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PRESERVATION OF PROX-REGULARITY OF SETS WITH APPLICATIONS TO CONSTRAINED OPTIMIZATION*

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Abstract. In this paper, we first provide counterexamples showing that sublevels of prox-regular functions and levels of differentiable mappings with Lipschitz derivatives may fail to be prox-regular. Then, we prove the uniform prox-regularity of such sets under usual verifiable qualification conditions. The preservation of uniform prox-regularity of intersection and inverse image under usual qualification conditions is also established. Applications to constrained optimization problems are given.

Key words. prox-regular set, hypomonotonicity, semiconvexity, metric regularity, constrained optimization, sweeping process

AMS subject classifications. 49J52, 49J53, 90C30, 26B25

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1. Introduction.

Nonlinear programming is a well-developed area of research with applications in many branches of sciences and engineering. Most problems encountered in constrained optimization involve inequality/equality constraints. Convex optimization, a special class of mathematical programming, is an important topic both theoretically and computationally. The convexity of an extended real-valued function $f : \mathcal{H} \to \overline{\mathbb{R}}$ from a Hilbert space $\mathcal{H}$ can be characterized via the epigraph of f, that is, the set $\operatorname{epi} f = \{(x, s) \in \mathcal{H} \times \mathbb{R} : \ f(x) \le s\}$ and not by using the sublevels of f, i.e., the sets $\{x \in \mathcal{H} : \ f(x) \le \lambda\}$ with $\lambda \in \mathbb{R}$. It is well-known that the sublevels of f are convex if and only if f is quasi-convex, which is the topic of generalized convexity analysis [8]. Many numerical algorithms in optimization used the projection operator over a set. This is the case, for example, of the proximal point algorithm, the gradient projection algorithm, the alternating projection algorithm (to name just a few). If the projection operator over a closed subset is single-valued on a suitable neighborhood of the set, then it is convenient for the choice numerically of the next iteration. The class of nonempty closed convex sets of a Hilbert space provides a good example. In order to go beyond the convexity, the class of uniform prox-regular sets was introduced, which is larger than the class of nonempty closed convex sets and shares with it many nice properties that are important in applications, in particular when projections are involved. The concept of a prox-regular set $C \subset \mathcal{H}$ at a point $\bar{x} \in C$ is somehow related to the hypomonotonicity of some truncation of the proximal normal cone mapping $N(C; \cdot)$ around this point $\bar{x}$. The class of prox-regular functions was introduced and studied thoroughly in [23] and such locally Lipschitz functions can be characterized via the prox-regularity of the epigraph. Many concrete problems in optimization and control involve intersection of prox-regular sets as well as sublevels of prox-regular functions (see, e.g., [1, 7, 25, 27]).

In addition to its role in optimization and control, the concept of prox-regular sets is of great interest also in the theory of Moreau sweeping process, in crowd motion, in

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second order analysis etc. (see, e.g., [15, 27, 25, 20]). Let J be a nonempty interval of $\mathbb{R}$ with $0 \in J$ as its left end point, let $C(t)_{t\in J}$ be a family of nonempty closed subsets of a Hilbert space $\mathcal{H}$, and let $\Phi : J \times \mathcal{H} \to \mathcal{H}$ be a mapping Lebesgue measurable in t and such that $\Phi(t,\cdot)$ is $\kappa(t)$-Lipschitzian with $\kappa(\cdot)$ Lebesgue integrable on J. The extended Moreau sweeping process, as involved in electrical circuit (see, e.g., [1, 7]) and in crowd motion (see, e.g., [20]), can be stated as the (measure) differential inclusion

$$(ESP) \quad du \in -N\big(C(t); u(t)\big) - \Phi(t, u(t)) \quad \text{and} \quad u(0) = u_0 \in C(0),$$

where $N(\cdot;\cdot)$ denotes a normal cone. The uniform r-prox-regularity of all the sets $C(t)$ is known to be the general condition under which (ESP) admits a (unique) solution with bounded variation (see, e.g., [1, 7, 15, 17]). Concrete problems are considered in [1, 7, 27] where the sets $C(t)$ are in the form either $C(t) = \{x \in \mathcal{H} : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0\}$ with $g_k(t,\cdot)$ prox-regular functions or $C(t) = C_1(t) \cap C_2(t)$ with $C_1(t)$ and $C_2(t)$ prox-regular subsets of $\mathcal{H}$. Counterexamples in [5] show that intersections of prox-regular sets can fail to be prox-regular. In addition, we provide in this paper various counterexamples where sublevel sets of smooth prox-regular functions (resp., sets of zeros of smooth mappings) are not prox-regular.

Our goal in this paper is then to establish, under various usual qualification conditions, the prox-regularity of sublevel sets of prox-regular functions as well as the preservation of prox-regularity under intersection and inverse image. Taking (ESP) into account, after some preliminaries in section 2 we work in sections 3 and 4 with the uniform prox-regularity of families $(C(t))_{t\in I}$ with

$$C(t) = \{x \in \mathcal{H} : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0\}$$

or $C(t) = \{x \in \mathcal{H} : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0, g_{m+1}(t,x) = 0, \dots, g_{m+n}(t,x) = 0\}$,

where the functions $g_k(t,\cdot)$ are, respectively, smooth and nonsmooth. The uniform prox-regularity of families $(C(t))_{t\in I}$ in the form (of intersection) $C(t) = C_1(t) \cap \cdots \cap C_m(t)$ is studied in section 5 and those in the form $C(t) = G_t^{-1}\big(D(t)\big)$ are developed in section 6. In doing so, we provide, besides [29, 13, 28, 1], new significant results with verifiable conditions for the uniform prox-regularity of families of sets in the above forms. Applications to optimization problems are given in section 7.

2. Notation and preliminaries. Our notation is quite standard. Throughout the paper, all vector spaces will be real vector spaces. For any normed space X, we denote by $\mathbb{B}_X$ the closed unit ball of X centered at zero, by $B(x,r)$ (resp., $B[x,r]$) the open (resp., closed) ball centered at $x \in X$ of radius $r > 0$ and by $X^\star$ the topological dual space of X. For a set $S \subset X$ (resp., $S \subset X^\star$), the notation $\mathrm{co}\,(S)$ (resp., $\overline{\mathrm{co}}^\star\,(S)$) stands for the convex hull (resp., the weak-$\star$ closed convex hull) of S, and $\mathrm{bdry}\, S$ for the boundary of S. By $d_S(\cdot)$ or $d(\cdot,S)$ we denote the distance function from S, i.e.,

$$d_S(x) := \inf_{s\in S} \|x - s\| \quad \text{for all } x \in X.$$

For any $x \in X$, the (possibly empty) set of all nearest points of x in S is defined by

$$\mathrm{Proj}_S(x) = \{y \in S : d_S(x) = \|x - y\|\}.$$

When $\mathrm{Proj}_S(x)$ contains one and only one vector $\overline{y}$, we set $\mathrm{proj}_S(x) := \overline{y}$.

A nonempty subset S of X is said to be closed near $x \in S$ whenever, there is a neighborhood V of x such that $S \cap V$ is closed in V with respect to the induced topology on V.

The Bouligand–Peano (resp., Clarke) tangent cone of S at $x \in S$ (see, e.g., [2, 22]) will be denoted by $T^B(S;x)$ (resp., $T^C(S;x)$); when $T^B(S;x) = T^C(S;x)$, the set S is called (Clarke) tangentially regular at x. If there is a neighborhood U of x such that S is tangentially regular at any point of $S \cap U$, the set S is said to be (Clarke) tangentially regular near x.

Similarly, the proximal (resp., Fréchet, Mordukhovich limiting, Clarke) normal cone of S at x (see, e.g., [21, 11]) is denoted by $N^P(S;x)$ (resp., $N^F(S;x)$, $N^L(S;x)$, $N^C(S;x)$). So, denoting by $\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$ the *epigraph* of an extended real-valued function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, its proximal (resp., Fréchet, Mordukhovich limiting, Clarke) subdifferential at $x \in X$ with $f(x) < +\infty$ is defined by saying $x^\star \in X^\star$ belongs to $\partial_P f(x)$ (resp., $\partial_F f(x)$, $\partial_L f(x)$, $\partial_C f(x)$) when $(x^\star, -1)$ belongs to the corresponding normal cone of $\operatorname{epi} f$ at $(x, f(x))$.

2.1. Prox-regular sets. In this subsection, S is a nonempty closed subset of a Hilbert space $\mathcal{H}$, and r is an extended real of $]0,+\infty]$. We will use the classical convention $\frac{1}{r} = 0$ whenever $r = +\infty$ and we denote by $U_r(S)$ the open r-enlargement of the set S, that is, $U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}$. We start with the definition of uniformly prox-regular sets.

DEFINITION 2.1. *The set S is said to be r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $\zeta \in N^P(S;x) \cap \mathbb{B}_{\mathcal{H}}$, and for all $t \in]0,r[$, one has $x \in \operatorname{Proj}_S(x + t\zeta)$.*

Some authors called such sets positively reached (see [16]), weakly convex (see [29]), p-convex (see [9]), $O(2)$-convex (see [26]), or proximally smooth (see [12]). We refer, for example, to [13] for historical comments.

The set S is said to be *prox-regular at* $\overline{x} \in S$ when the property in the above definition holds true for x near $\overline{x}$, that is, there is a real $\varepsilon > 0$ such that for all $x \in S \cap B(\overline{x}, \varepsilon)$, for all $\zeta \in N^P(S;x) \cap \mathbb{B}_{\mathcal{H}}$, and for all $t \in]0,r[$, one has $x \in \operatorname{Proj}_S(x + t\zeta)$.

THEOREM 2.2 (see [24]). *The following assertions are equivalent.*

(a) *The set S is r-prox-regular.*

(b) *For all $x_1, x_2 \in S$, for all $\zeta \in N^P(S;x_1)$, one has*

$$\langle \zeta, x_2 - x_1 \rangle \leq \frac{1}{2r} \|\zeta\| \, \|x_1 - x_2\|^2 .$$

(c) *For all $x_1, x_2 \in S$, for all $\zeta_1 \in N^P(S;x_1) \cap \mathbb{B}_{\mathcal{H}}$, and for all $\zeta_2 \in N^P(S;x_2) \cap \mathbb{B}_{\mathcal{H}}$, one has*

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2 .$$

(d) *The function d_S^2 is of class $C^{1,1}$ on $U_r(S)$, that is, it is differentiable on $U_r(S)$ and its derivative is locally Lipschitz therein.*

The features in the next proposition are fundamental (see, e.g., [24]).

PROPOSITION 2.3. *The following assertions hold true.*

(a) *If S is r-prox-regular, then for any $x \in \mathcal{H}$,*

$$N^P(S;x) = N^F(S;x) = N^L(S;x) = N^C(S;x) \quad \text{and} \quad T^B(S;x) = T^C(S;x).$$

(b) *If S is r-prox-regular, then for any $x \in U_r(S)$, the set $\operatorname{Proj}_S(x)$ is a singleton, i.e., $\operatorname{proj}_S(x)$ is well-defined.*

(c) *If S is r-prox-regular, the mapping* $\operatorname{proj}_S : U_r(S) \longrightarrow S$ *is well-defined and locally Lipschitz on* $U_r(S)$.

(d) *The set S is r-prox-regular if and only if any one of the properties* (b)–(c) *of Theorem* 2.2 *holds true with any one of the normal cones* $N^F(S;\cdot)$, $N^L(S;\cdot)$, $N^C(S;\cdot)$ *in place of* $N^P(S;\cdot)$.

According to the assertion (a) of the above proposition, whenever S is a uniformly prox-regular subset of $\mathcal{H}$ containing x, we will set

$$N(S;x) := N^P(S;x) = N^F(S;x) = N^L(S;x) = N^C(S;x),$$

$$T(S;x) := T^B(S;x) = T^C(S;x).$$

The property (c) of Theorem 2.2 means that the multimapping $N^P(S;\cdot) \cap \mathbb{B}_{\mathcal{H}}$ is $\frac{1}{r}$-*hypomonotone*. For the local prox-regularity, we know (see [13]) that S is prox-regular at $\overline{x} \in S$ if and only if there is a real $\delta > 0$ such that for all $x_1 \in B(\overline{x},\delta)\cap S$, for all $x_2 \in B(\overline{x},\delta)\cap S$, and for all $\zeta \in N^P(S;x_1)$ (or $N^F(S;x_1)$, $N^L(S;x_1)$, $N^C(S;x_1)$),

$$\langle \zeta, x_2 - x_1\rangle \le \frac{1}{2r}\|\zeta\|\,\|x_1 - x_2\|^2. \tag{2.1}$$

We now state another characterization of uniform prox-regularity which will be crucial in the development of this paper.

