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PERTURBED BV SWEEPING PROCESS INVOLVING PROX-REGULAR SETS

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Abstract. In this paper, we study the existence of solutions for a variant of discontinuous Moreau sweeping process in the infinite dimensional setting. The sets involved are assumed to be uniformly prox-regular and move with bounded variation. The sweeping process is perturbed by a sum of Lipschitz continuous single-valued mapping and a scalarly upper semicontinuous multimapping satisfying a linear growth condition with respect to a compact set.

1. Introduction

In 1971, J.J. Moreau introduced and developed the notion of sweeping process in the absolute continuous framework ([19, 20]). Given $T_0, T \in \mathbb{R}$ with $T_0 < T$, a Hilbert space $\mathcal{H}$ and a multimapping $C : [T_0, T] \rightrightarrows \mathcal{H}$ with nonempty convex values, a sweeping process consists to find an absolutely continuous mapping $u : [T_0, T] \to \mathcal{H}$ with $u(t) \in C(t)$ for all $t \in [T_0, T]$ satisfying

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \lambda\text{-a.e. } t \in [T_0, T] \\ u(T_0) \in C(T_0), \end{cases}$$

where for each $t \in [T_0, T]$, $N(C(t); u(t))$ denotes the (outward) normal cone to the set $C(t)$ at $u(t)$, in the sense of convex analysis. Such differential inclusions are of great interest in elastoplasticity, quasistatics and dynamics (see, e.g., [21, 24]). Motivated by unilateral mechanics where jumps could appear, J.J. Moreau considers in [23] the bounded variation sweeping process

$$\begin{cases} -du \in N(C(t); u(t)) \\ u(T_0) \in C(T_0). \end{cases} \tag{1.1}$$

Over the years, many variants of Moreau sweeping process have been studied in the literature, in particular

$$\begin{cases} -du \in N(C(t); u(t)) + G(t, u(t)) \\ u(T_0) \in C(T_0), \end{cases} \tag{1.2}$$

with $G : [T_0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ a multimapping, which is called perturbed sweeping process. The case where the moving set $C(\cdot)$ is convex has been extensively developed (see, e.g., [8, 9, 1, 28] and references therein). It is of interest in infinite dimensions to remove the convexity assumption of $C(\cdot)$ as in [30, 9], where $C(\cdot) = \mathbb{R}^n \setminus \mathrm{int}K(\cdot)$ with $K(\cdot)$ a convex moving set (the normal cone involved is in the sense of Clarke).

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Due to their good properties of metric projection, prox-regular sets ([26]) are well appropriate for the study of sweeping process in the nonconvex setting in infinite dimensions (see, e.g., [12, 6, 14, 3, 2]).

In [14], J.F. Edmond and L. Thibault showed in infinite dimensional Hilbert space that (1.2) with $G = F$ and $C(\cdot)$ a prox-regular moving set with bounded variation, has at least one solution, under a compact linear growth condition for the multimapping F, that is,

$$F(t,x) \subset \beta(t)(1 + \|x\|)\, K \quad \text{for all } (t,x) \in [T_0, T] \times \mathcal{H}, \tag{1.3}$$

where $\beta(\cdot) \in L^1([T_0, T], \mathbb{R}_+)$ and $K \subset \mathbb{B}$ is a compact set. More recently, under the bounded variation of $C(\cdot)$ the well-posedness (in the sense of existence and uniqueness of a solution) of (1.2) with $G = f$ single-valued satisfying a Lipschitz type condition, was stated and proved in the general setting of Hilbert space, for a convex moving set $C(\cdot)$ in [14] and for a prox-regular valued multimapping $C(\cdot)$ in [2]. Those works lead naturally to study the sweeping process (1.2) with the perturbation $G = F + f$. The existence of solutions for such a Moreau sweeping process was established in [3], but only in the absolutely continuous framework, that is

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + F(t, u(t)) + f(t, u(t)) & \lambda\text{-a.e.} t \in [T_0, T] \\ u(T_0) \in C(T_0), \end{cases} \tag{1.4}$$

with F a multimapping scalarly upper semicontinuous satisfying (1.3), f a Lipschitz single-valued mapping, and for a prox-regular set $C(t)$ moving in an absolutely continuous way in a general Hilbert space.

The aim of the present paper is to analyze the variant of Moreau sweeping process (1.4) in the bounded variation framework, that is, the discontinuous perturbed sweeping process (1.1) with $G = F + f$, a sum of a single-valued mapping f and a multimapping F, as the perturbation of the normal cone, and $C(t)$ is prox-regular with bounded variation.

The paper is organized as follows:
Section 2 is devoted to introduce notations and recall fundamental results for the study of discontinuous sweeping process. In Sections 3-4, we develop the concept of solution of our measure differential inclusion and then we state and prove an existence result. Some consequences are provided in the last part, in particular, it is shown that there is a solution $u(\cdot)$ which satisfies (as in [23, 2, 28])

$$\mathrm{proj}_{C(t)}(u(t^-)) = u(t) \quad \text{for all } t \in]T_0, T],$$

where $u(t^-) := \lim_{\tau \uparrow t} u(\tau)$.

