u_n(t) \to 0$ for all $t \in I$. Through (4.21), the latter convergence obviously entails

$$v_n(t) \to u(t) \quad \text{for all } t \in I.$$

Observe that for every integer $n \geq N'$, $i \in \{0, \ldots, 2^n - 1\}$, $v_n(\cdot)$ is differentiable at $t \in]t_i^n, t_{i+1}^n[$ and

$$\dot{v}_n(t) + z_i^n + f(t, u_i^n) = \frac{u_{i+1}^n - u_i^n + (t_{i+1}^n - t_i^n)z_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n)\mathrm{d}s}{t_{i+1}^n - t_i^n}. \tag{4.22}$$

For each integer $n \geq N'$, let us define the mapping $z_n : I \to \mathcal{H}$ by $z_n(T) := z_{2^n}$,

$$z_n(t) := z_i^n \quad \text{if } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0, \ldots, 2^n - 1\}.$$

Fix for a moment any integer $n \geq N'$. Thanks to (4.11), note that

$$\|z_n(t)\| \leq \beta(1+l) \quad \text{for all } t \in I. \tag{4.23}$$

Combining (4.8), (4.22), (4.23) and the assumption (i), we see that

$$\|\dot{v}_n(t) + z_n(t) + f(t, u_n(\delta_n(t)))\| \leq \eta + \vartheta =: \omega \quad \lambda\text{-a.e. } t \in I. \tag{4.24}$$

According to (4.23), we may assume for a mapping $z(\cdot) \in L^1(I, \mathcal{H}, \lambda)$ that

$$z_n(\cdot) \to z(\cdot) \quad \text{weakly in } L^1(I, \mathcal{H}, \lambda). \tag{4.25}$$

Set for λ-almost every $t \in I$ and all integer $n \geq N'$,

$$\zeta_n(t) := \omega^{-1}(\dot{v}_n(t) + z_n(t) + f(t, u_n(\delta_n(t)))) \quad \text{and}$$
$$\zeta(t) := \omega^{-1}(\dot{u}(t) + z(t) + f(t, u(t))).$$

Putting (2.4), (4.7), (4.22) and (4.24) together, we observe that for λ-almost every $t \in I$ and for all integer $n \geq N'$,

$$\zeta_n(t) \in -N\Big(C\big(\theta_n(t), u_n(\theta_n(t))\big); u_n(\theta_n(t))\Big) \cap \mathbb{B}, \tag{4.26}$$

where $\theta_n : I \to I$ is the function defined by $\theta_n(0) := t_1^n$ and

$$\theta_n(t) := t_{i+1}^n \quad \text{for all}\, t \in]t_i^n, t_{i+1}^n] \,\text{with}\, i \in \{0, \ldots, 2^n - 1\}.$$

By virtue of (2.6), the inclusion (4.26) can be rewritten as:

$$\zeta_n(t) \in -\partial_P d_{C\left(\theta_n(t), u_n(\theta_n(t))\right)}(u_n(\theta_n(t))) \quad \lambda\text{-a.e.}\, t \in I,\, n \geq N'.$$

On the other hand, the inequality (see (4.24))

$$\|\dot{v}_n(t)\| \leq \omega + (\alpha + \beta)(1 + l) \quad \lambda\text{-a.e.}\, t \in I,\, n \geq N',$$

allows us to assume that $(\dot{v}_n(\cdot))_{n \geq N'}$ weakly converges in $L^2(I, \mathcal{H}, \lambda)$ to some $h(\cdot) \in L^2(I, \mathcal{H}, \lambda)$. The absolute continuity of each $v_n(\cdot)$ with $n \geq N'$ guarantees

$$v_n(t) = v_n(0) + \int_0^t \dot{v}_n(s)\mathrm{d}s \quad \text{for all}\, n \geq N', t \in I,$$

so passing to the limit gives

$$u(t) = u(0) + \int_0^t h(s)\mathrm{d}s,$$

hence $\dot{u}(\cdot) = h(\cdot)$ λ-a.e. on I. In particular, we have

$$\dot{v}_n(\cdot) \to \dot{u}(\cdot) \quad \text{weakly in}\, L^2(I, \mathcal{H}, \lambda)$$

and this yields

$$\dot{v}_n(\cdot) \to \dot{u}(\cdot) \quad \text{weakly in}\, L^1(I, \mathcal{H}, \lambda). \tag{4.27}$$