Proposition 2.4. *Let s,t be two extended reals in* $]0,+\infty]$. *The set S is* $\min\{s,t\}$*-prox-regular whenever for all $x,x' \in S$ with $\|x - x'\| < 2t$ and for all* $\zeta \in N^P(S;x)\cap\mathbb{B}_{\mathcal{H}}$, $\langle \zeta, x'-x\rangle \le \frac{1}{2s}\|x'-x\|^2$.

2.2. Metric regularity. Various results related to the prox-regularity of intersection and preimage will involve the concept of metric regularity of multimappings.

Definition 2.5. *Let X,Y be two normed spaces and let $M : X \rightrightarrows Y$ be a multimapping,* $(\overline{x},\overline{y}) \in \operatorname{gph} M := \{(x,y)\in X\times Y : y \in M(x)\}$. *One says that M is metrically regular at $\overline{x}$ for $\overline{y}$ whenever there are a real $\gamma \ge 0$ and neighborhoods U and V of $\overline{x}$ and $\overline{y}$, respectively, such that*

$$d(x, M^{-1}(y)) \le \gamma d(y, M(x)) \quad \textit{for all}\,(x,y)\in U\times V.$$

Given two normed spaces, X,Y, a multimapping $M : X \rightrightarrows Y$ and $(x,y)\in X\times Y$, one defines $T^B M(x,y) : X \rightrightarrows Y$, called the *Bouligand–Peano tangential derivative of M at* (x,y), as the multimapping $T^B M(x,y) : X \rightrightarrows Y$ which satisfies

$$\operatorname{gph} T^B M(x,y) = T^B(\operatorname{gph} M; (x,y)).$$

So, for all $(u,v)\in X\times Y$, one has

$$(u,v) \in T^B(\operatorname{gph} M;(x,y)) \Longleftrightarrow v \in T^B M(x,y)(u). \tag{2.2}$$

We recall the following result (see, [2, Theorem 5.4.3]), which ensures the metric regularity of a multimapping, under a tangential condition.

Theorem 2.6 (Aubin tangential condition for metric regularity). *Let X,Y be two Banach spaces, $M : X \rightrightarrows Y$ a multimapping,* $(\overline{x},\overline{y}) \in \operatorname{gph} M$. *Assume the following:*

(i) $\operatorname{gph} M$ *is closed near* $(\overline{x},\overline{y})$*;*

(ii) *there exist a real $s > 0$ and neighborhoods U and V of $\overline{x}$ and $\overline{y}$ such that*

$$s\mathbb{B}_Y \subset T^B M(x,y)(\mathbb{B}_X) \quad \textit{for all } (x,y) \in (U \times V) \cap \operatorname{gph} M.$$

Then, M is metrically regular at $\overline{x}$ for $\overline{y}$.

According to [22, Lemma 6.7], it is not difficult to prove the following result.

PROPOSITION 2.7. *Let X, Y be two Asplund spaces and let $f : X \longrightarrow Y$ be a mapping which is strictly Fréchet differentiable at $\overline{x} \in f^{-1}(D)$, where D is a nonempty subset of Y closed near $f(\overline{x})$. Assume that there exist two reals $\gamma, \delta > 0$ such that*

$$d(x, f^{-1}(D)) \leq \gamma d(f(x), D) \quad \textit{for all } x \in B(\overline{x}, \delta).$$

Then, one has

$$N^L(f^{-1}(D); \overline{x}) \subset \left\{ y^\star \circ Df(\overline{x}) : y^\star \in N^L(D; f(\overline{x})) \right\}.$$

3. Prox-regularity of set with smooth constraints. In general, the prox-regularity of sets is unfortunately not perserved under operations without additional qualification conditions, as shown in the following examples.

Example 1. A first simple example of a smooth (polynomial) function $g : \mathbb{R}^2 \to \mathbb{R}$ whose sublevel (resp., level) set $\{(x,y) \in \mathbb{R}^2 : g(x,y) \leq 0\}$ (resp., $\{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$) is not prox-regular (see Figure 1) is furnished by the polynomial function defined by $g(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$. Concerning a bounded non-prox-regular sublevel set of a smooth function (see Figure 2 (resp., Figure 3)), we can consider the set $\{(x,y) \in \mathbb{R}^2 : g(x,y) \leq 0\}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is the classical function whose zero level is Bernouilli's lemniscate (resp., is the function whose zero sublevel is the union of the closed balls of $\mathbb{R}^2$ of radius 1 centered, respectively, at $(-1,0)$ and $(1,0)$), that is, for all $(x,y) \in \mathbb{R}^2$

$$g(x,y) = (x^2+y^2)^2 - 2(x^2-y^2) \quad (\text{resp.}, g(x,y) = \big((x-1)^2+y^2-1\big)\big((x+1)^2+y^2-1\big)). \quad \square$$

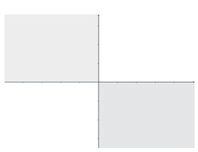

FIG. 1.

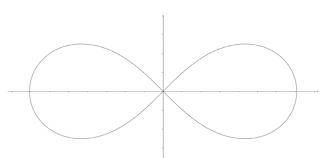

FIG. 2.

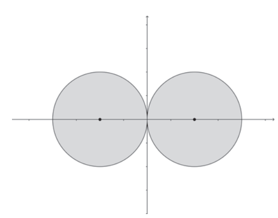

FIG. 3.

Example 2. In regard to the stability under intersection, we invoke [5]. Consider first the closed set of the Euclidean space $\mathbb{R}^2$ defined in [5] in the following way. For each $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers, $n = 1, \ldots$) denote by D_n the closed ball with radius $r = 1/4$ (independent of n) in $\mathbb{R}^2$ with the points $(1/2^{n-1}, 0)$ and $(1/2^n, 0)$ on its boundary and whose ordinate of its center is nonpositive. With $R = 1/2$ the suitable closed set in [5] is defined as

$$Q := \left\{ (x,y) \in \mathbb{R}^2 : y \geq 0, \ \left(x - \frac{1}{2}\right)^2 + y^2 \leq R^2 \right\} \setminus \bigcup_{n \in \mathbb{N}} \operatorname{int} D_n,$$

and clearly it is r-prox-regular; see Figure 4. Denoting by E the vector subspace given by the axis of abscissa, that is, $E := \mathbb{R} \times \{0\}$, as noted in [5] the intersection $Q \cap E$ fails to be prox-regular at $(0,0)$, in particular $Q \cap E$ is not uniformly prox-regular, that is, there is no $r' \in]0,+\infty]$ such that $Q \cap E$ is r'-prox-regular.

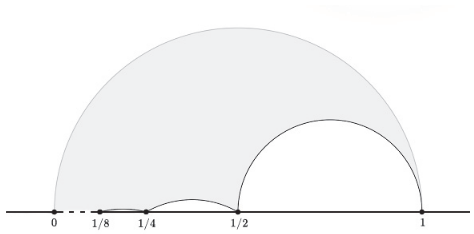

FIG. 4. *Intersection of prox-regular sets which fails to be prox-regular.*

We also observe, with the linear mapping $A : \mathbb{R} \to \mathbb{R}^2$ defined by $Ax := (x,0)$ for all $x \in \mathbb{R}$, that the susbset $A^{-1}(Q)$ is not prox-regular in $\mathbb{R}$. With the above construction at hand, we can naturally provide (in addition to Example 1) another example of a sublevel set of a smooth function which is not prox-regular. Indeed, consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) := d^2\big((x,y),Q\big) + d^2\big((x,y),E\big) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

From Theorem 2.2(d) the function g is of class $C^{1,1}$ on the open set $U_r(Q)$ of $\mathbb{R}^2$, its derivative is Lipschitz on $U_{r_0}(Q)$ for any $0 < r_0 < r$, so in particular it is *a smooth prox-regular function* on $U_{r_0}(C)$ (see [23, 4]). Nevertheless, the sublevel set

$$\{g \le 0\} := \{(x,y) \in \mathbb{R}^2 : g(x,y) \le 0\} = Q \cap E$$

is not prox-regular according to the first observation above; see also section 7, for other examples.

Remark 1. Concerning the Bouligand–Peano and Clarke tangent cones, it is obvious that

$$T^B(Q \cap E; (0,0)) = [0,+\infty[\times\{0\} \quad \text{and} \quad T^C(Q \cap E; (0,0)) = \{(0,0)\},$$

so the set $Q \cap E$ is not even tangentially regular. This says in particular that, without any qualification condition, the intersection of two subsmooth sets (see [3, 14] for the definition) may fail to be subsmooth. Similarly, with the above linear mapping A and the above smooth functions g, the sets $A^{-1}(Q)$ and $\{g \le 0\}$ are not tangentially regular at 0 and $(0,0)$, respectively. Consequently, without any qualification condition, neither the subsmoothness property is preserved under inverse image by (continuous) linear mapping nor sublevel sets of $C^{1,1}$ smooth functions are subsmooth.

The above example illustrates that, without qualification condition, the prox-regularity of sets is not preserved under intersection and inverse image, and sublevel sets of $C^{1,1}$ (hence prox-regular) functions may fail to be prox-regular. As a simple positive case, we recall that a sublevel set of smooth real-valued function with Lipschitz gradient, nonvanishing at boundary points, is prox-regular (see, e.g., [1, 13, 28, 29]). Our aim in this section and the next ones is to show that with additional usual constraint qualifications the prox-regularity is preserved. In order to state and prove results for the stability of local prox-regularity, the approach with the *normal cone intersection property* and *normal cone inverse image property* is introduced and developed in [13]. One can see [14] for the use of those concepts in the study of the preservation of subsmoothness under operations on sets. Given two normal cones $N(\cdot;\cdot)$ and $\mathcal{N}(\cdot;\cdot)$, recall that the normal cone intersection property of $\mathcal{N}(\cdot;\cdot)$ with respect to $N(\cdot;\cdot)$ for two sets S_1, S_2 in a normed space X amounts to requiring some real $\beta > 0$ such that

$$\mathcal{N}(S_1 \cap S_2; x) \cap \mathbb{B}_{X^*} \subset N(S_1; x) \cap \beta\mathbb{B}_{X^*} + N(S_2; x) \cap \beta\mathbb{B}_{X^*},$$

and similarly the normal cone inverse image property for a set S in a normed space Y and a differentiable mapping $g : X \to Y$ means that, for some real $\beta > 0$,

$$\mathcal{N}(g^{-1}(S); x) \cap \mathbb{B}_{X^*} \subset Dg(x)^\star\big(N(S; g(x)) \cap \beta\mathbb{B}_{Y^*}\big),$$

where $Dg(x)^*$ denotes the adjoint of the continuous linear mapping $Dg(x) : X \to Y$. In this paper, in view of applications to the theory of Sweeping Process (see, e.g., [1, 7]) we provide and develop, for the stability of uniform global prox-regularity, new verifiable quantitative conditions, and this is done in dealing with families $\big(C(t)\big)_{t\in I}$ of prox-regular sets as involved in the theory of Sweeping Process.

We start by recalling a result from [1] establishing, through some verifiable quantitative conditions, the uniform prox-regularity of constraint sets with finitely many smooth inequalities. The prox-regularity of sublevel sets of smooth functions is clearly a particular case.