2. Preliminaries

Throughout, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real-line, $\mathbb{R}_+ = [0, +\infty[$ is the set of the nonnegative reals, $\mathbb{N}$ is the set of the positive integers, $n = 1, \ldots$, $I := [T_0, T]$ is an interval of $\mathbb{R}$ with $T_0 < T$ and λ is the Lebesgue measure on I. In all the paper, $\mathcal{H}$ is a real Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, the associated norm $\|\cdot\|$ and $\mathbb{B}$ the closed unit ball centered at zero. For any subset S

of $\mathcal{H}$, $\operatorname{co} S$ (resp., $\overline{\operatorname{co}}\, S$) stands for the convex (resp., closed convex) hull of S and $d_S(\cdot)$ (or $d(\cdot, S)$) is the *distance function* to S, i.e.,

$$d_S(x) := \inf_{s \in S} \|x - s\| \quad \text{for all } x \in \mathcal{H}.$$

2.1. **Nonsmooth analysis.** In this subsection, S is a nonempty susbset of the real Hilbert space $\mathcal{H}$, U is a nonempty open subset of $\mathcal{H}$ and $f : U \to \overline{\mathbb{R}}$ is a function.

For any $x \in \mathcal{H}$, the possibly empty set of all *nearest points* of x in S is defined by

$$\operatorname{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

If $\operatorname{Proj}_S(x) = \{\overline{y}\}$ for some $\overline{y} \in S$, one says that $\operatorname{proj}_S(x)$ (or $P_S(x)$) is well-defined and in such a case one sets $\operatorname{proj}_S(x) := \overline{y}$ (or $P_S(x) := \overline{y}$).

The *proximal normal cone* to S at $x \in S$ is the set

$$N^P(S; x) := \{v \in \mathcal{H} : \exists r > 0, x \in \operatorname{Proj}_S(x + rv)\},$$

which is obviously a cone containing 0. By convention, one sets

$$N^P(S; x) = \emptyset \quad \text{for all } x \in \mathcal{H} \setminus S.$$

It is readily seen that for $v \in \mathcal{H}$ such that $\operatorname{Proj}_S(v) \neq \emptyset$,

$$v - w \in N^P(S; w) \quad \text{for all } w \in \operatorname{Proj}_S(v). \tag{2.1}$$

One defines the *proximal subdifferential* $\partial_P f(x)$ of f at $x \in U$ as the set

$$\partial_P f(x) = \left\{v \in \mathcal{H} : (v, -1) \in N^P\big(\operatorname{epi} f; (x, f(x))\big)\right\}, \tag{2.2}$$

where $\operatorname{epi} f$ is the *epigraph* of f, i.e.,

$$\operatorname{epi} f := \{(u, r) \in \mathcal{H} \times \mathbb{R} : u \in U, f(u) \leq r\}$$

and where $\mathcal{H} \times \mathbb{R}$ is endowed with the usual product structure. In particular, note that $\partial_P f(x) = \emptyset$ if f is not finite at $x \in U$.

The *Clarke tangent cone* to S at $x \in S$, $T^C(S; x)$, is the set of $h \in \mathcal{H}$ such that for every sequence $(x_n)_{n \in \mathbb{N}}$ of S with $x_n \to x$, for every sequence $(t_n)_{n \in \mathbb{N}}$ of positive reals with $t_n \to 0$, there is a sequence $(h_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$ with $h_n \to h$ satisfying

$$x_n + t_n h_n \in S \quad \text{for all } n \in \mathbb{N}.$$

It is known that this set is a closed convex cone containing 0. The *Clarke normal cone* of S at $x \in S$ is denoted by $N^C(S; x)$ and defined as the polar cone of $T^C(S; x)$, i.e.,

$$N^C(S; x) := \left\{v \in \mathcal{H} : \langle v, h \rangle \leq 0, \forall h \in T^C(S; x)\right\}.$$

By convention again, one puts

$$T^C(S; x) = N^C(S; x) = \emptyset \quad \text{for all } x \in \mathcal{H} \setminus S.$$

It is not difficult to check that

$$N^P(S; x) \subset N^C(S; x) \quad \text{for all } x \in \mathcal{H}. \tag{2.3}$$

As for the proximal subdifferential, one defines the *Clarke subdifferential* $\partial_C f(x)$ of f at $x \in U$ by

$$\partial_C f(x) := \left\{ v \in \mathcal{H} : \, (v, -1) \in N^C\big(\mathrm{epi} f; (x, f(x))\big) \right\}, \tag{2.4}$$

so $\partial_C f(x) = \emptyset$ whenever f is not finite at $x \in U$. According to (2.2), (2.4) and (2.3), it is straightforward that

$$\partial_P f(x) \subset \partial_C f(x) \quad \text{for all}\, x \in U.$$

If f is γ-Lipschitz near $x \in U$ for some real $\gamma \geq 0$, it is well-known that $\partial_C f(x) \subset \gamma\mathbb{B}$. In particular, this yields

$$\partial_C d_S(y) \subset \mathbb{B} \quad \text{for all}\, y \in \mathcal{H}.$$

Furthermore, if S is closed, the following relations between the proximal (resp., Clarke) subdifferential of the distance function of S and the proximal (resp., Clarke) normal cone to S hold true for all $x \in S$ (see, e.g., [6]):

$$\partial_P d_S(x) = N^P(S; x) \cap \mathbb{B} \tag{2.5}$$

and

$$\partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}. \tag{2.6}$$

For more details, we refer the reader to [27, 17, 10].