Thanks to the continuity assumption on f, note that

$$f(t, u_n(t)) \to f(t, u(t)) \quad \text{for all}\, t \in I.$$

The latter pointwise convergence and the inequality provided by (i) entail through the Lebesgue dominated convergence theorem

$$f(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot)) \quad \text{strongly in}\, L^1(I, \mathcal{H}, \lambda). \tag{4.28}$$

Now, we apply a classical technique due to Castaing [7]. By virtue of (4.25), (4.27) and (4.28), observe first that

$$\zeta_n(\cdot) \to \zeta(\cdot) \quad \text{weakly in}\, L^1(I, \mathcal{H}, \lambda).$$

Thanks to Mazur's lemma, there exists a sequence $(\xi_n(\cdot))_{n \geq N'}$ which converges strongly in $L^1(I, \mathcal{H}, \lambda)$ to $\zeta(\cdot)$ with

$$\xi_n(\cdot) \in \mathrm{co}\,\{\zeta_k(\cdot) : k \geq n\} \quad \text{for all}\, n \geq N'.$$

Extracting a subsequence if necessary, we may suppose that

$$\xi_n(t) \to \zeta(t) \quad \lambda\text{-a.e.}\, t \in I$$

which allows us to write

$$\zeta(t) \in \bigcap_{n \geq N'} \overline{\mathrm{co}}\,\{\zeta_k(t) : k \geq n\} \quad \lambda\text{-a.e. } t \in I$$

and such an inclusion yields for λ-almost every $t \in I$ that

$$\langle \xi, \zeta(t) \rangle \leq \inf_{n \geq N'} \sup_{k \geq n} \langle \xi, \zeta_k(t) \rangle \quad \text{for all } \xi \in \mathcal{H}.$$

It follows that, for λ-almost every $t \in I$,

$$\langle \xi, \zeta(t) \rangle \leq \limsup_{n \to +\infty} \sigma\big(\xi, -\partial_P d_{C(\theta_n(t), u_n(\theta_n(t)))}(u_n(\theta_n(t)))\big) \quad \text{for all } \xi \in \mathcal{H}.$$

Hence, according to Proposition 3.2, for λ-almost every $t \in I$,

$$\langle \xi, \zeta(t) \rangle \leq \sigma\big(\xi, \partial_C d_{C(t,u(t))}(u(t))\big) \quad \text{for all } \xi \in \mathcal{H}.$$

Thanks to (2.1), we have

$$\{\zeta(t)\} \subset \overline{\mathrm{co}}\,\big(-\partial_C d_{C(t,u(t))}(u(t))\big) \quad \lambda\text{-a.e. } t \in I$$

or equivalently (since the Clarke subdifferential is always closed and convex)

$$\zeta(t) \in -\partial_C d_{C(t,u(t))}(u(t)) \quad \lambda\text{-a.e. } t \in I.$$

This inclusion and (2.6) furnish

$$\zeta(t) \in -N(C(t, u(t)); u(t)) \quad \lambda\text{-a.e. } t \in I.$$

Coming back to the definition of $\zeta(\cdot)$, we arrive to the inclusion

$$\omega^{-1}\big(\dot{u}(t) + z(t) + f(t, u(t))\big) \in -N(C(t, u(t)); u(t)) \quad \lambda\text{-a.e. } t \in I,$$

or equivalently

$$\dot{u}(t) + z(t) + f(t, u(t)) \in -N(C(t, u(t)); u(t)) \quad \lambda\text{-a.e. } t \in I. \tag{4.29}$$

Now, we show that, $z(t) \in F(t, u(t))$ for λ-almost every $t \in I$. Thanks to the fact that $z_k(\cdot)$ converges to $z(\cdot)$ weakly in $L^1(I, \mathcal{H}, \lambda)$ (see (4.25)), via Mazur's lemma again, extracting a subsequence if necessary, we may write

$$z(t) \in \bigcap_{n \geq N'} \overline{\mathrm{co}}\,\{z_k(t) : k \geq n\} \quad \lambda\text{-a.e. } t \in I.$$