THEOREM 3.1. *Let I be a nonempty set, let $\mathcal{H}$ be a Hilbert space, and let $m \in \mathbb{N}$ and $g_k : I \times \mathcal{H} \to \mathbb{R}$ with $k \in \{1, \ldots, m\}$ be functions such that, for each $t \in I$, the set*

$$C(t) = \{x \in \mathcal{H} : g_1(t, x) \le 0, \ldots, g_m(t, x) \le 0\}$$

is nonempty. Assume that there exists an extended real $\rho \in]0, +\infty]$ such that

(i) *for all $t \in I$, for all $k \in \{1, \ldots, m\}$, $g_k(t, \cdot)$ is strictly Hadamard differentiable on $U_\rho(C(t))$;*

(ii) *there exists a real $\gamma \ge 0$ such that for all $t \in I$, for all $k \in \{1, \ldots, m\}$, and for all $x, y \in U_\rho(C(t))$,*

$$\langle \nabla g_k(t, \cdot)(x) - \nabla g_k(t, \cdot)(y), x - y\rangle \ge -\gamma \|x - y\|^2, \tag{3.1}$$

that is, $\nabla g_k(t, \cdot)$ is γ-hypomonotone on $U_\rho(C(t))$.
Assume also that there is a real $\delta > 0$ such that for all $(t, x) \in I \times \mathcal{H}$ with $x \in \mathrm{bdry}\, C(t)$, there exists $\overline{v} \in \mathbb{B}_{\mathcal{H}}$ satisfying, for all $k \in \{1, \ldots, m\}$,

$$\langle \nabla g_k(t, \cdot)(x), \overline{v}\rangle \le -\delta. \tag{3.2}$$

Then, for all $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\big\{\rho, \frac{\delta}{\gamma}\big\}$.

With functions g_k independent of t (so, the set C is independent of t as well), in Theorem 3.1 note that conditions (i) and (ii) are obviously fulfilled whenever the functions $g_1, \ldots, g_m$ are differentiable on $U_\rho(C)$ and γ-Lipschitz continuous on $U_\rho(C)$. This leads us to provide an example of a real-valued function g of class C^1, satisfying (3.2) in the preceding theorem but not the hypomonotonicity property (3.1) and such that the set $\{g \le 0\}$ is not uniformly prox-regular.

Example 3. Let us define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^{\frac{3}{2}}$ if $x \ge 0$ and $f(x) = -(-x)^{\frac{3}{2}}$ otherwise. Let us show first that epi f is not prox-regular at $(0, 0)$. Since f is C^1 on

$\mathbb{R}$ with $\nabla f(0) = 0$, it is easily seen that $N^F(\text{epi } f; (0,0)) = \{0\} \times]-\infty, 0]$. Suppose that epi$f$ is prox-regular at $(0,0)$. By (2.1) there exist two reals $r, \delta > 0$ such that for all $(x^\star, r^\star) \in N^F(\text{epi } f; (0,0))$ and for all $(x,s) \in \text{epi } f \cap B((0,0),\delta)$,

$$\langle (x^\star, r^\star), (x,s) - (0,0)\rangle \le \frac{1}{2r} \|(x^\star, r^\star)\| \, \|(x,s)\|^2 .$$

Fix any real $r^\star < 0$. Choose some real $\varepsilon > 0$ with $\varepsilon < \min\{1, r^2\}$ such that $(-\varepsilon, f(-\varepsilon)) \in B((0,0),\delta)$. Taking $(0, r^\star) \in N^F(\text{epi } f; (0,0))$ and $(x,s) = (-\varepsilon, f(-\varepsilon))$ in the latter inequality, we obtain

$$\frac{\varepsilon^2 + \varepsilon^3}{2r} \ge \varepsilon^{\frac{3}{2}}.$$

Since $\varepsilon < 1$, we have $\frac{\varepsilon^2}{r} \ge \varepsilon^{\frac{3}{2}}$, i.e., $\frac{\varepsilon^4}{r^2} \ge \varepsilon^3$. It follows that $\varepsilon^3(\frac{\varepsilon}{r^2} - 1) \ge 0$, thus $\varepsilon \ge r^2$ and this cannot hold true, according to the choice of ε. As a consequence, the function (which is obviously C^1 on $\mathbb{R}^2$, so strictly Hadamard differentiable on $\mathbb{R}^2$) $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$(x,s) \longmapsto g(x,s) := f(x) - s \tag{3.3}$$

has its sublevel set $C := \{g \le 0\} = \text{epi } f$ not prox-regular at $(0,0)$. On the other hand, observing that $\nabla g(x,s) = (\frac{3}{2}\sqrt{|x|}, -1)$ for all $(x,s) \in \mathbb{R}^2$, we see with $\overline{v} = (-\frac{1}{2}, \frac{1}{2}) \in \mathbb{B}_{\mathbb{R}^2}$ that, for any $(x,s) \in \mathbb{R}^2$,

$$\langle \nabla g(x,s), \overline{v}\rangle = -\frac{3}{4}\sqrt{|x|} - \frac{1}{2} \le -\frac{1}{2} < 0,$$

hence g satisfies (3.2) in Theorem 3.1. Finally, let us verify that ∇g is hypomonotone on no open enlargement of C. Suppose that ∇g is hypomonotone on some open enlargement of C. Since $(0,0) \in C$, there exist two reals $\gamma, \varepsilon > 0$ such that

$$\langle \nabla g(x,s) - \nabla g(0,0), (0,0) - (x,s)\rangle \le \gamma \|(x,s)\|^2 = \gamma(x^2 + s^2) \quad \text{for all } x, s \in]-\varepsilon, \varepsilon[.$$

Thus, with $s = 0$ we get $-\frac{3}{2}x\sqrt{|x|} \le \gamma x^2$ for all $x \in]-\varepsilon, \varepsilon[$. In particular, we get $\frac{3}{2} \le \gamma\sqrt{|x|}$ for all $x \in]-\varepsilon, 0[$ and this inequality cannot hold true.

It is readily seen that a differentiable function g, with its gradient Lipschitz continuous, satisfies assumption (3.1) in Theorem 3.1; that is, g has its gradient hypomonotone. The next example shows that the converse is not true in general.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|^{\frac{3}{2}}$. It is straightforward that f is C^1 and convex on $\mathbb{R}$, thus ∇f is monotone (in particular, hypomonotone) on $\mathbb{R}$. However, ∇f is not even Lipschitz near zero.

Now, given a subset S of $\mathcal{H}$ and $x, y \in S$ with $\|x - y\| < 2\rho$, where $\rho \in]0, +\infty]$, for any real $\tau \in [0,1]$ and $z_\tau := x + \tau(y - x)$, we have

$$d(z_\tau, S) \le \|z_\tau - x\| = \tau\|x - y\| \quad \text{and} \quad \mathrm{d}(z_\tau, S) \le \|z_\tau - y\| = (1-\tau)\|x - y\|,$$

hence $d(z_\tau, S) \le \min\{\tau, 1-\tau\}\|x - y\| \le \frac{1}{2}\|x - y\| < \rho$. We have then established the following lemma.

LEMMA 3.2. *Let S be a subset of a Hilbert space $\mathcal{H}$ and $x, y \in S$ with $\|x-y\| < 2\rho$, where $\rho \in]0, +\infty]$. Then, for any $\tau \in [0,1]$ one has*

$$x + \tau(y - x) \in U_\rho(S).$$

The next result concerns the uniform prox-regularity of constraint sets with infinitely many equalities.

THEOREM 3.3. *Let I be a nonempty set, let $\mathcal{H}$ be a Hilbert space, and let $G : I\times\mathcal{H} \to Y$ be a mapping from $I\times\mathcal{H}$ into a Banach space Y such that, for each $t \in I$, the set*

$$C(t) := \{x \in \mathcal{H} : G(t, x) = 0\}$$

is nonempty. Assume that, there exists an extended real $\rho \in]0, +\infty]$ such that

(i) *for each $t \in I$, the mapping $G(t, \cdot)$ is differentiable on $U_\rho(C(t))$;*

(ii) *there is a real $\gamma \geq 0$ such that for every $t \in I$ the mapping $x \mapsto DG(t, \cdot)(x)$ is γ-Lipschitz on $U_\rho(C(t))$, i.e., for all $x_1, x_2 \in U_\rho(C(t))$,*

$$\|DG(t, \cdot)(x_1) - DG(t, \cdot)(x_2)\| \leq \gamma\|x_1 - x_2\|.$$

Assume also that there is some real $\delta > 0$ such that

$$\delta\mathbb{B}_Y \subset DG(t, \cdot)(x)\big(\mathbb{B}_{\mathcal{H}}\big) \quad \textit{for all}\ t \in I, x \in \operatorname{bdry} C(t). \tag{3.4}$$

Then for every $t \in I$, the set $C(t)$ is r-prox-regular with $r := \min\{\rho, \frac{\delta}{\gamma}\}$.

Proof. Fix any $t \in I$ and fix also any $x \in \operatorname{bdry}C(t)$ and $u \in C(t)$ with $\|u-x\| < 2\rho$, so by the above lemma $x + s(u - x) \in U_\rho(C(t))$ for all $s \in [0, 1]$. First, we note that the set $C(t)$ is closed since the mapping $G(t, \cdot)$ is continuous on the open set $U_\rho(C(t))$. Further, the C^1 property of the mapping $G(t, \cdot)$ near x along with the surjectivity of $DG(t, \cdot)(x)$ according to (3.4) implies (see, e.g., [21, Theorem 1.14]) that

$$N^C(C(t); x) = \{y^* \circ A : y^* \in Y^*\},\ \text{ where } A := DG(t, \cdot)(x).$$

Take any $x^* \in N^C(C(t); x)$ and choose by the latter equality some $y^* \in Y^*$ such that $x^* = y^* \circ A$. Let $y \in \mathbb{B}_Y$. Using the inclusion (3.4), there exists $v \in \mathbb{B}_{\mathcal{H}}$ such that $\delta y = A(v)$. Thus, $\delta\,\|y^*(y)\| = \|A^*(y^*)(v)\| \leq \|A^*(y^*)\| = \|x^*\|$ and this implies

$$\delta\,\|y^*\| \leq \|A^*(y^*)\| = \|x^*\|\,. \tag{3.5}$$

Consequently, we have

$$\begin{aligned}
0 &= \langle y^*, G(t, u) - G(t, x)\rangle \\
&= \int_0^1 \langle y^* \circ DG(t, \cdot)(x + s(u - x)), u - x\rangle\, ds \\
&= \langle y^* \circ A, u - x\rangle + \int_0^1 \langle y^* \circ DG(t, \cdot)(x + s(u - x)) - y^* \circ A, u - x\rangle\, ds,
\end{aligned}$$

hence

$$\begin{aligned}
\langle x^*, u - x\rangle &= \int_0^1 \langle y^* \circ DG(t, \cdot)(x) - y^* \circ DG(t, \cdot)(x + s(u - x)), u - x\rangle\, ds \\
&\leq \gamma\|y^*\|\,\|u - x\|^2 \int_0^1 s\, ds \\
&= \frac{\gamma}{2}\|y^*\|\,\|u - x\|^2.
\end{aligned}$$

Using (3.5), we deduce that

$$\langle x^*, u - x\rangle \leq \frac{\gamma}{2\delta}\|x^*\|\,\|u - x\|^2.$$

Putting $r := \min\left\{\rho, \frac{\delta}{\gamma}\right\}$, it results that for all $x \in \operatorname{bdry} C(t)$ and $u \in C(t)$ with $\|u-x\| < 2r$ and all $x^* \in N^C(C(t); x)$, $\langle x^*, u-x\rangle \leq \frac{1}{2r}\|x^*\|\,\|u-x\|^2$, which translates the r-prox-regularity of $C(t)$, according to Proposition 2.4. $\square$

Remark 2. As in Theorem 3.1, the result fails with a mapping G of class C^1 with DG not Lipschitz continuous. Indeed, let us consider again the function $G := g$ in (3.3) and define $C = \{G = 0\}$. Applying [25, Theorem 6.14], we get $N^C(C; (0,0)) = \{0\} \times \mathbb{R}$. Arguing as in Example 3, one can show that the set $\{g = 0\}$ is not prox-regular at $(0,0)$. However, condition (3.4) is fulfilled. Indeed, fix any $x \in C$. Take any $b' \in [-1,1]$ and put $b = (0, -b') \in \mathbb{B}_{\mathbb{R}^2}$. Since $DG(x)(b) = \langle \nabla G(x), b\rangle = b'$, we see that

$$[-1,1] \subset DG(x)(\mathbb{B}_{\mathbb{R}^2}).$$

This says that G satisfies condition (3.4) of Theorem 3.3 as claimed above.