2.2. **Prox-regular sets.** In this paper, we deal with the concept of uniform prox-regularity in the Hilbert setting, which is due to R.A. Poliquin, R.T. Rockafellar and L.Thibault ([26]). In this subsection, r is an extended real of $]0, +\infty]$. Whenever $r = +\infty$, we set by convention, $\frac{1}{r} := 0$.

Definition 2.1. Let S be a nonempty closed subset of $\mathcal{H}$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P(S; x) \cap \mathbb{B}$ and for all $t \in]0, r[$, one has $x \in \mathrm{Proj}_S(x + tv)$.

The following theorems provide some useful characterizations and properties of uniform prox-regularity (see, e.g., [11]).

Theorem 2.2. *Let S be a nonempty closed subset of $\mathcal{H}$. The following assertions are equivalent.*

(a) *The set S is r-prox-regular.*

(b) *For all $x_1, x_2 \in S$, for all $v \in N^P(S; x_1)$, one has*

$$\langle v, x_2 - x_1 \rangle \leq \frac{1}{2r} \|v\| \, \|x_1 - x_2\|^2 .$$

(c) *For all $x_1, x_2 \in S$, for all $v_1 \in N^P(S; x_1)$, for all $v_2 \in N^P(S; x_2)$, one has*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{2}\left(\frac{\|v_1\|}{r} + \frac{\|v_2\|}{r}\right) \|x_1 - x_2\|^2 .$$

Theorem 2.3. *Let S be an r-prox-regular subset of $\mathcal{H}$.*

(a) *For any $x \in S$, one has*

$$N^P(S; x) = N^C(S; x) \quad \textit{and} \quad \partial_P d_S(x) = \partial_C d_S(x).$$

(b) *For any $x \in U_r(S) := \{u \in \mathcal{H} : d_S(u) < r\}$, $\mathrm{proj}_S(x)$ is well-defined.*

(c) *The well-defined mapping* $\text{proj}_S : U_r(S) \to S$ *is locally Lipschitz on* $U_r(S)$.

As in [2], according to (a) of Theorem 2.3, we put

$$N(S;x) := N^P(S;x) = N^C(S;x) \quad \text{for all } x \in S,$$

whenever S is a uniform prox-regular set of the real Hilbert space $\mathcal{H}$.

In order to prove that our perturbed sweeping process has a solution, we need the following proposition. We refer to [14] for the proof.

Proposition 2.4. *Let* S *be an* r*-prox-regular subset of* $\mathcal{H}$, $x \in S$, $v \in \partial_P d_S(x)$. *Then, for all* $z \in \mathcal{H}$ *such that* $d_S(z) < r$, *one has*

$$\langle v, z - x\rangle \le \frac{1}{2r}\|z - x\|^2 + \frac{1}{2r}d_S^2(z) + \left(\frac{1}{r}\|z - x\| + 1\right) d_S(z).$$

The last result of this subsection deals with the nearest points of a uniformly prox-regular set (see [2] for the proof).

Proposition 2.5. *Let* S *be an* r*-prox-regular subset of* $\mathcal{H}$ *and let* $x, x' \in \mathcal{H}$. *If* $x - x' \in N(S;x')$ *and* $\|x - x'\| \le r$ *(resp.,* $\|x - x'\| < r$*) then* $x' \in \text{Proj}_S(x)$ *(resp.,* $x' = \text{proj}_S(x)$*).*

2.3. **Scalar upper semicontinuity.** For any subset S of the real Hilbert space $\mathcal{H}$, its *support function* $\sigma(\cdot, S)$ is defined by

$$\sigma(v, S) := \sup_{x\in S} \langle v, x\rangle \quad \text{for all } v \in \mathcal{H}.$$

Thanks to the Hahn-Banach separation Theorem, we know that for any two closed convex subsets S_1, S_2 of $\mathcal{H}$, one has

$$S_1 \subset S_2 \Leftrightarrow \sigma(\cdot, S_1) \le \sigma(\cdot, S_2). \tag{2.7}$$

Recall that a multimapping $F : \mathcal{T} \rightrightarrows X$ from a real Hausdorff topological space $\mathcal{T}$ to a topological space X is said to be *scalarly upper semicontinuous* whenever, for any $\xi \in X$, the extended real-valued function $\sigma(\xi, F(\cdot))$ is upper semicontinuous.

The following scalar upper semicontinuity property will be useful (see [2] for the proof).