On the other hand, it can be checked (with the help of (4.11)) that for every integer $n \geq N'$ and all $t \in I$,

$$z_n(t) \in F(\delta_n(t), u_n(t)).$$

Thus, for λ-almost every $t \in I$, we have

$$\langle \xi, z(t) \rangle \leq \limsup_{n \to +\infty} \sigma\big(\xi, F(\delta_n(t), u_n(t))\big) \quad \text{for all } \xi \in \mathcal{H}$$

and this entails (through the fact that $F(\cdot, \cdot)$ is scalarly upper semicontinuous) that for ν-almost every $t \in I$

$$\langle \xi, z(t) \rangle \leq \sigma(\xi, F(t, u(t))) \quad \text{for all } \xi \in \mathcal{H}.$$

Since $F(t, u(t))$ is closed and convex for all $t \in I$, we have (thanks to (2.1))

$$z(t) \in F(t, u(t)) \quad \lambda\text{-a.e. } t \in I. \tag{4.30}$$

It remains to put together (4.15), (4.16), (4.29) and (4.30) to complete the proof. $\square$

An important consequence of the latter theorem with $\alpha = 0$ and $\beta = 0$ corresponds to the unperturbed case (i.e., $F, f \equiv 0$).

Corollary 4.4. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$, $u_0 \in \mathcal{H}$ with $u_0 \in C(0, u_0)$, $\rho_0 \in]\, \|u_0\|, +\infty[$. Assume that:*

(i) *there exist a real $L_1 \geq 0$, a real $L_2 \in [0, 1[$ and an extended real $\rho \geq \rho_0 + L_1 T(1 - L_2)^{-1} + r$ such that*

$$\mathrm{haus}_\rho(C(t, x), C(\tau, y)) \leq L_1 |t - \tau| + L_2 \|x - y\| \quad \textit{for all}\, t, \tau \in I, x, y \in \mathcal{H};$$

(ii) *there exists a real $\delta > \|u_0\| + L_1 T(1 - L_2)^{-1}$ such that for every bounded subset B of $\mathcal{H}$ with $\gamma(B) > 0$,*

$$\gamma(C(t, B)) \cap \delta\mathbb{B}) < \gamma(B).$$

Then, there exists a Lipschitz continuous mapping $u : I \to \mathcal{H}$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t, u(t)); u(t)) & \lambda\text{-}a.e.\, t \in I, \\ u(t) \in C(t, u(t)) & \textit{for all}\, t \in I, \\ u(0) = u_0 \end{cases}$$

along with

$$\|\dot{u}(t)\| \leq \frac{L_1}{1 - L_2} \quad \lambda\text{-}a.e.\, t \in I.$$

Remark 4.5. It is worth pointing out that we recover the first general result of the theory of state-dependent sweeping process ([18, Theorem 3.3]) whenever $r = \rho = +\infty$ in the latter corollary (that is, $C(\cdot, \cdot)$ is nonempty closed convex valued and Lipschitz with respect to the Hausdorff–Pompeiu distance). $\square$

5. Semi-implicit algorithm for bounded state-dependent sweeping process

To deal with the evolution problem $(\mathcal{P})$ described in the introduction, Haddad developed [14] the semi-implicit scheme (with $G = F$)

$$t_i^n := i\frac{T}{2^n}, \quad u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \mathrm{proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n - (t_{i+1}^n - t_i^n) f_i^n), \tag{5.1}$$

where $f_i^n \in F(t_{i+1}^n, u_{i+1}^n)$. The existence of solutions for $(\mathcal{P})$ is established under the Lipschitz control (1.2) on the prox-regular moving set $C(\cdot, \cdot)$, the existence of a fixed strong compact set $\mathcal{K}$ satisfying

$$C(t, x) \subset \mathcal{K} \quad \text{for all}\, (t, x) \in I \times \mathcal{H}$$

and under the growth linear condition (1.6) for the semicontinuous convex weakly compact valued multimapping $F(\cdot, \cdot)$. The algorithm (5.1) has been also recently used in [16] for the same differential inclusion $(\mathcal{P})$ associated with a subsmooth moving set $C(\cdot,\cdot)$. Recall that the class of subsmooth sets has been introduced by Aussel, Daniilidis and Thibault [3] and contains strictly the class of prox-regular sets. However, subsmooth sets do not possess in general any property on nearest points and this leads the authors of [16] to assume a property of ball-compactness of the moving set in order to get $\mathrm{Proj}_{C(t,y)}(x) \neq \emptyset$.