Before stating the next result, let us recall (see, e.g., [6, Corollary 2.91 and (2.191)]) the description, under the Mangasarian–Fromovitz condition, of the Clarke normal cone of a constraint set with finitely many inequality and equality constraints.

THEOREM 3.4. *Let X be a Banach space, $m, n \in \mathbb{N}$, and*

$$S := \{x \in X : g_1(x) \leq 0, \ldots, g_m(x) \leq 0, g_{m+1}(x) = 0, \ldots, g_{m+n}(x) = 0\},$$

where $g_1, \ldots, g_{m+n} : X \to \mathbb{R}$ are functions of class C^1 near a point $\bar{x} \in S$. Assume that the Mangasarian–Fromovitz qualification condition is satisfied at $\bar{x}$, that is:

The vectors $Dg_{m+1}(\bar{x}), \ldots, Dg_{m+n}(\bar{x})$ are linearly independent and there is a vector $\bar{v} \in X$ such that $\langle Dg_{m+1}(\bar{x}), \bar{v}\rangle = 0, \ldots, \langle Dg_{m+n}(\bar{x}), \bar{v}\rangle = 0$ and

$$\langle Dg_k(\bar{x}), \bar{v}\rangle < 0 \quad \text{for all } k \in K^{\leq}(\bar{x}),$$

where $K^{\leq}(\bar{x}) := \{k \in K^{\leq} : g_k(\bar{x}) = 0\}$ and $K^{\leq} := \{1, \ldots, m\}$. Then the Clarke and Fréchet normal cones of S at $\bar{x}$ coincide and, with $K^{=} := \{m+1, \ldots, m+n\}$,

$$N^C(S; \bar{x}) = \Bigg\{ \sum_{k=1}^{m+n} \lambda_k Dg_k(\bar{x}) : \lambda_k \in \mathbb{R} \ \text{for all } k \in K^{=},$$
$$\lambda_k \geq 0,\ \lambda_k g_k(\bar{x}) = 0 \ \text{for all } k \in K^{\leq} \Bigg\}.$$

The next theorem deals with the prox-regularity of sets defined by finitely many smooth inequality and equality constraints.

THEOREM 3.5. *Let I be a nonempty set, let $\mathcal{H}$ be a Hilbert space, $m, n \in \mathbb{N}$, and let $g_k : I \times \mathcal{H} \to \mathbb{R}$ with $k \in \{1, \ldots, m+n\}$ (resp., $k = \{1, \ldots, m\}$) be functions such that, for each $t \in I$, the set*

$$C(t) := \{x \in \mathcal{H} : g_1(t,x) \leq 0, \ldots, g_m(t,x) \leq 0, g_{m+1}(t,x) = 0, \ldots, g_{m+n}(t,x) = 0\}$$

$$(\textit{resp.}, C(t) := \{x \in \mathcal{H} : g_1(t,x) \leq 0, \ldots, g_m(t,x) \leq 0\})$$

is nonempty. Assume that there is an extended real $\rho \in]0, +\infty]$ such that

(i) *for each $t \in I$, for all $k \in \{1, \ldots, m+n\}$ (resp., $k \in \{1, \ldots, m\}$) the functions $g_k(t, \cdot)$ are C^1 on $U_\rho(C(t))$;*

(ii) *there exists a real $\gamma \geq 0$ such that for all $t \in I$ and for all $x, y \in U_\rho(C(t))$,*

$$\langle \nabla g_k(t,\cdot)(x) - \nabla g_k(t,\cdot)(y), x - y \rangle \geq -\gamma \|x - y\|^2 \quad \text{for all } k \in \{1, \ldots, m\} \tag{3.6}$$

and

$$\|\nabla g_k(t,\cdot)(x) - \nabla g_k(t,\cdot)(y)\| \leq \gamma \|x - y\| \quad \text{for all } k \in \{m+1, \ldots, m+n\}$$

(resp., (3.6) holds). Assume also that there exists a real $\delta > 0$ such that for all $x \in \operatorname{bdry} C(t)$

$$[-\delta, \delta]^{p_{t,x}} \times [-\delta, \delta]^n \subset A_{t,x}(\mathbb{B}_{\mathcal{H}}) + \mathbb{R}_+^{p_{t,x}} \times \{0_{\mathbb{R}^n}\} \tag{3.7}$$

$$(\textit{resp.},\ [-\delta, \delta]^{p_{t,x}} \subset A_{t,x}(\mathbb{B}_{\mathcal{H}}) + \mathbb{R}_+^{p_{t,x}}),$$

where $p_{t,x} = \operatorname{Card}\{k \in \{1, \ldots, m\} : g_k(t,x) = 0\}$,

$$A_{t,x} := \big(Dg_{i_1}(t,\cdot)(x), \ldots, Dg_{i_{p_{t,x}}}(t,\cdot)(x), Dg_{m+1}(t,\cdot)(x), \ldots, Dg_{m+n}(t,\cdot)(x)\big)$$

$$(\textit{resp.},\ A_{t,x} := \big(Dg_{i_1}(t,\cdot)(x), \ldots, Dg_{i_{p_{t,x}}}(t,\cdot)(x)\big))$$

and $\{i_1, \ldots, i_{p_{t,x}}\} = \{k \in \{1, \ldots, m\} : g_k(t,x) = 0\}$. Then for every $t \in I$, the set $C(t)$ is r-prox-regular with $r := \min\{\rho, \frac{\delta}{\gamma}\}$.

Proof. Clearly, it suffices to prove the result with $n \geq 1$. All the sets $C(t)$ are obviously closed according to the continuity of the functions $g_k(t, \cdot)$ over $U_\rho(C(t))$. Fix any $t \in I$, $x, y \in C(t)$ with $\|x - y\| < 2\rho$ and $x \in \operatorname{bdry} C(t)$. Let $\lambda_{m+1}, \ldots, \lambda_{m+n}$ be reals such that

$$\sum_{i=1}^{n} \lambda_i \nabla g_{m+i}(t,\cdot)(x) = 0. \tag{3.8}$$

Fix any real $\alpha_{m+1} \in [-\delta, \delta] \setminus \{0\}$. According to (3.7), there is $u \in \mathbb{B}_{\mathcal{H}}$ such that

$$(\alpha_{m+1}, 0, \ldots, 0) = (Dg_{m+1}(t,\cdot)(x)(u), \ldots, Dg_{m+n}(t,\cdot)(x)(u)).$$

Using (3.8) and the latter equality, we obtain

$$\lambda_{m+1} \langle \nabla g_{m+1}(t,\cdot)(x), u \rangle = \lambda_{m+1}\alpha_{m+1} = 0.$$

Since $\alpha_{m+1} \neq 0$, we get $\lambda_{m+1} = 0$. In the same way, we obtain $\lambda_{m+2} = \cdots = \lambda_{m+n} = 0$. Thus, we see that the first part of the Mangasarian–Fromovitz qualification condition is satisfied at x, i.e., the vectors $\nabla g_{m+1}(t,\cdot)(x), \ldots, \nabla g_{m+n}(t,\cdot)(x)$ are linearly independent. From inclusion (3.7) and from

$$(-\delta, \ldots, -\delta, 0, \ldots, 0) \in [-\delta, \delta]^{p_{t,x}} \times [-\delta, \delta]^n$$

it is easily seen as above that the second part of the Mangasarian–Fromovitz qualification condition is satisfied at x. Consequently, we have

$$N = \left\{ \sum_{k\in K} \lambda_k \nabla g_k(t,\cdot)(x) + \sum_{k=m+1}^{m+n} \lambda_k \nabla g_k(t,\cdot)(x) : \lambda_k \geq 0 \; for \; k \in K, \lambda_k \in \mathbb{R} \; for \; k = m+1,\dots,m+n \right\},$$

where $N := N^C(C(t);x)$ and $K := \{k \in \{1,\dots,m\} : g_k(t,x) = 0\}$. Take any $\zeta \in N^C(C(t);x) \setminus \{0\}$ (if $N^C(C(t);x) = \{0\}$, it is straightforward). We can write

$$\zeta = \sum_{k\in K} \lambda_k \nabla g_k(t,\cdot)(x) + \sum_{k=m+1}^{m+n} \lambda_k \nabla g_k(t,\cdot)(x) \tag{3.9}$$

with some reals $\lambda_k \geq 0$ for $k \in K$ and $\lambda_k \in \mathbb{R}$ for $k = m+1,\dots,m+n$. Fix for a moment $k \in K \cup \{m+1,\dots,m+n\}$. By Lemma 3.2, we know that for all $s \in [0,1]$, $x + s(y-x) \in U_\rho(C(t))$. Using assumption (ii), one has for all $s \in [0,1]$,

$$\langle \nabla g_k(t,\cdot)(x+s(y-x)) - \nabla g_k(t,\cdot)(x), y-x\rangle \geq -\gamma s \|y-x\|^2 .$$

One observes that

$$\begin{aligned} 0 &\geq g_k(t,y) - g_k(t,x) \\ &= \int_0^1 \langle \nabla g_k(t,\cdot)(x+s(y-x)), y-x\rangle \, ds \\ &= \langle \nabla g_k(t,\cdot)(x), y-x\rangle + \int_0^1 \langle \nabla g_k(t,\cdot)(x+s(y-x)) - \nabla g_k(t,\cdot)(x), y-x\rangle \, ds \\ &\geq \langle \nabla g_k(t,\cdot)(x), y-x\rangle - \gamma \|y-x\|^2 \int_0^1 s \, ds, \end{aligned}$$

and hence one has

$$\langle \nabla g_k(t,\cdot)(x), y-x\rangle \leq \frac{\gamma}{2} \|y-x\|^2 . \tag{3.10}$$

Fix now any $k \in \{m+1,\dots,m+n\}$. Again, using assumption (ii), one has

$$\begin{aligned} 0 &= g_k(t,x) - g_k(t,y) \\ &= \int_0^1 \langle \nabla g_k(t,\cdot)(y+s(x-y)), x-y\rangle \, ds \\ &= \langle \nabla g_k(t,\cdot)(x), x-y\rangle + \int_0^1 \langle \nabla g_k(t,\cdot)(y+s(x-y)) - \nabla g_k(t,\cdot)(x), x-y\rangle \, ds. \end{aligned}$$

It follows that

$$\langle \nabla g_k(t,\cdot)(x), x-y\rangle = \int_0^1 \langle \nabla g_k(t,\cdot)(x) - \nabla g_k(t,\cdot)(y+s(x-y)), x-y\rangle \, ds,$$

and then we have

$$\langle \nabla g_k(t,\cdot)(x), x-y\rangle \leq \frac{\gamma}{2} \|x-y\|^2 .$$

Thanks to (3.10), we get

$$|\langle \nabla g_k(t,\cdot)(x), y-x\rangle| \le \frac{\gamma}{2}\|y-x\|^2.$$

It ensues that

$$\begin{aligned}\langle \zeta, y-x\rangle &= \sum_{k\in K}\lambda_k \langle \nabla g_k(t,\cdot)(x), y-x\rangle + \sum_{k=m+1}^{m+n}\lambda_k \langle \nabla g_k(t,\cdot)(x), y-x\rangle \\ &\le \Big(\sum_{k\in K}\lambda_k\Big)\frac{\gamma}{2}\|y-x\|^2 + \Big(\sum_{k=m+1}^{m+n}|\lambda_k|\Big)\frac{\gamma}{2}\|y-x\|^2,\end{aligned}$$

or equivalently

$$\langle \zeta, y-x\rangle \le \Big(\sum_{k\in K}|\lambda_k| + \sum_{m+1}^{m+n}|\lambda_k|\Big)\frac{\gamma}{2}\|y-x\|^2. \tag{3.11}$$

With $K=\{i_1,\dots,i_p\}$ (so, $p := \operatorname{Card} K = p_{t,x}$) consider the continuous linear mapping $A:\mathcal{H}\longrightarrow \mathbb{R}^p\times\mathbb{R}^n$ given for all $h\in\mathcal{H}$ by

$$Ah := \big(Dg_{i_1}(t,\cdot)(x)h,\dots,Dg_{i_p}(t,\cdot)(x)h, Dg_{m+1}(t,\cdot)(x)h,\dots,Dg_{m+n}(t,\cdot)(x)h\big),$$

and note that (keeping in mind (3.9)) $\langle\zeta,\cdot\rangle = (y^*\circ A)(\cdot)$, where the linear functional y^* is defined on $\mathbb{R}^p\times\mathbb{R}^n$ by

$$\langle y^*, v\rangle = \sum_{k\in K}\lambda_k v_k + \sum_{k=m+1}^{m+n}\lambda_k v_k \quad \text{for all } v=(v_1,\dots,v_p,v_{m+1},\dots,v_{m+n})\in\mathbb{R}^p\times\mathbb{R}^n.$$