Proposition 2.6. *Let* $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ *be a multimapping satisfying:*

(i) *there exists* $r \in]0, +\infty]$ *such that* $C(t)$ *is* r*-prox-regular for all* $t \in I$;

(ii) *there exists* μ *a positive measure on* I *such that, for all* $s_1, s_2 \in I$ *with* $s_1 \le s_2$, *for all* $y \in \mathcal{H}$,

$$d_{C(s_2)}(y) - d_{C(s_1)}(y) \le \mu(]s_1, s_2]).$$

Let $(t_n)_{n\in\mathbb{N}}$ *be a sequence of* I *converging to some* $t \in I$ *with* $t_n \ge t$ *for all* $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ *a sequence of* $\mathcal{H}$ *converging to some* $x \in C(t)$ *with* $x_n \in C(t_n)$ *for all* $n \in \mathbb{N}$. *If there exists* $N \in \mathbb{N}$ *with* $\mu(]t, t_N]) < +\infty$, *then for any* $z \in \mathcal{H}$, *one has*

$$\limsup_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n)}(x_n)) \le \sigma(z, \partial_P d_{C(t)}(x)).$$

2.4. **Differential measure and BV mappings.** For the convenience of the reader, let us recall some preliminaries about the measure theory that will be required by the main result of the paper. One can also see [13, 14, 1, 2, 28].

For a set $A \subset I$, the notation $\mathbf{1}_A$ stands for the *characteristic function* (in the sense of measure theory) of A relative to I, that is, for all $x \in I$

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let ν be a positive measure on I, $p \geq 1$ be a real. We denote by $L^p(I, \mathcal{H}, \nu)$ the real space of (classes of) Bochner measurable mappings from I to $\mathcal{H}$ for which the p-th power of their norm value is integrable with respect to the measure ν.

Let ν and $\hat{\nu}$ be two positive Radon measures on I. We recall (see, e.g., [15]) that, with $I(t,r) := I \cap [t-r, t+r]$ ($r > 0$ and $t \in I$) the limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\hat{\nu}(I(t,r))}{\nu(I(t,r))} \tag{2.8}$$

(with the convention $\frac{0}{0} = 0$) exists and is finite for ν-almost every $t \in I$. The (nonnegative Borel) function $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is called the *derivative of the measure* $\hat{\nu}$ *with respect to* ν. Moreover, the measure $\hat{\nu}$ is *absolutely continuous with respect to* ν if and only if $\hat{\nu} = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$ (i.e., $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a density relative to ν). If the latter equality holds, a mapping $u(\cdot) : I \to \mathcal{H}$ is $\hat{\nu}$-integrable on I if and only if $u(\cdot)\frac{d\hat{\nu}}{d\nu}(\cdot)$ is ν-integrable on I. In such a case, one has

$$\int_I u(t) d\hat{\nu}(t) = \int_I u(t) \frac{d\hat{\nu}}{d\nu}(t) d\nu(t). \tag{2.9}$$

If the two Radon measures ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other one, one says that ν and $\hat{\nu}$ are *absolutely continuously equivalent.*

It is worth pointing out that the relation (2.8) gives

$$\frac{d\lambda}{d\nu}(t) = \frac{\lambda(\{t\})}{\nu(\{t\})} = 0 \quad \text{for all } t \in I \text{ with } \nu(\{t\}) > 0,$$

hence

$$\frac{d\lambda}{d\nu}(t)\nu(\{t\}) = 0 \quad \nu\text{-a.e. } t \in I. \tag{2.10}$$

Let $u : [T_0, T] \to \mathcal{H}$ be a mapping. Any finite sequence $\sigma = (t_0, \dots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that $T_0 = t_0 < \cdots < t_k = T$ is called a *subdivision* σ of $[T_0, T]$. One associates to such a subdivision σ, the real $S_\sigma := \sum_{i=1}^k \|u(t_i) - u(t_{i-1})\|$. The *variation* of u on $[T_0, T]$ is defined as the extended real

$$V(u; T_0, T) := \sup_{\zeta \in \mathcal{S}} S_\zeta,$$

where $\mathcal{S}$ is the set of all subdivisions of $[T_0, T]$. The mapping u is said to be of *bounded variation on* $[T_0, T]$ if $V(u; T_0, T) < +\infty$.

It is well-known that $u(\cdot)$ has one sided limits at each point of I whenever it is of bounded variation on I. In such a case, one defines

$$u(\tau^-) := \lim_{t\uparrow\tau} u(t) \quad \text{for all } \tau \in]T_0, T],$$

where in the whole paper, $t \uparrow \tau$ means $t \to \tau$ with $t < \tau$.