The main objective here is to provide, besides Theorem 4.3, another existence result for $(\mathcal{SDSP})$ described by a prox-regular bounded moving set, say $C(t,x) \subset \rho\mathbb{B}$, with the help of the semi-implicit scheme

$$u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \mathrm{proj}_{C(t_i^n,u_i^n)} \left(u_i^n - (t_{i+1}^n - t_i^n) f_i^n - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)\mathrm{d}\tau \right).$$

Doing so, we will relax the control on $C(\cdot,\cdot)$ in (1.2) through the Hausdorff–Pompeiu excess; more precisely, we will assume that for every $\tau < t$, for all $x, y \in \mathcal{H}$,

$$\mathrm{exc}(C(\tau,x), C(t,y)) \le L_1(t-\tau) + L_2 \|x-y\|,$$

for some reals $L_1 \ge 0$ and $L_2 \in [0,1[$. The compactness hypothesis on $C(\cdot,\cdot)$ and the involved perturbations f, F of the normal cone will be of the same type as in Theorem 4.3.

Theorem 5.1. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be an r-prox-regular-valued multimapping for some $r \in]0,+\infty]$, $u_0 \in \mathcal{H}$ with $u_0 \in C(0,u_0)$. Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping, $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping such that:*

(i) *$f(\cdot,x)$ (resp., $f(t,\cdot)$) is measurable for each $x \in \bigcup_{(t,a)\in I\times\mathcal{H}} C(t,a)$ (resp., continuous for each $t \in I$) and there exists a real $\alpha \ge 0$ such that*

$$\|f(t,x)\| \le \alpha(1+\|x\|) \quad \textit{for all}\,(t,x) \in I \times \bigcup_{(t,a)\in I\times\mathcal{H}} C(t,a);$$

(ii) *the multimapping $F(\cdot,\cdot)$ is nonempty closed convex valued and scalarly upper-semicontinuous and there exists a real $\beta \ge 0$ such that*

$$F(t,x) \subset \beta(1+\|x\|)\mathbb{B} \quad \textit{for all}\, t \in I, x \in \bigcup_{(t,a)\in I\times\mathcal{H}} C(t,a).$$

Assume that:

(iii) *there exist three reals $\rho > 0$, $L_1 \ge 0$ and $0 \le L_2 < 1$ such that for every $(t,x) \in \mathcal{H} \times I$,*

$$C(t,x) \subset \rho\mathbb{B}$$

and for every $x, y \in \mathcal{H}$ and every $\tau, t \in I$ with $\tau < t$,

$$\mathrm{exc}(C(\tau,x), C(t,y)) \le L_1(t-\tau) + L_2 \|x-y\|;$$

(iv) *for every bounded subset B of $\mathcal{H}$ with $\gamma(B) > 0$,*

$$\gamma(C(t,B)) < \gamma(B).$$

Then, there exists a Lipschitz continuous mapping $u : I \to \mathcal{H}$ satisfying $(\mathcal{SDSP})$ along with

$$\|\dot{u}(t)\| \leq \frac{L_1 + 2(\alpha+\beta)(1+\rho)}{1-L_2} \quad \lambda\text{-}a.e.\, t \in I.$$

Proof. Set $\kappa := \frac{L_1+2(\alpha+\beta)(1+\rho)}{1-L_2}$. For each $n \in \mathbb{N}$, let us consider (as above) the partition of I

$$t_i^n := i\frac{T}{2^n} \quad \text{for all}\, i \in \{0, \dots, 2^n\}.$$

Since $\frac{T}{2^n} \to 0$, we can choose (keeping in mind that $0 \leq L_2 < 1$) $N \in \mathbb{N}$ such that for all integer $n \geq N$,

$$\frac{T}{2^n}(L_1 + 2(\alpha+\beta)(1+\rho)) \sum_{j=0}^{+\infty} L_2^j = \frac{T}{2^n}\kappa < r. \tag{5.2}$$