Setting $\bar v := \big(\delta,\dots,\delta,\operatorname{sign}(\lambda_{m+1})\delta,\dots,\operatorname{sign}(\lambda_{m+n})\delta\big)$ and noting that $-\bar v\in[-\delta,\delta]^p\times[-\delta,\delta]^n$, the inclusion (3.7) yields some $b\in\mathbb{B}_{\mathcal{H}}$ and $q=(q_{i_1},\dots,q_{i_p},q_{m+1},\dots,q_{m+n})$, with $q_{i_1}\ge 0,\dots,q_{i_p}\ge 0$ and $q_{m+1}=\dots=q_{m+n}=0$, such that $-\bar v = A(-b)+q$, that is, $\bar v = A(b)-q$. This and the definition of y^* and q combined with the inequalities $\lambda_k\ge 0$, for all $k\in K$, give

$$\langle y^*,\bar v\rangle = (y^*\circ A)(b) - \sum_{k\in K}\lambda_k q_k \le (y^*\circ A)(b) \le \|y^*\circ A\|. \tag{3.12}$$

On the other hand, we have from the definitions of y^* and $\overline{v}$

$$\langle y^*,\bar v\rangle = \delta\Big(\sum_{k\in K}|\lambda_k| + \sum_{k=m+1}^{m+n}|\lambda_k|\Big),$$

thus (thanks to (3.12))

$$\delta\Big(\sum_{k\in K}|\lambda_k| + \sum_{m+1}^{m+n}|\lambda_k|\Big) \le \|y^*\circ A\| = \|\zeta\|.$$

This combined with (3.11) guarantees that

$$\langle\zeta, y-x\rangle \le \frac{\gamma}{2\delta}\|\zeta\|\,\|y-x\|^2.$$

Consequently, for all $x,y\in C(t)$ with $x\in \operatorname{bdry} C(t)$ and $\|y-x\|<2r$ and for all $\zeta\in N^C(C(t);x)$, we obtain $\langle\zeta,y-x\rangle\le\frac{1}{2r}\|\zeta\|\,\|y-x\|^2$, which justifies the r-prox-regularity of the set $C(t)$, according to Proposition 2.4. ∎

4. Prox-regularity of nonsmooth sublevel sets. In Theorem 3.1, we recalled a result related to the uniform prox-regularity of sublevel sets of smooth functions. This section is concerned with the situation of sublevel sets of finitely/infinitely many nonsmooth functions. Its first theorem says in particular that, under a generalized Slater qualification condition, sublevel sets of locally Lipschitz *prox-regular functions* are prox-regular sets.

THEOREM 4.1. *Let I be a nonempty set and let $\mathcal{H}$ be a Hilbert space, $m \in \mathbb{N}$, $g_1, \ldots, g_m : I \times \mathcal{H} \longrightarrow \mathbb{R}$ such that, for each $t \in I$, the set*

$$C(t) = \{x \in \mathcal{H} : g_1(t,x) \leq 0, \ldots, g_m(t,x) \leq 0\}$$

is nonempty. Assume that there is an extended real $\rho \in]0,+\infty]$ such that

(i) *for each $t \in I$ and for all $k \in \{1, \ldots, m\}$, $g_k(t,\cdot)$ is locally Lipschitz continuous on $U_\rho(C(t))$;*

(ii) *there is a real $\gamma \geq 0$ such that for all $t \in I$ and $k \in \{1, \ldots, m\}$, for all $x_1, x_2 \in U_\rho(C(t))$, and for all $v_1 \in \partial_C g_k(t,\cdot)(x_1)$ and all $v_2 \in \partial_C g_k(t,\cdot)(x_2)$,*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\gamma \|x_1 - x_2\|^2. \tag{4.1}$$

Assume also that there is a real $\delta > 0$ such that for all $(t,x) \in I \times \mathcal{H}$ with $x \in$ $\mathrm{bdry}\, C(t)$, there exists $\overline{v} \in \mathbb{B}_{\mathcal{H}}$ satisfying for all $k \in \{1, \ldots, m\}$ and for all $\xi \in$ $\partial_C g_k(t,\cdot)(x)$,

$$\langle \xi, \overline{v} \rangle \leq -\delta.$$

Then, for all $t \in I$, $C(t)$ is r-prox-regular with $r = \min\left\{\rho, \frac{\delta}{\gamma}\right\}$.

Proof. Set $K = \{1, \ldots, m\}$ and fix any $t \in I$. The set $C(t)$ is closed in $\mathcal{H}$, thanks to the continuity of each $g_k(t,\cdot)$ on $U_\rho(C(t))$ with $k \in K$. For each $x \in \mathcal{H}$ put

$$g(t,x) = \max_{k \in K} g_k(t,x) \quad \text{and} \quad K(t,x) = \{k \in K : g_k(t,x) = g(t,x)\}.$$

Obviously, one observes that $C(t) = \{x \in \mathcal{H} : g(t,x) \leq 0\}$. Using [11, Proposition 2.3.12] and our assumption (i), one has

$$\partial_C g(t,\cdot)(x) \subset \mathrm{co}\left(\bigcup_{k \in K(t,x)} \partial_C g_k(t,\cdot)(x) \right) \quad \text{for all } x \in C(t). \tag{4.2}$$

It is readily seen that the latter inclusion and the existence of $\overline{v}$ in (iii) give us

$$0 \notin \partial_C g(t,\cdot)(x) \quad \text{for all } x \in \mathrm{bdry}\, C(t).$$

According to Corollary 1 of [11, Theorem 2.4.7], one has

$$N^C(C(t);x) \subset \mathbb{R}_+ \partial_C g(t,\cdot)(x) \quad \text{for all } x \in \mathrm{bdry}\, C(t).$$

Fix now any $x, y \in C(t)$ with $x \in \mathrm{bdry}\, C(t)$ and $\|x - y\| < 2\rho$. For all $s \in [0,1]$, one has by Lemma 3.2, $x + s(y - x) \in U_\rho(C(t))$. Further, for $\zeta \in \partial_C g(t,\cdot)(x)$ and $\xi \in \partial_C g(t,\cdot)(y)$, from (4.2) there are $\zeta_k \in \partial_C g_k(t,\cdot)(x)$ and $\xi_k \in \partial_C g_k(t,\cdot)(y)$, and $\lambda_k, \mu_k \geq 0$ with $\sum_{k \in K} \lambda_k = \sum_{k \in K} \mu_k = 1$ such that $\zeta = \sum_{k \in K} \lambda_k \zeta_k$ and $\xi = \sum_{k \in K} \mu_k \xi_k$. It ensues that

$$\langle \zeta - \xi, x - y \rangle = \sum_{j \in K} \sum_{k \in K} \mu_j \lambda_k \langle \zeta_k - \xi_j, x - y \rangle \geq -\gamma \|x - y\|^2.$$

Define the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\varphi(\tau) := g(t, x + \tau(y - x))$$

and observe that it is Lipschitz continuous on $[0, 1]$. As a consequence, there exists a Lebesgue negligible subset N of $[0, 1]$ such that φ is derivable on $[0, 1] \setminus N$ and

$$0 \geq g(t, y) - g(t, x) = \int_0^1 \varphi'(\tau) d\tau.$$

Fix for a moment any $s \in [0, 1] \setminus N$ and define the affine mapping $G : \mathbb{R} \longrightarrow \mathcal{H}$ with

$$G(\tau) := x + \tau(y - x).$$

By assumption (i), the mapping $g(t, \cdot)$ is locally Lipschitz continuous on $U_\rho(C(t))$, in particular it is Lipschitz continuous near $G(s)$. Since φ is differentiable at s, one has

$$\varphi'(s) \in \partial_C (g(t, \cdot) \circ G)(s).$$

Using the chain rule in [11, Theorem 2.3.10], we obtain some $\zeta_s \in \partial_C g(t, \cdot)(x_s)$ such that $\varphi'(s) = \langle \zeta_s, y - x \rangle$, where $x_s := x + s(y - x)$. Consequently, we have

$$\begin{aligned} 0 &\geq \int_0^1 \varphi'(s) ds = \int_0^1 \langle \zeta_s, y - x \rangle \, ds \\ &= \int_0^1 \langle \zeta_s - \zeta, y - x \rangle \, ds + \langle \zeta, y - x \rangle \\ &= \int_0^1 \frac{1}{s} \langle \zeta_s - \zeta, x_s - x \rangle \, ds + \langle \zeta, y - x \rangle \\ &\geq - \int_0^1 \frac{1}{s} \gamma \|x_s - x\|^2 \, ds + \langle \zeta, y - x \rangle \\ &= -\gamma \|y - x\|^2 \int_0^1 s \, ds + \langle \zeta, y - x \rangle = -\frac{\gamma}{2} \|y - x\|^2 + \langle \zeta, y - x \rangle. \end{aligned}$$

From this we deduce $\langle \zeta, y - x \rangle \leq \frac{\gamma}{2} \|y - x\|^2$. It is straightforward that the inclusion $\overline{v} \in \mathbb{B}_{\mathcal{H}}$ and the inequality $\langle \zeta, \overline{v} \rangle > -\delta$ give us $\|\zeta\| \geq \delta > 0$. Then, we can write

$$\langle \zeta, y - x \rangle \leq \frac{\gamma}{2\delta} \|\zeta\| \, \|y - x\|^2 .$$

Proposition 2.4 ensures that for all $t \in I$, $C(t)$ is r-prox-regular with $r = \min \left\{ \rho, \frac{\delta}{\gamma} \right\}$. ∎

Remark 3. The latter result obviously encompasses Theorem 3.1. Nevertheless, due to the lack of differentiability of the constraints functions g_k, the proof of Theorem 4.1 is quite different from those of [1, Theorem 9.1].

Remark 4. Let U be a nonempty open subset of a normed space X. It can be verified that a locally Lipschitz (resp., lower semicontinuous) function g from U into $\mathbb{R}$ (resp., into $\mathbb{R} \cup \{+\infty\}$), which has its Clarke subdifferential γ-hypomonotone on U (that is, g satisfies (ii) of the above theorem) for some real $\gamma \geq 0$, is γ-semiconvex on U in the sense ([10])

$$g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y) + \frac{1}{2} \gamma t(1 - t) \|x - y\|^2$$

for all $x, y \in U$ and all $t \in]0, 1[$. It is worth pointing out that the γ-hypomonotonicity of the Clarke subdifferential of a lower semicontinuous function f is shown in [18] to be equivalent to the γ-paraconvexity of f whenever $\gamma > 1$ (see [18] for more details). Semiconvex functions are also called weakly convex in [29]. Further, if X is a Hilbert space, the local semiconvexity of a locally Lipschitz function g (i.e., the semiconvexity on a neighborhood of each point) means that g is prox-regular on U (see, e.g., [23, 4]). Such functions have been proved to be Clarke tangentially regular (Clarke subdifferentially regular) in [18]. ▯

The study of sets structured by infinitely many nonsmooth inequalities is a consequence of the latter theorem.