Let $u(\cdot) : I \to \mathcal{H}$ be a mapping of bounded variation and right continuous on $I = [T_0, T]$. Then, there exists a vector measure du on I with values in $\mathcal{H}$ associated with $u(\cdot)$ (see N. Dinculeanu [13] and J.J. Moreau [18]). This measure is called the *differential measure* (or the *Stieltjes measure*) of $u(\cdot)$ and it satisfies for all $s, t \in I$ with $s \le t$,

$$u(t) = u(s) + \int_{]s,t]} du.$$

Now, consider ν a positive Radon measure on I, $u(\cdot) : I \to \mathcal{H}$ a mapping and $\tilde{u}(\cdot) \in L^1(I, \mathcal{H}, \nu)$. If, for any $t \in I$,

$$u(t) = u(T_0) + \int_{]T_0,t]} \tilde{u}\, d\nu,$$

then $u(\cdot)$ is of bounded variation, right continuous on I and

$$du = \tilde{u}\, d\nu.$$

In such a case, the mapping $\tilde{u}(\cdot)$ is said to be a *density of the measure du relative to* ν. According to J.J. Moreau and M.Valadier ([25]), for ν-almost every $t \in I$,

$$\tilde{u}(t) = \frac{du}{d\nu}(t) := \lim_{r\downarrow 0} \frac{du(I(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{du(I^+(t,r))}{\nu(I^+(t,r))} = \lim_{r\downarrow 0} \frac{du(I^-(t,r))}{\nu(I^-(t,r))},$$

where $I^-(t,r) = [t-r, t] \cap I$ and $I^+(t,r) = [t, t+r] \cap I$ for each $t \in I$ and each real $r > 0$. It follows from this

$$\frac{du}{d\nu}(t) = \lim_{s\uparrow t} \frac{du(]s,t] \cap I)}{\nu(]s,t] \cap I)} \quad \nu\text{-a.e. } t \in I. \tag{2.11}$$

The following proposition, due to J.J. Moreau ([18]), is fundamental in the paper.

Proposition 2.7. *Let ν be a positive Radon measure on $I = [T_0, T]$, $u(\cdot) : I \to \mathcal{H}$ be a right continuous mapping of bounded variation such that the differential measure du has a density $\frac{du}{d\nu}$ relative to ν. Then, the function $\Phi(\cdot) = \|u(\cdot)\|^2 : I \to \mathbb{R}$ is a right continuous function of bounded variation whose differential measure $d\Phi$ satisfies, in the sense of the ordering of real measures,*

$$d\Phi \le 2 \left\langle u(\cdot), \frac{du}{d\nu}(\cdot) \right\rangle d\nu.$$

The last result of this section is a variant of Gronwall Lemma which is due to M.D.P. Monteiro Marques ([16]).

Lemma 2.8. *Let ν be a positive Radon measure on $[T_0, T]$, $g, \varphi : [T_0, T] \to \mathbb{R}_+$ two functions such that:*

(*i*) *For some fixed* $\theta \in \mathbb{R}_+$*, one has, for all* $t \in]T_0, T]$*,*

$$0 \leq g(t)\nu(\{t\}) \leq \theta < 1$$

and $g \in L^1([T_0, T], \mathbb{R}_+, \nu)$*.*

(*ii*) *For some fixed* $\alpha \in \mathbb{R}_+$*, one has, for all* $t \in [T_0, T]$*,*

$$\varphi(t) \leq \alpha + \int_{]T_0,t]} g(s)\varphi(s)d\nu(s)$$

and $\varphi \in L^\infty([T_0, T], \mathbb{R}_+, \nu)$*.*

Then, one has

$$\varphi(t) \leq \alpha \exp\Big(\frac{1}{1-\theta}\int_{]T_0,t]} g(s)d\nu(s)\Big) \quad \textit{for all}\, t \in [T_0, T].$$

3. Concept of solution

Following [14, 1, 2, 28], one defines the concept of solution for our measure differential inclusion.

Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping, $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping, $C : I \rightrightarrows \mathcal{H}$ be a r-prox-regular valued multimapping for some extended real $r \in]0, +\infty]$. Assume that there exists a finite positive Radon measure μ on I such that

$$|d(y, C(s)) - d(y, C(t))| \leq \mu(]s, t]) \quad \text{for all } y \in \mathcal{H}, \text{ for all } s, t \in I \text{ with } s \leq t.$$

Given $u_0 \in C(T_0)$, a mapping $u : I \to \mathcal{H}$ is a solution of the measure differential inclusion

$$(\mathcal{P}) \begin{cases} -du \in N(C(t); u(t)) + F(t, u(t)) + f(t, u(t)) \\ u(T_0) = u_0, \end{cases}$$

whenever:

(*a*) the mapping $u(\cdot)$ is of bounded variation on I, right continuous on I and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;

(*b*) there exist a λ-integrable mapping $z(\cdot) : I \to \mathcal{H}$ with $z(t) \in F(t, u(t))$ for λ-almost every $t \in I$ and a positive Radon measure ν on I, absolutely continuously equivalent to $\lambda + \mu$ and with respect to which the differential measure du of u is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1(I, \mathcal{H}, \nu)$-density and

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \nu\text{-a.e. } t \in I. \tag{3.1}$$

As in [14, 1, 2, 28], the concept of solution does not depend on the measure ν in the sense that a mapping $u(\cdot) : I \to \mathcal{H}$ satisfying (*a*) above is a solution of $(\mathcal{P})$ if and only if (3.1) holds for any positive Radon measure ν which is absolutely continuously equivalent to $\lambda + \mu$. Indeed, let $u(\cdot) : I \to \mathcal{H}$ be a solution of $(\mathcal{P})$ and let ν_1, given by the definition of a solution to $(\mathcal{P})$ be an associated Radon measure absolutely continuous equivalent to $\lambda + \mu$ for which

$$\frac{du}{d\nu_1}(t) + z(t)\frac{d\lambda}{d\nu_1}(t) + f(t, u(t))\frac{d\lambda}{d\nu_1}(t) \in -N(C(t); u(t)) \quad \nu_1\text{-a.e. } t \in I. \tag{3.2}$$