Fix for a moment any $n \in \mathbb{N}$ with $n \geq N$. For each $(t,x) \in I \times \mathcal{H}$, choose $\zeta(t,x) \in F(t,x) \neq \emptyset$. Set $u_0^n := u_0$. By induction, let us construct a sequence $(u_i^n)_{i \in \{1,\dots,2^n\}}$ such that for each $i \in \{1, \dots, 2^n\}$,

$$u_i^n := \mathrm{proj}_{C(t_i^n, u_{i-1}^n)}\left(u_{i-1}^n - (t_i^n - t_{i-1}^n)\zeta(t_{i-1}^n, u_{i-1}^n) - \int_{t_{i-1}^n}^{t_i^n} f(\tau, u_{i-1}^n)\mathrm{d}\tau\right)$$

and

$$\left\|u_i^n - u_{i-1}^n\right\| \leq \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) \sum_{j=0}^{i-1} L_2^j \leq \frac{T}{2^n}\kappa. \tag{5.3}$$

From (i), (ii), the inclusion $u_0^n \in C(t_0^n, u_0^n)$, (iii) and (5.2), we see that

$$\begin{aligned}
&d_{C(t_1^n,u_0^n)}\left(u_0^n - (t_1^n - t_0^n)\zeta(t_0^n,u_0^n) - \int_{t_0^n}^{t_1^n} f(\tau,u_0^n)\mathrm{d}\tau\right) \\
&\quad \leq d_{C(t_1^n,u_0^n)}(u_0^n) + (t_1^n - t_0^n)\left\|\zeta(t_0^n,u_0^n)\right\| + \int_{t_0^n}^{t_1^n} \left\|f(\tau,u_0^n)\right\| \mathrm{d}\tau \\
&\quad \leq \mathrm{exc}(C(t_0^n,u_0^n), C(t_1^n,u_0^n)) + (t_1^n - t_0^n)(\alpha+\beta)(1+\|u_0^n\|) \\
&\quad \leq (t_1^n - t_0^n)\big(L_1 + (\alpha+\beta)(1+\rho)\big) = \frac{T}{2^n}\big(L_1 + (\alpha+\beta)(1+\rho)\big) \\
&\quad < r.
\end{aligned} \tag{5.4}$$

Hence, the r-prox-regularity of $C(t_0^n, u_0^n)$ allows us to set

$$u_1^n := \mathrm{proj}_{C(t_1^n,u_0^n)}\left(u_0^n - (t_1^n - t_0^n)\zeta(t_0^n,u_0^n) - \int_{t_0^n}^{t_1^n} f(\tau,u_0^n)\mathrm{d}\tau\right). \tag{5.5}$$

Using (5.5), (i), (ii), (5.4) and (iii), it is clear that

$$\begin{aligned}\|u_1^n - u_0^n\| &\leq \left\| u_1^n - \left(u_0^n - (t_1^n - t_0^n)\zeta(t_0^n, u_0^n) - \int_{t_0^n}^{t_1^n} f(\tau, u_0^n)\mathrm{d}\tau \right) \right\| \\ &\quad + (t_1^n - t_0^n)\, \|\zeta(t_0^n, u_0^n)\| + \int_{t_0^n}^{t_1^n} \|f(\tau, u_0^n)\|\, \mathrm{d}\tau \\ &\leq d_{C(t_1^n, u_0^n)} \left(u_0^n - (t_1^n - t_0^n)\zeta(t_0^n, u_0^n) - \int_{t_0^n}^{t_1^n} f(\tau, u_0^n)\mathrm{d}\tau \right) \\ &\quad + (t_1^n - t_0^n)(\alpha + \beta)(1 + \|u_0^n\|) \\ &\leq \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) \leq \frac{T}{2^n}\kappa.\end{aligned}$$