COROLLARY 4.2. *Let I be a nonempty set, let $(W, \mathcal{O})$ be a Hausdorff topological space, let $\mathcal{H}$ be a Hilbert space. For each $w \in W$, let $g_w : I \times \mathcal{H} \longrightarrow \mathbb{R}$ be a function such that for each $t \in I$, the set*

$$C(t) = \left\{ x \in \mathcal{H} : \sup_{w \in W} g_w(t, x) \le 0 \right\}$$

is nonempty. Let ρ be an extended real of $]0, +\infty]$, and for each $t \in I$ and each $x \in U_\rho(C(t))$, let

$$M_t(x) := \left\{ w \in W : g_w(t, x) = \sup_{w' \in W} g_{w'}(t, x) \right\}.$$

Assume that there is an extended real $\rho \in]0, +\infty]$ such that for each $t \in I$,

(i) *the functions $g_w(t, \cdot)$, $w \in W$, are locally equi-Lipschitz on $U_\rho(C(t))$;*

(ii) *for each $x \in U_\rho(C(t))$, the function $w \mapsto g_w(t, x)$ is upper semicontinuous on W;*

(iii) *for each $t \in I$ and for each $\overline{x} \in U_\rho(C(t))$, there exist a neighborhood $U \subset U_\rho(C(t))$ of $\overline{x}$ and a compact set $K_{t,\overline{x}} \subset W$ such that*

$$\bigcup_{x \in U_\rho(C(t))} M_t(x) \subset K_{t,\overline{x}};$$

and $M_t(x) \neq \emptyset$ for all $x \in U$;

(iv) *the multimapping $(w, x) \mapsto \partial_C g_w(t, x)$ from $W \times U_\rho(C(t))$ into $\mathcal{H}$ has its graph which is $\mathcal{O} \times \|\cdot\| \times w$-closed;*

Assume also that

(v) *there is a real $\gamma \ge 0$ such that for all $t \in I$, for all $x_1, x_2 \in U_\rho(C(t))$, for all $v_1 \in \bigcup_{w \in M_t(x_1)} \partial_C g_w(t, \cdot)(x_1)$, and for all $v_2 \in \bigcup_{w \in M_t(x_2)} \partial_C g_w(t, \cdot)(x_2)$*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \ge -\gamma \|x_1 - x_2\|^2;$$

(vi) *there is a real $\delta > 0$ satisfying for all $(t, x) \in I \times \mathcal{H}$ with $x \in \operatorname{bdry} C(t)$, there is $\overline{v} \in \mathbb{B}_\mathcal{H}$ satisfying for all $\xi \in \bigcup_{w \in M_t(x)} \partial_C g_w(t, \cdot)(x)$,*

$$\langle \xi, \overline{v} \rangle \le -\delta.$$

Then, for all $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\{\rho, \frac{\delta}{\gamma}\}$.

Proof. Fix any $t \in I$ and note that $C(t) = \{x \in \mathcal{H} : f(t, x) \le 0\}$, where

$$f(t, x) := \sup_{w \in W} g_w(t, x) \quad \text{for all } x \in \mathcal{H}.$$

From (i)–(iv) and following the arguments in the proof of [11, Theorem 2.8.2], one obtains that the function $f(t,\cdot)$ is locally Lipschitz continuous on $U_\rho(C(t))$ and that the inclusion

$$\partial_C f(t,\cdot)(x) \subset \overline{\mathrm{co}}^w \left(\bigcup_{w\in M_t(x)} \partial_C g_w(t,\cdot)(x) \right)$$

holds true for all $x \in U_\rho(C(t))$. From the latter inclusion and the assumption (vi), it is easily seen that, for every $x \in \mathrm{bdry}\, C(t)$,

$$\langle x^\star, \overline{v}\rangle \le -\delta \quad \text{for all } x^\star \in \partial_C f(t,\cdot)(x).$$

Further, from (v) it is also easily seen that for all $x_1, x_2 \in U_\rho(C(t))$, for all $v_1 \in \partial_C f(t,\cdot)(x_1)$, and for all $v_2 \in \partial_C f(t,\cdot)(x_2)$,

$$\langle v_1 - v_2, x_1 - x_2\rangle \ge -\gamma \left\| x_1 - x_2 \right\|^2.$$

Applying Theorem 4.1, the set $C(t)$ is r-prox-regular with $r = \min\{\rho, \frac{\delta}{\gamma}\}$. ∎

5. Intersection of prox-regular subsets. Given two prox-regular subsets S_1 and S_2 of a Hilbert space X, one natural question would be to check for the prox-regularity of the intersection $S_1 \cap S_2$. In order to study the prox-regularity of the intersection of sets, given two subsets S_1, S_2 of a normed space X, let us consider the multimapping $M(\cdot) = (S_1 - \cdot) \times (S_2 - \cdot) : X \rightrightarrows X \times X$ defined by

$$x \longmapsto (S_1 - x) \times (S_2 - x). \tag{5.1}$$

The following lemma describes the Bouligand–Peano tangent cone of the graph of $M(\cdot) = (S_1 - \cdot) \times (S_2 - \cdot)$. Its proof is omitted; it follows directly from the definitions of Bouligand–Peano and Clarke tangent cones.

LEMMA 5.1. *Let X be a normed space and let S_1, S_2 be two subsets of X, $M(\cdot) = (S_1 - \cdot) \times (S_2 - \cdot)$, $(x,y,z) \in X^3$ with $x + y \in S_1$ and $x + z \in S_2$. Then, one has*

$$\begin{aligned}&T^B(\mathrm{gph}\, M; (x,y,z))\\ &\subset \left\{(u,v,w) \in X^3 : u + v \in T^B(S_1; x+y),\ u + w \in T^B(S_2; x+z)\right\}.\end{aligned}$$

If, in addition, either S_1 is Clarke tangentially regular at $x + y$ or S_2 is Clarke tangentially regular at $x + z$, then the inclusion is an equality.

The next result is crucial in the development of this section.

PROPOSITION 5.2. *Let X be a normed space and let S_1, S_2 be two nonempty subsets of X, $\overline{x} \in X$, $M(\cdot) = (S_1 - \cdot) \times (S_2 - \cdot)$. Consider the following assertions.*

$\big(A_{S_1,S_2}(s)\big)$*: There exist a real $s > 0$ and a neighborhood U of $\overline{x}$ such that for all $x_1 \in U \cap S_1$ and for all $x_2 \in U \cap S_2$,*

$$s\mathbb{B}_X \subset T^B(S_1; x_1) \cap \mathbb{B}_X - T^B(S_2; x_2) \cap \mathbb{B}_X.$$

$\big(A_M(s)\big)$*: There exist a real $s > 0$, a neighborhood U of $\overline{x}$ and a neighborhood V of 0 such that, for all $(x,y,z) \in \mathrm{gph}\, M \cap U \times V \times V$,*

$$s(\mathbb{B}_X \times \mathbb{B}_X) \subset T^B M(x,y,z)(\mathbb{B}_X).$$

Then, the implication $\big(A_M(s)\big) \Rightarrow \big(A_{S_1,S_2}(\frac{2s}{s+1})\big)$ *holds true. If, in addition, either* S_1 *or* S_2 *is Clarke tangentially regular near* $\overline{x}$*, then* $\big(A_{S_1,S_2}(s)\big) \Rightarrow \big(A_M(\frac{s}{s+2})\big)$.

Proof. $\big(A_M(s)\big) \Rightarrow \big(A_{S_1,S_2}(\frac{2s}{s+1})\big)$. Fix any real $\eta > 0$ such that $B[0,\eta] \subset V$. Let $\zeta \in s\mathbb{B}_X$, so $(\zeta,-\zeta) \in s(\mathbb{B}_X \times \mathbb{B}_X)$. For each $i \in \{1,2\}$, let $x_i \in B[\overline{x},\eta] \cap S_i$. In particular, we have $x_i - \overline{x} \in S_i - \overline{x}$. Then, $(\overline{x}, x_1 - \overline{x}, x_2 - \overline{x}) \in \operatorname{gph} M$ and $x_i - \overline{x} \in V$ for each $i \in \{1,2\}$. The inclusion given by $(A_M(s))$ entails that

$$(\zeta,-\zeta) \in T^B M(\overline{x}, x_1-\overline{x}, x_2-\overline{x})(\mathbb{B}_X),$$

which gives some $u \in \mathbb{B}_X$ such that

$$(\zeta,-\zeta) \in T^B M(\overline{x}, x_1-\overline{x}, x_2-\overline{x})(u),$$

that is (thanks to (2.2))

$$(u,\zeta,-\zeta) \in T^B(\operatorname{gph} M; (\overline{x}, x_1-\overline{x}, x_2-\overline{x})).$$

Then, we can apply Lemma 5.1 to get

$$u+\zeta \in T^B(S_1; \overline{x}+x_1-\overline{x}) = T^B(S_1;x_1)$$

and

$$u-\zeta \in T^B(S_2; \overline{x}+x_2-\overline{x}) = T^B(S_2;x_2).$$

Further, we have

$$\max\{\|u+\zeta\|, \|u-\zeta\|\} \le \|u\| + \|\zeta\| \le 1+s.$$

Since $\zeta = \frac{1}{2}(u+\zeta) - \frac{1}{2}(u-\zeta)$, we see that

$$\zeta \in T^B(S_1;x_1) \cap \frac{1+s}{2}\mathbb{B}_X - T^B(S_2;x_2) \cap \frac{1+s}{2}\mathbb{B}_X.$$

Hence (keeping in mind that $\zeta \in s\mathbb{B}_X$)

$$\frac{2s}{s+1}\mathbb{B}_X \subset T^B(S_1;x_1) \cap \mathbb{B}_X - T^B(S_2;x_2) \cap \mathbb{B}_X.$$

$\big(A_{S_1,S_2}(s)\big) \Rightarrow \big(A_M(\frac{s}{s+2})\big)$. Assume that S_1 or S_2 is Clarke tangentially regular near $\overline{x}$. Without loss of generality, we may suppose that S_1 is tangentially regular at any points of $S_1 \cap U$. Choose any real $\eta > 0$ such that $B[\overline{x},\eta] \subset U$. Fix any $(x,y,z) \in \operatorname{gph} M$ with $x \in \overline{x} + \frac{\eta}{2}\mathbb{B}_X$, $y \in \frac{\eta}{2}\mathbb{B}_X$, $z \in \frac{\eta}{2}\mathbb{B}_X$. Let $(v,w) \in \frac{s}{2}(\mathbb{B}_X \times \mathbb{B}_X)$, so $w - v \in s\mathbb{B}_X$. We have $x+y \in S_1$ and $x+z \in S_2$. On the other hand

$$\|x+y-\overline{x}\| \le \eta \quad \text{and} \quad \|x+z-\overline{x}\| \le \eta.$$

Hence, by $\big(A_{S_1,S_2}(s)\big)$, there are $b_1 \in T^B(S_1;x+y) \cap \mathbb{B}_X$ and $b_2 \in T^B(S_2;x+z) \cap \mathbb{B}_X$ such that $v - w = b_1 - b_2$. Putting $u := b_1 - v$, we have

$$u + v \in T^B(S_1; x+y). \tag{5.2}$$

The equality $u + w = b_2$ gives us

$$u + w \in T^B(S_2; x+z). \tag{5.3}$$

Combining (5.2), (5.3), and Lemma 5.1 (thanks to the fact that S_1 is tangentially regular at $x+y$), we have $(u,v,w) \in T^B(\text{gph}M;(x,y,z))$. It is readily seen that $\|u\| \leq 1+\frac{s}{2} = \frac{2+s}{2}$. Since $T^B(\text{gph}M;(x,y,z))$ is a cone, we get from the latter inclusion

$$\frac{2}{2+s}(u,v,w) \in T^B(\text{gph}M;(x,y,z)),$$

i.e., $\frac{2}{2+s}(v,w) \in T^B M(x,y,z)(\frac{2}{2+s}u)$. As a consequence, we have

$$\frac{s}{s+2}(\mathbb{B}_X \times \mathbb{B}_X) \subset T^B M(x,y,z)(\mathbb{B}_X). \qquad \square$$

Now, given two subsets S_1 and S_2 of an Asplund space with $S_1 \cap S_2 \ni \overline{x}$, our aim is to prove that we have the following inclusion:

$$N^L(S_1 \cap S_2; x) \subset N^L(S_1; x) + N^L(S_2; x)$$

under an openness assumption on the Bouligand–Peano tangent cones of S_1 and S_2. Note that the set gph M (where M is the multimapping defined as in (5.1)) is closed near $(\overline{x},0,0)$ whenever S_1 and S_2 are closed near $\overline{x}$.