Fix any other Radon measure ν_2 absolutely continuously equivalent to $\lambda + \mu$. Then, the measures ν_1 and ν_2 are absolutely continuously equivalent. Consequently, $\frac{d\nu_1}{d\nu_2}(\cdot)$

and $\frac{d\nu_2}{d\nu_1}(\cdot)$ exist as densities and for $\frac{du}{d\nu_2}(\cdot)$ and the derivative $\frac{d\lambda}{d\nu_2}(\cdot)$ the following equalities hold

$$\frac{du}{d\nu_2}(t) = \frac{du}{d\nu_1}(t)\frac{d\nu_1}{d\nu_2}(t), \quad \frac{d\lambda}{d\nu_2}(t) = \frac{d\lambda}{d\nu_1}(t)\frac{d\nu_1}{d\nu_2}(t) \quad \nu_2\text{-a.e. } t \in I.$$

This yields according to (3.2)

$$\frac{du}{d\nu_2}(t) + z(t)\frac{d\lambda}{d\nu_2}(t) + f(t,u(t))\frac{d\lambda}{d\nu_2}(t) \in -N(C(t);u(t)) \quad \nu_2\text{-a.e. } t \in I.$$

4. Existence result

In this section, we prove under assumptions on f, F, C and μ that $(\mathcal{P})$ has at least one solution.

Theorem 4.1. *Let $C(\cdot) : I \rightrightarrows \mathcal{H}$ be an r-prox-regular valued multimapping for some extended real $r \in]0,+\infty]$, for which there exists a finite positive Radon measure on I with $\sup_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$ such that for all $y \in \mathcal{H}$, for all $s,t \in I$ with $s < t$,*

$$|d(y,C(t)) - d(y,C(s))| \le \mu(]s,t]). \tag{4.1}$$

Let $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping with nonempty convex compact values such that:

(i) $F(\cdot,\cdot)$ is scalarly upper-semicontinuous.

(ii) There exist some compact set $K \subset \mathbb{B}$ and a function $\beta : I \to \mathbb{R}_+$ with $\beta(\cdot) \in L^1(I,\mathbb{R},\lambda)$ such that

$$F(t,x) \subset \beta(t)(1+\|x\|)K \quad \text{for all } t \in I, \text{ for all } x \in \bigcup_{s\in I} C(s).$$

Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping such that:

(iii) For each real $s > 0$, there exists a function $L_s : I \to \mathbb{R}_+$ with $L_s \in L^1(I,\mathbb{R},\lambda)$ such that

$$\|f(t,x) - f(t,y)\| \le L_s(t)\,\|x-y\| \quad \text{for all } t \in I, \text{ for all } x,y \in s\mathbb{B}.$$

(iv) $f(\cdot,x)$ is Lebesgue measurable for all $x \in \mathcal{H}$ and there exists a function $\alpha : I \to \mathbb{R}_+$ with $\alpha \in L^1(I,\mathbb{R},\lambda)$ such that

$$\|f(t,x)\| \le \alpha(t)(1+\|x\|) \quad \text{for all } t \in I, \text{ for all } x \in \bigcup_{s\in I} C(s).$$

Then, for each $u_0 \in C(T_0)$, the following perturbed sweeping process

$$\begin{cases} -du \in N(C(t);u(t)) + F(t,u(t)) + f(t,u(t)) \\ u(T_0) = u_0 \end{cases}$$

has at least one solution satisfying

$$\left\|u(t) - u(t^-)\right\| \le 2\mu(\{t\}) \quad \text{for all } t \in]T_0,T].$$

Proof. Fix any $u_0 \in C(T_0)$. We denote by $(\mathcal{P})$ the perturbed sweeping process

$$(\mathcal{P}) \begin{cases} -du \in N(C(t);u(t)) + F(t,u(t)) + f(t,u(t)) \\ u(T_0) = u_0. \end{cases}$$

As in [14, 3], we suppose, without loss of generality, that K is convex and contains 0 (if not so, we may replace it by $\overline{\text{co}}(K \cup \{0\})$).