Fix any $k \in \{1, \ldots, 2^n\}$. We may assume that $k > 1$ otherwise there is nothing to establish. Suppose that the steps $1, \ldots, k-1$ of the induction have been completed, i.e., we have $u_1^n, \ldots, u_{k-1}^n \in \mathcal{H}$ such that for each $i \in \{1, \ldots, k-1\}$,

$$u_i^n := \mathrm{proj}_{C(t_i^n, u_{i-1}^n)} \left(u_{i-1}^n - (t_i^n - t_{i-1}^n)\zeta(t_{i-1}^n, u_{i-1}^n) - \int_{t_{i-1}^n}^{t_i^n} f(\tau, u_{i-1}^n)\mathrm{d}\tau \right).$$

and

$$\left\|u_i^n - u_{i-1}^n\right\| \leq \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) \sum_{j=0}^{i-1} L_2^j \leq \frac{T}{2^n}\kappa. \tag{5.6}$$

Let us construct u_k^n. Using the inclusion $u_{k-1}^n \in C(t_{k-1}^n, u_{k-2}^n)$, the assumptions (i)–(iii) and (5.2), (5.6), we have

$$\begin{aligned}&d_{C(t_k^n, u_{k-1}^n)} \left(u_{k-1}^n - (t_k^n - t_{k-1}^n)\zeta(t_{k-1}^n, u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(\tau, u_{k-1}^n)\mathrm{d}\tau \right) \\ &\quad \leq d_{C(t_k^n, u_{k-1}^n)}(u_{k-1}^n) + (t_k^n - t_{k-1}^n) \left\|\zeta(t_{k-1}^n, u_{k-1}^n)\right\| + \int_{t_{k-1}^n}^{t_k^n} \left\|f(\tau, u_{k-1}^n)\right\| \mathrm{d}\tau \\ &\quad \leq \mathrm{exc}\big(C(t_{k-1}^n, u_{k-2}^n), C(t_k^n, u_{k-1}^n)\big) + (t_k^n - t_{k-1}^n)(\alpha+\beta)(1+\rho) \\ &\quad \leq (t_k^n - t_{k-1}^n)\big(L_1 + (\alpha+\beta)(1+\rho)\big) + L_2 \left\|u_{k-1}^n - u_{k-2}^n\right\| \\ &\quad \leq \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) \left(1 + \sum_{j=0}^{k-2} L_2^{j+1} \right) \\ &\quad = \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) \sum_{j=0}^{k-1} L_2^j < r\end{aligned} \tag{5.7}$$

and (thanks to the r-prox-regularity of $C(t_k^n, u_{k-1}^n)$) we can set

$$u_k^n := \mathrm{proj}_{C(t_k^n, u_{k-1}^n)} \left(u_{k-1}^n - (t_k^n - t_{k-1}^n)\zeta(t_{k-1}^n, u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(\tau, u_k^n)\mathrm{d}\tau \right).$$

To complete the induction, it remains to note from the latter equality, (i)–(iii), (5.6) and (5.7) that

$$\begin{aligned}
\left\|u_k^n - u_{k-1}^n\right\| &\le \left\|u_k^n - \left(u_{k-1}^n - (t_k^n - t_{k-1}^n)\zeta(t_{k-1}^n, u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(\tau, u_{k-1}^n)\mathrm{d}\tau\right)\right\| \\
&\quad + (t_k^n - t_{k-1}^n)\left\|\zeta(t_{k-1}^n, u_{k-1}^n)\right\| + \int_{t_{k-1}^n}^{t_k^n} \left\|f(\tau, u_{k-1}^n)\right\| \mathrm{d}\tau \\
&\le d_{C(t_k^n, u_{k-1}^n)}\left(u_{k-1}^n - (t_k^n - t_{k-1}^n)\zeta(t_{k-1}^n, u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(\tau, u_{k-1}^n)\mathrm{d}\tau\right) \\
&\quad + (t_k^n - t_{k-1}^n)(\alpha+\beta)(1+\rho) \\
&\le (t_k^n - t_{k-1}^n)\big(L_1 + 2(\alpha+\beta)(1+\rho)\big) + L_2\left\|u_{k-1}^n - u_{k-2}^n\right\| \\
&\le \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big)\left(1 + \sum_{j=0}^{k-2} L_2^{j+1}\right) \\
&\le \frac{T}{2^n}\big(L_1 + 2(\alpha+\beta)(1+\rho)\big)\sum_{j=0}^{k-1} L_2^{j} \le \frac{T}{2^n}\kappa.
\end{aligned}$$