PROPOSITION 5.3. *Let X be an Asplund space and let S_1, S_2 be two nonempty subsets closed near $\overline{x} \in S_1 \cap S_2$. Assume the following:*

(i) *either S_1 or S_2 is Clarke tangentially regular near $\overline{x}$;*

(ii) *there exist a real $s > 0$ and a neighborhood U of $\overline{x}$ such that for all $x_1 \in U \cap S_1$ and for all $x_2 \in U \cap S_2$,*

$$s\mathbb{B}_X \subset T^B(S_1;x_1) \cap \mathbb{B}_X - T^B(S_2;x_2) \cap \mathbb{B}_X.$$

Then, one has

$$N^L(S_1 \cap S_2; \overline{x}) \subset N^L(S_1; \overline{x}) + N^L(S_2; \overline{x}).$$

Proof. Set $M(\cdot) = (S_1 - \cdot) \times (S_2 - \cdot)$. Combining (i), (ii), and Proposition 5.2, there exist two reals $s', \eta > 0$ such that for all $(x,y,z) \in \text{gph } M$ with $x \in \overline{x} + \eta\mathbb{B}_X$, $y \in \eta\mathbb{B}_X$, $z \in \eta\mathbb{B}_X$,

$$s'(\mathbb{B}_X \times \mathbb{B}_X) \subset T^B M(x,y,z)(\mathbb{B}_X).$$

According to Theorem 2.6, M is metrically regular at $\overline{x}$ for $(0,0)$, where X^2 is endowed with the norm defined by $\|(u,v)\| = \|u\| + \|v\|$ for all $(u,v) \in X^2$. The metric regularity gives a real $\gamma \geq 0$, an open neighborhood V of $\overline{x}$ in X, such that

$$d(x, M^{-1}(0,0)) \leq \gamma d((0,0), M(x)) \quad \text{for all } x \in V.$$

As a consequence, we have

$$d(x, S_1 \cap S_2) \leq \gamma\big(d(x,S_1) + d(x,S_2)\big) \quad \text{for all } x \in V.$$

Using [22, Theorem 6.44], we get

$$N^L(S_1 \cap S_2; \overline{x}) \subset N^L(S_1; \overline{x}) + N^L(S_2; \overline{x}).$$

This completes the proof. $\square$

Remark 5. As pointed out by one of the referees, the conclusion of the latter proposition could be seen as a consequence of [19, Corollary 3.4] (whose proof is still valid in Asplund space), which is slightly more general than Proposition 5.3. For the reader's convenience, we prefer to give a direct proof (which is short). □

Now, we can state and prove the main result of this section.

THEOREM 5.4. *Let I be a nonempty set, let $\mathcal{H}$ be a Hilbert space, and for each $t \in I$, let $C_1(t)$, $C_2(t)$ be two r-prox-regular subsets of $\mathcal{H}$ with $r \in]0, +\infty[$ such that $C_1(t) \cap C_2(t) \neq \emptyset$ for all $t \in I$.*

Assume that there is a real $s > 0$ such that for every $t \in I$ and for every $x \in \operatorname{bdry}(C_1(t) \cap C_2(t))$, there is a neighborhood U_t of x in $\mathcal{H}$ such that for all $x_1 \in U_t \cap C_1(t)$ and for all $x_2 \in U_t \cap C_2(t)$,

$$s\mathbb{B}_{\mathcal{H}} \subset T(C_1(t); x_1) \cap \mathbb{B}_{\mathcal{H}} - T(C_2(t); x_2) \cap \mathbb{B}_{\mathcal{H}}. \tag{5.4}$$

Then, for all $t \in I$, $C_1(t) \cap C_2(t)$ is $\frac{rs}{2}$-prox-regular.

Proof. Fix any $t \in I$ and $x, x' \in C_1(t) \cap C_2(t)$ with $x \in \operatorname{bdry}(C_1(t) \cap C_2(t))$ and fix any $\zeta \in N^L(C_1(t) \cap C_2(t); x)$. Applying Proposition 5.3 (thanks to the fact that $C_1(t)$ and $C_2(t)$ are Clarke tangentially regular near x), we have

$$N^L(C_1(t) \cap C_2(t); x) \subset N^L(C_1(t); x) + N^L(C_2(t); x).$$

Let us choose $\zeta_i \in N^L(C_i(t); x)$ for each $i \in \{1, 2\}$ such that, $\zeta = \zeta_1 + \zeta_2$. Fix any $v \in \mathbb{B}_{\mathcal{H}}$. Using assumption (5.4), for each $i \in \{1, 2\}$, there exists $v_i \in T(C_i(t); x) \cap \mathbb{B}_{\mathcal{H}}$ satisfying $sv = v_1 - v_2$. We then have

$$\begin{aligned} s\langle \zeta_1, v\rangle &= \langle \zeta_1, v_1\rangle - \langle \zeta_1, v_2\rangle \\ &\leq -\langle \zeta_1, v_2\rangle = -\langle \zeta, v_2\rangle + \langle \zeta_2, v_2\rangle \\ &\leq \langle \zeta, -v_2\rangle \leq \|\zeta\|, \end{aligned}$$

where the first (resp., second) inequality is due to the fact that $\langle \zeta_1, v_1\rangle \leq 0$ (resp., $\langle \zeta_2, v_2\rangle \leq 0$). It follows that $s\|\zeta_1\| \leq \|\zeta\|$. In a similar way, we get $s\|\zeta_2\| \leq \|\zeta\|$. Since $C_1(t)$ and $C_2(t)$ are r-prox-regular sets, we have

$$\begin{aligned} \langle \zeta, x' - x\rangle &= \langle \zeta_1, x' - x\rangle + \langle \zeta_2, x' - x\rangle \\ &\leq \frac{1}{2r}(\|\zeta_1\| + \|\zeta_2\|)\|x' - x\|^2 \\ &\leq \frac{1}{rs}\|\zeta\|\|x' - x\|^2 = \frac{1}{2(\frac{1}{2}rs)}\|\zeta\|\|x' - x\|^2, \end{aligned}$$

and this combined with Theorem 2.2(b) ensures that the set $C_1(t) \cap C_2(t)$ is $\frac{1}{2}rs$-prox-regular. □

6. Preimage of prox-regular sets. This section is concerned with general verifiable conditions ensuring the uniform prox-regularity of the preimage of a prox-regular set. For the direct image, that is, the problem of finding sufficient conditions to ensure the uniform prox-regularity of $g(D)$ where $g : \mathcal{H} \longrightarrow \mathcal{H}'$ is a mapping between two Hilbert spaces and D is a uniformly prox-regular set of $\mathcal{H}$, we refer to [13, Proposition 37].

Let X, Y be two normed spaces. For a subset D of Y and a mapping $f : X \longrightarrow Y$, one denotes by $M(\cdot) = f(\cdot) - D : X \rightrightarrows Y$ the multimapping defined by

$$x \longmapsto f(x) - D. \tag{6.1}$$

We need to describe the Bouligand–Peano tangent cone of the graph of $M(\cdot) = f(\cdot) - D$. The following lemma is in this sense, its proof is easy and will be omitted.

LEMMA 6.1. *Let X, Y be two normed spaces, let $f : X \longrightarrow Y$ be a mapping, let D be a nonempty subset of Y, and let $M(\cdot) = f(\cdot) - D$, $(x, y) \in \operatorname{gph} M$. Assume that f is Gâteaux differentiable at x. Then, one has*

$$\left\{(u, v) \in X \times Y : v \in Df(x)(u) - T^B(D; f(x) - y)\right\} \subset T^B(\operatorname{gph} M; (x, y)).$$

If f is Hadamard differentiable at x, then the latter inclusion is an equality.

The next result is a consequence of Lemma 6.1 and will be useful.

LEMMA 6.2. *Let $f : X \longrightarrow Y$ be a mapping between two normed spaces X and Y, let D be a nonempty subset of Y, and let $M(\cdot) := f(\cdot) - D$. Assume that f is Gâteaux differentiable at $\overline{x} \in f^{-1}(D)$ and that there exist two reals $s, \eta > 0$ such that for all $(x, y) \in (\overline{x} + \eta\mathbb{B}_X) \times \eta\mathbb{B}_Y \cap \operatorname{gph} M$,*

$$s\mathbb{B}_Y \subset Df(x)(\mathbb{B}_X) - T^B(D; f(x) - y). \tag{6.2}$$

Then, for all $(x, y) \in (\overline{x} + \eta\mathbb{B}_X) \times \eta\mathbb{B}_Y \cap \operatorname{gph} M$,

$$s\mathbb{B}_Y \subset T^B M(x, y)(\mathbb{B}_X).$$

Proof. Fix any $(x, y) \in \operatorname{gph} M$ with $x \in (\overline{x} + \eta\mathbb{B}_X)$ and $y \in \eta\mathbb{B}_Y$. Let $v \in \mathbb{B}_Y$. According to the assumption (6.2), there are $u \in \mathbb{B}_X$, $w \in T^B(D; f(x) - y)$ such that $sv = Df(x)(u) - w$. Using Lemma 6.1, we get $(u, sv) \in T^B(\operatorname{gph} M; (x, y))$, i.e., $sv \in T^B M(x, y)(u)$. As a consequence, we have $s\mathbb{B}_Y \subset T^B M(x, y)(\mathbb{B}_X)$. ∎

Before stating the next proposition, observe that the graph of the multimapping M in (6.1) is closed near $(x, 0) \in \operatorname{gph} M$ with $x \in f^{-1}(D)$, whenever D is closed near $f(x)$ and f is continuous near x.

PROPOSITION 6.3. *Let X, Y be two Asplund spaces, let D be a nonempty subset of Y, let $f : X \longrightarrow Y$ be a mapping which is strictly Fréchet differentiable at $\overline{x} \in f^{-1}(D)$, and let $M(\cdot) = f(\cdot) - D$. Assume that D is closed near $f(\overline{x})$. Assume also that there are two reals $s, \eta > 0$ such that for all $(x, y) \in (\overline{x} + \eta\mathbb{B}_X) \times \eta\mathbb{B}_Y \cap \operatorname{gph} M$,*

$$s\mathbb{B}_Y \subset Df(x)(\mathbb{B}_X) - T^B(D; f(x) - y).$$

Then, one has

$$N^L(f^{-1}(D); \overline{x}) \subset \left\{y^\star \circ Df(\overline{x}) : y^\star \in N^L(D; f(\overline{x}))\right\}.$$

Proof. According to Lemma 6.2 and Theorem 2.6, the multimapping $M(\cdot) = f(\cdot) - D$ is metrically regular at $\overline{x}$ for 0. By the metric regularity there are $\gamma, \delta > 0$ two reals such that

$$d(x, M^{-1}(y)) \leq \gamma d(y, M(x)) \quad \text{for all } x \in B(\overline{x}, \delta),\ y \in B(0, \delta).$$

This entails

$$d(x, f^{-1}(D + y)) \leq \gamma d(f(x) - y, D) \quad \text{for all } x \in B(\overline{x}, \delta),\ y \in B(0, \delta).$$

In particular, we have $d(x, f^{-1}(D)) \leq \gamma d(f(x), D)$ for all $x \in B(\overline{x}, \delta)$. By Proposition 2.7, we get

$$N^L(f^{-1}(D); \overline{x}) \subset \left\{y^\star \circ Df(\overline{x}) : y^\star \in N^L(D; f(x))\right\}. \qquad ∎$$

Remark 6. Remark 5 is valid for Proposition 6.3.

With the above results at hand, we can state and prove the theorem on uniform prox-regularity of preimage set.