Case 1: Assume that

$$\int_{T_0}^{T} (\beta(s)+1)d\lambda(s) \le \frac{1}{8} \quad \text{and} \quad \int_{T_0}^{T} \alpha(s)d\lambda(s) \le \frac{1}{8} \tag{4.2}$$

Set

$$l = 2\Big(\mu(]T_0,T]) + \|u_0\| + \frac{3}{2}\Big) \tag{4.3}$$

and define the positive Radon measure on I

$$\nu = \mu + l(\beta(\cdot)+1+\alpha(\cdot))\lambda. \tag{4.4}$$

Let us consider the function $v(\cdot) : I \to \mathbb{R}$ defined by

$$v(t) = \nu(]T_0,t]) \quad \text{for all}\, t \in I$$

and set

$$V = v(T) = \nu(]T_0,T]).$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $\varepsilon_n \downarrow 0$. Following J.J. Moreau in [23], choose for each $n \in \mathbb{N}$, $0 = V_0^n < V_1^n < \ldots < V_{q_n}^n = V$ (with $q_n \in \mathbb{N}$) such that

(a) for all $j \in \{0,\ldots,q_n-1\}$, $V_{j+1}^n - V_j^n \le \varepsilon_n$;

(b) for all $k \in \mathbb{N}$, $\{V_0^k,\ldots,V_{q_k}^k\} \subset \{V_0^{k+1},\ldots,V_{q_{k+1}}^{k+1}\}$.

For each $n \in \mathbb{N}$, set $V_{1+q_n}^n := V + \varepsilon_n$ and consider the partition $(J_j^n)_{j\in\{0,\ldots,q_n-1\}}$ of I where for each $j \in \{0,\ldots,q_n-1\}$

$$J_j^n := v^{-1}(\big[V_j^n, V_{j+1}^n\big[) = \{t \in I : V_j^n \le \nu(]T_0,t]) < V_{j+1}^n\}.$$

Note that $(J_j^m)_{0\le j\le q_m}$ is a refinement of $(J_j^n)_{0\le j\le q_n}$ for all $m,n \in \mathbb{N}$ with $m \ge n$. Since $v(\cdot)$ is nondecreasing and right continuous on I, it is easy to see that, for each $n \in \mathbb{N}$, $j \in \{0,\ldots,q_n-1\}$, the set J_j^n is either empty or an interval of the form $[a,b[$ with $a<b$. Furthermore, we have $J_{q_n}^n = \{T\}$ for all $n \in \mathbb{N}$. This gives for each $n \in \mathbb{N}$ an integer $p(n) \in \mathbb{N}$ and a finite sequence

$$T_0 = t_0^n < \ldots < t_{p(n)}^n = T$$

such that for each $i \in \{0,\ldots,p(n)-1\}$, there is some $j \in \{0,\ldots,q_n-1\}$ satisfying $J_j^n = \big[t_i^n, t_{i+1}^n\big[$. Observe that $(p(n))_{n\in\mathbb{N}}$ is an nondecreasing sequence. For all $n \in \mathbb{N}$, put $E_n = \{0,\ldots,p(n)-1\}$. Fix for a moment any $n \in \mathbb{N}$. For each $i \in E_n$, put

$$\eta_i^n = t_{i+1}^n - t_i^n, \quad \alpha_i^n = \int_{t_i^n}^{t_{i+1}^n} \alpha(s)d\lambda(s) \quad \text{and} \quad \beta_i^n = \int_{t_i^n}^{t_{i+1}^n} (\beta(s)+1)d\lambda(s).$$

Set also,

$$\Delta_n = \max_{i\in E_n}(t_{i+1}^n - t_i^n) \quad \text{and} \quad \xi_n = \max_{i\in E_n}(\beta_i^n + \alpha_i^n).$$

For all $i \in E_n$ and $t \in \big[t_i^n, t_{i+1}^n\big[$, one has

$$\nu(]t_i^n,t]) = v(t) - v(t_i^n) \le \varepsilon_n,$$

so

$$\mu(\big]t_i^n, t_{i+1}^n\big[) \le \nu(\big]t_i^n, t_{i+1}^n\big[) \le \varepsilon_n. \tag{4.5}$$

Hence (since $\lambda \le \nu$), one has

$$\eta_i^n = t_{i+1}^n - t_i^n \le \varepsilon_n \quad \text{for all } i \in E_n. \tag{4.6}$$

As a consequence, we observe that $\lim_{k\to+\infty} \Delta_k = 0$ and hence, $\lim_{k\to+\infty} \xi_k = 0$. Fix any $n_0 \in \mathbb{N}$ such that for all integers $n \ge n_0$,

$$\xi_n + \varepsilon_n < \frac{r}{2l}. \tag{4.7}$$

Fix any integer $n \ge n_0$. Let us define $(s_i^n)_{0\le i\le p(n)-1}$ as follows. If $p(n) = 1$, choose $s_0^n \in [T_0, T]$ such that

$$\beta(s_0^n) \le \inf_{s\in[T_0,T]} \beta(s) + 1.$$

If $p(n) > 1$, choose (as in [14]) for each $i \in \{0, \dots, p(n)-2\}$ some $s_i^n \in [t_i^n, t_{i+1}^n[$ satisfying

$$\beta(s_i^n) \le \inf_{s\in[t_i^n, t_{i+1}^n[} \beta(s) + 1,$$

and some $s_{p(n)-1}^n \in [t_{p(n)-1}^n, t_{p(n)}^n]$ such that

$$\beta(s_{p(n)-1}^n) \le \inf_{s\in[t_{p(n)-1}^n, t_{p(n)}^n]} \beta(s) + 1.$$

Let us define $\kappa_n(\cdot) : I \to I$ by

$$\kappa_n(t) = \begin{cases} s_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in E_n, \\ s_{p(n)-1}^n & \text{if } t = T. \end{cases}$$