Now, for every integer $n \ge N$, note that

$$u_i^n \in C(t_i^n, u_{i-1}^n) \subset \rho\mathbb{B} \quad \text{for each } i \in \{1, \ldots, 2^n\}, \tag{5.8}$$

set

$$z_i^n := \zeta(t_i^n, u_i^n) \in \beta(1+\rho)\mathbb{B} \quad \text{for each } i \in \{0, \ldots, 2^n\}$$

and define the mapping $u_n : I \to \mathcal{H}$ by $u_n(T) = u_{2^n}^n$ and

$$u_n(t) := u_i^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0, \ldots, 2^n - 1\}.$$

Combining the latter definition, the inclusion $u_0 \in C(0, u_0)$ and (5.8), we observe that for all integer $n \ge N$ and all $t \in I$,

$$\|u_n(t)\| \le \rho.$$

As in Theorem 4.3, concerning the variation of u_n with $n \ge N$, note that (see (5.3))

$$\sup_{n \ge N} \operatorname{var}(u_n; I) \le \kappa T.$$

Consequently, we may suppose through Theorem 3.1 the existence of a mapping $u(\cdot) : I \to \mathcal{H}$ with bounded variation on I such that

$$u_n(t) \xrightarrow{w} u(t) \quad \text{for all } t \in I. \tag{5.9}$$

Again, as in Theorem 4.3, we establish that $u(\cdot)$ is Lipschitz continuous on I and

$$\|\dot{u}(t)\| \le \kappa = \frac{L_1 + 2(\alpha+\beta)(1+\rho)}{1 - L_2} \quad \lambda\text{-a.e. } t \in I.$$

We are going to establish that $u(t) \in C(t, u(t))$ for all $t \in I$. For each integer $n \ge N$, define the mapping $\delta_n : I \to I$ by $\delta_n(T) := T$ and

$$\delta_n(t) := t_i^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0, \ldots, 2^n - 1\}.$$

Fix any $n \in \mathbb{N}$ with $n \geq N$ and $\tau \in]0, T]$. There is $j \in \{0, \ldots, 2^n - 1\}$ such that $\tau \in]t_j^n, t_{j+1}^n]$. Assume for a moment that $\tau \neq t_{j+1}^n$, i.e., $\tau \in]t_j^n, t_{j+1}^n[$. If $j = 0$, set $\varphi(\tau) = 0$ otherwise set $\varphi(\tau) = t_{j-1}^n$. From the definition of $u_n(\cdot)$ and (5.3), we have

$$\|u_n(\tau) - u_n(\varphi(\tau))\| \leq \frac{T}{2^n}\kappa \tag{5.10}$$

and from (5.8) and the definition of $\delta_n(\cdot)$,

$$u_n(\tau) \in C(\delta_n(\tau), u_n(\varphi(\tau))). \tag{5.11}$$

Now, suppose that $\tau = \tau_{j+1}^n$. Setting $\varphi(\tau) = t_j^n$, we see that (5.10) and (5.11) still holds.

Let $n \in \mathbb{N}$ with $n \geq N$ and $t \in]0, T]$. Set $t' := \varphi(t) \in I$. By virtue of (iii) and (5.10), we see that

$$\mathrm{exc}[C(\delta_n(t), u_n(t')), C(t, u_n(t))] < \frac{L_1 T}{2^n} + L_2 \|u_n(t) - u_n(t')\| \leq \frac{T}{2^n}(L_1 + L_2\kappa),$$

which entails by (2.2)

$$C(\delta_n(t), u_n(t')) \subset C(t, u_n(t)) + \frac{T}{2^n}(L_1 + L_2\kappa)\mathbb{B}.$$