THEOREM 6.4. *Let I be a nonempty set and let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces, and for each $t \in I$, let $D(t)$ be an r-prox-regular subset of $\mathcal{H}'$ with $r \in]0, +\infty]$ and let $G_t : \mathcal{H} \to \mathcal{H}'$ be a mapping such that $C(t) := G_t^{-1}(D(t)) \neq \emptyset$ for each $t \in I$. Assume that there is an extended real $\rho \in]0, +\infty]$ such that*

(i) *for all $t \in I$, G_t is differentiable on $U_\rho(C(t))$;*

(ii) *there is a real $K > 0$ such that for all $t \in I$ and for all $x, y \in C(t)$ with $\|x - y\| < 2\rho$,*

$$\|G_t(x) - G_t(y)\| \le K \|x - y\| ;$$

(iii) *there is a real $\gamma \ge 0$ such that for all $t \in I$ and for all $x, y \in U_\rho(C(t))$,*

$$\|DG_t(x) - DG_t(y)\| \le \gamma \|x - y\| ;$$

(iv) *there is a real $s > 0$ for which, for all $t \in I$ and for all $\overline{x} \in \operatorname{bdry} C(t)$, there is a real $\eta > 0$ such that for all $(x, y) \in (\overline{x} + \eta\mathbb{B}_{\mathcal{H}}) \times \eta\mathbb{B}_{\mathcal{H}'}$ with $G_t(x) - y \in D(t)$,*

$$s\mathbb{B}_{\mathcal{H}'} \subset DG_t(x)(\mathbb{B}_{\mathcal{H}}) - T(D(t); G_t(x) - y).$$

Then, for all $t \in I$, the set $C(t)$ is r'-prox-regular with

$$r' := \min \left\{ \rho, s \left(\frac{K^2}{r} + \gamma \right)^{-1} \right\}.$$

Proof. Fix any $t \in I$, $x, x' \in C(t)$ with $x \in \operatorname{bdry} C(t)$ and $\|x - x'\| < 2\rho$. For all $\tau \in [0, 1]$, by Lemma 3.2, we have $x + \tau(x' - x) \in U_\rho(C(t))$. Note that, by (i) and (iii), G_t is of class $C^{1,1}$ on $U_\rho(C(t))$. Thus, in particular, G_t is strictly Fréchet differentiable at x. Using Proposition 6.3, we get

$$N^L(C(t); x) \subset \left\{ y^\star \circ DG_t(x) : y^\star \in N^L(D(t); G_t(x)) \right\}.$$

Take any $\zeta \in N^L(C(t); x)$ and choose $\xi \in N^L(D(t); G_t(x)) = N^C(D(t); G_t(x))$ satisfying $\zeta = \xi \circ DG_t(x)$. By the r-prox-regularity of $D(t)$, we obtain

$$\begin{aligned}
\langle \zeta, x' - x \rangle &= \langle \xi, DG_t(x)(x' - x) \rangle \\
&= \left\langle \xi, G_t(x') - G_t(x) - \int_0^1 \big(DG_t(x + \tau(x' - x)) - DG_t(x)\big)(x' - x)\, d\tau \right\rangle \\
&= \langle \xi, G_t(x') - G_t(x) \rangle \\
&\qquad - \left\langle \xi, \int_0^1 \big(DG_t(x + \tau(x' - x)) - DG_t(x)\big)(x' - x)\, d\tau \right\rangle \\
&\le \frac{\|\xi\|}{2r} \|G_t(x') - G_t(x)\|^2 \\
&+ \|\xi\| \int_0^1 \left\| \big(DG_t(x + \tau(x' - x)) - DG_t(x)\big)(x' - x) \right\| \, d\tau,
\end{aligned}$$

hence using (ii) and (iii) it results that

$$
\begin{aligned}
\langle \zeta, x' - x \rangle &\leq \frac{\|\xi\|}{2r} K^2 \|x' - x\|^2 \\
&\quad + \|\xi\| \|x' - x\| \int_0^1 \left\| \left(DG_t(x + \tau(x' - x)) - DG_t(x) \right) \right\| d\tau \\
&\leq \frac{\|\xi\|}{2r} K^2 \|x' - x\|^2 + \gamma \|\xi\| \|x' - x\|^2 \int_0^1 \tau \, d\tau \\
&\leq \|\xi\| \left(\frac{K^2}{2r} + \frac{\gamma}{2} \right) \|x' - x\|^2 . \qquad (6.3)
\end{aligned}
$$

Consider any $v \in \mathbb{B}_{\mathcal{H}'}$. The assumption (iv) gives some $v' \in T^C(D(t); G_t(x))$ and some $u \in \mathbb{B}_{\mathcal{H}}$, such that $sv = v' - DG_t(x)(u)$. Since $\langle \xi, v' \rangle \leq 0$, it follows that

$$s \langle \xi, v \rangle = \langle \xi, v' \rangle - \langle \xi \circ DG_t(x), u \rangle \leq \langle \zeta, -u \rangle \leq \|\zeta\| ,$$

thus $s \|\xi\| \leq \|\zeta\|$. Combining this with (6.3), we get

$$\langle \zeta, x' - x \rangle \leq \frac{\|\zeta\|}{s} \Big(\frac{K^2}{2r} + \frac{\gamma}{2} \Big) \|x' - x\|^2 .$$

In conclusion, Proposition 2.4 tells us that the set $C(t)$ is r'-prox-regular with $r' := \min \left\{ \rho, s(\frac{K^2}{r} + \gamma)^{-1} \right\}$. ∎

Remark 7. Since a nonempty closed subset of a Hilbert space is convex if and only if it is ∞-prox-regular, the preceding result gives that $G_t^{-1}(D(t))$ is $\min \left\{ \rho, \frac{s}{\gamma} \right\}$-prox-regular whenever $D(t)$ is a nonempty closed convex set for each $t \in I$.

Remark 8. Theorem 6.4 holds true *with* (i') *instead of* (i) *and* (iv') *instead of* (iv)*, where*

(i') *G_t is differentiable on $U_\rho(C(t))$ and $DG_t(x) : X \longrightarrow Y$ is surjective for all $x \in$ bdry $C(t)$.*

(iv') *there is a real $s > 0$ such that for all $t \in I$ and for all $x \in$ bdry $C(t)$,*

$$s\mathbb{B}_{\mathcal{H}'} \subset DG_t(x)(\mathbb{B}_{\mathcal{H}}) - T(D(t); G_t(x)).$$

Indeed, according to [21, Theorem 1.17]*, for all $x \in$ bdry $C(t)$, we have*

$$N^L(C(t); x) = \{ y^\star \circ DG_t(x) : y^\star \in N^L(D(t); G_t(x)) \}.$$

We conclude as in the proof of Theorem 6.4*.*

From this remark, we derive the following result. Given a continuous linear mapping $A : \mathcal{H} \to \mathcal{H}'$, whose range is closed, let $A_0 : \mathcal{H} \longrightarrow A(\mathcal{H})$ with $A_0(x) = A(x)$ for all $x \in \mathcal{H}$. Let $s > 0$ be the Banach constant of A_0, i.e.,

$$s := \sup \{ s' > 0 : s' \mathbb{B}_{\mathcal{H}'} \cap A(\mathcal{H}) \subset A(\mathbb{B}_{\mathcal{H}}) \}.$$

Then, if $D(t) \subset A(\mathcal{H})$ is r-prox-regular for each $t \in I$, the above theorem entails that $A^{-1}(D(t))$ is $\|A\|^{-2} r s'$-prox-regular for each $s' \in]0, s[$, $t \in I$. It results that $A^{-1}(D(t))$ is $\|A\|^{-2} rs$-prox-regular for each $t \in I$. Otherwise stated, we have established the following corollary which is in the line of [27, Lemma 2.7].

COROLLARY 6.5. *Let $\mathcal{H}$, $\mathcal{H}'$ be two Hilbert spaces, let $A : \mathcal{H} \longrightarrow \mathcal{H}'$ be a continuous linear mapping whose range is closed, and let $s > 0$ be the Banach constant of the induced linear mapping from $\mathcal{H}$ onto $A(\mathcal{H})$. Let $(D(t))_{t \in I}$ be a family of r-prox-regular subsets of $\mathcal{H}'$ with $r \in]0, +\infty]$ and satisfying $D(t) \subset A(\mathcal{H})$ for all $t \in I$. Then, for every $t \in I$, the set $A^{-1}(D(t))$ is r'-prox-regular with $r' = \frac{rs}{\|A\|^2}$.*

7. Prox-regularity in semiconvex constrained optimization. In this section, we give an application of Theorem 3.5 to constrained optimization. First, consider the C^2 function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^6(1 - \cos(1/x))$ if $x \neq 0$ and $f(0) = 0$. With $f_0 := f$, the constrained optimization problem

$$\text{Minimize } f_0(x) \quad \text{subject to } -x \leq 0$$

admits as a set of solutions $S := \{0\} \cup \{1/(2k\pi) : k \in \mathbb{N}\}$ which fails to be r-prox-regular for any extended real $r \in]0, +\infty]$. On the other hand, in addition to Example 2, with $g := f$, the set $C := \{x \in \mathbb{R} : g(x) \leq 0\} = \{0\} \cup \{1/(2k\pi) : k \in \mathbb{Z} \setminus \{0\}\}$ is neither prox-regular. Conditions are then needed for the uniform prox-regularity of feasible sets and solution sets of optimization problems with even C^2-smooth functions.

Let $f_0, \ldots, f_m : \mathcal{H} \longrightarrow \mathbb{R}$ be real-valued functions on a Hilbert space $\mathcal{H}$. The constrained optimization problem is defined by

$$(\mathcal{P}) \begin{cases} \text{Minimize } f_0(x), \\ \text{subject to : } f_1(x) \leq 0, \ldots, f_m(x) \leq 0. \end{cases}$$

Set $C := \{x \in \mathcal{H} : f_1(x) \leq 0, \ldots, f_m(x) \leq 0\}$, $\mu := \inf_C f_0$, $K = \{1, \ldots, m\}$. Assume that $\mu \in \mathbb{R}$ and that the set of global solutions $S := \operatorname{Argmin}_C f_0$ is nonempty. For reals $\delta > 0$, $\gamma \geq 0$, and for an extended real $\rho \in]0, +\infty]$, consider the following conditions:

(i) $f_1, \ldots, f_m$ are of class C^1 on $U_\rho(C)$; (i') $f_0, \ldots, f_m$ are of class C^1 on $U_\rho(S)$;

(ii) $f_1, \ldots, f_m$ are γ-semiconvex on $U_\rho(C)$; (ii') $f_0, f_1, \ldots, f_m$ are γ-semiconvex on $U_\rho(S)$;

(iii) for all $x \in \operatorname{bdry} C$,

$$[-\delta, \delta]^{p_x} \subset A_x(\mathbb{B}_{\mathcal{H}}) + \mathbb{R}_+^{p_x},$$

where $p_x = \operatorname{Card}\{k \in K : f_k(x) = 0\}$, $A_x = (Df_{i_1}(x), \ldots, Df_{i_{p_x}}(x))$, and $\{i_1, \ldots, i_{p_x}\} = \{k \in K : f_k(x) = 0\}$;

(iii') for all $x \in \operatorname{bdry} S$,

$$[-\delta, \delta]^{p_x+1} \subset \Lambda_x(\mathbb{B}_{\mathcal{H}}) + \mathbb{R}_+^{p_x+1},$$

where p_x is as above and $\Lambda_x = (Df_{i_1}(x), \ldots, Df_{i_{p_x}}(x), Df_0(x))$.

PROPOSITION 7.1. *Let* $r := \min\left\{\rho, \frac{\delta}{\gamma}\right\}$. *The following hold:*

(a) *Under* (i), (ii), *and* (iii) *the feasible set* C *of* $(\mathcal{P})$ *is* r*-prox-regular.*

(b) *Under* (i'), (ii'), *and* (iii') *the set of global solutions* S *of* $(\mathcal{P})$ *is* r*-prox-regular.*

Proof. The set C fulfills the assumptions of the part of Theorem 3.5 involving only inequalities with $g_1 := f_1, \ldots, g_m := f_m$, hence C is r-prox-regular as stated in the assertion (a). Concerning (b), observe that

$$S = \operatorname*{Argmin}_C f_0 = \{x \in \mathcal{H} : f_1(x) \leq 0, \ldots, f_m(x) \leq 0, f_0(x) - \mu \leq 0\}.$$

Put $g_1 := f_1, \ldots, g_m := f_m$ and $g_0 := f_0 - \mu$, and note that $g_0(x) = 0$ at each point $x \in \operatorname{bdry} S$ (since this holds at each $x \in S$). Then, applying again the part of Theorem 3.5 related to inequalities yields the r-prox-regularity of the solution set S. ∎

8. Concluding remarks. On the one hand, we provided examples illustrating that sublevel sets of (smooth) prox-regular functions may fail to be prox-regular and that the prox-regularity of sets is not preserved under usual operations as intersection, preimage, etc. On the other hand, in the context of Hilbert spaces we showed that the desired above uniform prox-regularity properties are guaranteed whenever additional usual verifiable qualification conditions are required. The study of the preservation of the prox-regularity under other operations like the Minkowski sum or the projection operator will be the subject of future work.

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