Then, it follows from (5.11)

$$u_n(t) \in C(t, u_n(t)) + \frac{T}{2^n}(L_1 + L_2\kappa)\mathbb{B} \tag{5.12}$$

and this inclusion still holds if $t = 0$. Now, assume that $t \in [0, T]$. By contradiction assume that $U(t) := \{u_k(t) : k \geq N\}$ is not relatively compact, i.e., $\gamma(U(t)) > 0$. Using assumption (iv), we get

$$\gamma(U(t)) > \gamma\big(C(t, U(t))\big)$$

and this gives a real $\Delta > 0$ such that

$$\gamma(U(t)) - \gamma\big(C(t, U(t))\big) \geq 2\Delta > 0. \tag{5.13}$$

Choose any integer $n_0 \geq N$ such that $\frac{T}{2^{n_0 - 1}}(L_1 + L_2\kappa) < \Delta$. According to Proposition 2.7, the inclusion provided by (5.12), the inequality (2.8) and the choice of n_0, we have

$$\begin{aligned}
\gamma(U(t)) = \gamma(\{u_k(t) : k \geq n_0\}) &\leq \gamma\left(C(t, U(t)) + \frac{T}{2^{n_0}}(L_1 + L_2\kappa)\mathbb{B}\right) \\
&\leq \gamma\big(C(t, U(t))\big) + \gamma\left(\frac{T}{2^{n_0}}(L_1 + L_2\kappa)\mathbb{B}\right) \\
&\leq \gamma\big(C(t, U(t))\big) + \frac{T}{2^{n_0 - 1}}(L_1 + L_2\kappa) \\
&\leq \gamma\big(C(t, U(t))\big) + \Delta.
\end{aligned} \tag{5.14}$$

Putting together (5.13) and (5.14), we arrive to

$$\gamma(U(t)) \leq \gamma(U(t)) - 2\Delta + \Delta = \gamma(U(t)) - \Delta,$$

which is the desired contradiction. Consequently, for all $t \in I$, $U(t)$ is relatively compact in $\mathcal{H}$, so the weak convergence in (5.9) holds for the strong topology, i.e.,

$$u_k(\tau) \to u(\tau) \quad \text{for all } \tau \in I. \tag{5.15}$$

Using assumption (iii) again, we have

$$\begin{aligned}\operatorname{exc}[C(\delta_n(t), u_n(t')), C(t, u(t))] &\leq L_1(t - \delta_n(t)) + L_2 \|u_n(t') - u(t)\| \\ &< \frac{L_1 T}{2^n} + L_2 \|u_n(t') - u(t)\|\end{aligned}$$

and this ensures by (2.2) and (5.11)

$$u_n(t) \in C(\delta_n(t), u_n(t')) \subset C(t, u(t)) + \left(\frac{L_1 T}{2^n} + L_2 \|u_n(t') - u(t)\|\right)\mathbb{B}.$$

In particular, we have

$$d_{C(t,u(t))}(u_n(t)) \leq \frac{L_1 T}{2^n} + L_2 \|u_n(t') - u(t)\|.$$

Since $\|u_n(t') - u_n(t)\| \to 0$, the latter inequality entails

$$d_{C(t,u(t))}(u_n(t)) \to 0.$$

We deduce from the latter convergence and the fact that $C(\cdot,\cdot)$ is closed-valued

$$u(t) \in C(t, u(t)) \quad \text{for all } t \in I. \tag{5.16}$$

The rest of the proof is similar to those of Theorem 4.3. $\square$

We derive from the latter result the unperturbed (i.e., $F, f \equiv 0$) bounded prox-regular case.

Corollary 5.2. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some $r \in]0, +\infty]$. Assume that:*

(i) *there exist three reals $\rho > 0$, $L_1 \geq 0$ and $0 \leq L_2 < 1$ such that for every $(t, x) \in \mathcal{H} \times I$,*

$$C(t, x) \subset \rho\mathbb{B}$$

and for every $x, y \in \mathcal{H}$ and every $\tau, t \in I$ with $\tau < t$,

$$\operatorname{exc}(C(\tau, x), C(t, y)) \leq L_1(t - \tau) + L_2 \|x - y\|;$$

(ii) *for every bounded subset B of $\mathcal{H}$ with $\gamma(B) > 0$,*

$$\gamma(C(t, B)) < \gamma(B).$$

Then, for each $u_0 \in \mathcal{H}$ with $u_0 \in C(0, u_0)$, there exists a Lipschitz continuous mapping $u : I \to \mathcal{H}$ satisfying $(\mathcal{SDSP})$ and

$$\|\dot{u}(t)\| \leq \frac{L_1}{1 - L_2} \quad \lambda\text{-a.e. } t \in I.$$

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