For each $(t, x) \in I \times \mathcal{H}$, choose (thanks to the fact that F takes nonempty values) $\zeta(t, x) \in F(t, x)$. Let us set $u_0^n = u_0$ and, as in [3], let us construct by induction a sequence $(u_k^n)_{0\le k\le p(n)}$ such that, for all $k \in \{1, \dots, p(n)\}$,

$$1 + \|u_{k-1}^n\| < l,$$

$$\begin{aligned} &d_{C(t_k^n)}\Big(u_{k-1}^n - \eta_{k-1}^n \zeta(\kappa_n(t_{k-1}^n), u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(s, u_{k-1}^n) d\lambda(s)\Big) \\ &\le \mu(]t_{k-1}^n, t_k^n]) + l\beta_{k-1}^n + l\alpha_{k-1}^n < r, \end{aligned}$$

and

$$u_k^n := \mathrm{P}_{C(t_k^n)}\Big(u_{k-1}^n - \eta_{k-1}^n \zeta(\kappa_n(t_{k-1}^n), u_{k-1}^n) - \int_{t_{k-1}^n}^{t_k^n} f(s, u_{k-1}^n) d\lambda(s)\Big).$$

Step 1: Construction of the finite sequence.
It is obvious that $1+\|u_0\| < l$. Using (ii), the inclusion $\zeta(\kappa_n(t_0^n), u_0^n) \in F(\kappa_n(t_0^n), u_0^n)$, the latter inequality and the fact $u_0^n = u_0 \in C(t_0^n)$, we get

$$\|\zeta(\kappa_n(t_0^n), u_0^n)\| \le \beta(\kappa_n(t_0^n))(1 + \|u_0\|) \le l\beta(\kappa_n(t_0^n)).$$

This ensures, thanks to the equality $s_0^n = \kappa_n(t_0^n)$,

$$\begin{aligned}\eta_0^n \left\|\zeta(\kappa_n(t_0^n), u_0^n)\right\| &\leq l \int_{t_0^n}^{t_1^n} \beta(\kappa_n(t_0^n)) d\lambda(s) \\ &\leq l \int_{t_0^n}^{t_1^n} (\beta(s)+1) d\lambda(s) \\ &= l\beta_0^n. \end{aligned} \tag{4.8}$$

By (iv) and the inclusion $u_0^n \in C(t_0^n)$, we have

$$\|f(s, u_0^n)\| \leq \alpha(s)(1 + \|u_0\|) \leq l\alpha(s) \quad \text{for all } s \in I.$$

Hence, we obtain

$$\begin{aligned}\left\| \int_{t_0^n}^{t_1^n} f(s, u_0^n) d\lambda(s) \right\| &\leq \int_{t_0^n}^{t_1^n} \|f(s, u_0^n)\| \, d\lambda(s) \\ &\leq l \int_{t_0^n}^{t_1^n} \alpha(s) d\lambda(s) \\ &= l\alpha_0^n. \end{aligned} \tag{4.9}$$

According to the assumption on the variation of $C(\cdot)$ in (4.1), (4.5), the fact that $u_0^n \in C(t_0^n)$, (4.8), (4.9), the definition of ξ_n, the inequality $l \geq 1$ and (4.7), one has

$$\begin{aligned} & d_{C(t_1^n)}\Big(u_0^n - \eta_0^n \zeta(\kappa_n(t_0^n), u_0^n) - \int_{t_0^n}^{t_1^n} f(s, u_0^n) d\lambda(s)\Big) \\ \leq & \mu(]T_0, t_1^n]) + d_{C(t_0^n)}\Big(u_0^n - \eta_0^n \zeta(\kappa_n(t_0^n), u_0^n) - \int_{t_0^n}^{t_1^n} f(s, u_0^n) d\lambda(s)\Big) \\ \leq & \varepsilon_n + \eta_0^n \left\|\zeta(\kappa_n(t_0^n), u_0^n)\right\| + \left\| \int_{t_0^n}^{t_1^n} f(s, u_0^n) d\lambda(s) \right\| + \mu(\{t_1^n\}) \\ \leq & \varepsilon_n + l\beta_0^n + l\alpha_0^n + \frac{r}{2} \\ \leq & \varepsilon_n + l\xi_n + \frac{r}{2} \\ \leq & l(\varepsilon_n + \xi_n) + \frac{r}{2} \\ < & r. \end{aligned}$$

Since $C(t_1^n)$ is r-prox-regular,

$$u_1^n := \mathrm{P}_{C(t_1^n)}\Big(u_0^n - \eta_0^n \zeta(\kappa_n(t_0^n), u_0^n) - \int_{t_0^n}^{t_1^n} f(s, u_0^n) d\lambda(s)\Big),$$

is well-defined according to Theorem 2.3. Now, assume that $p(n) > 1$ (otherwise, the induction is complete). Fix any $k \in \{1, \ldots, p(n)-1\}$. Suppose that all the steps of the induction from 1 to k have been realized. Let $q \in \{0, \ldots, k-1\}$. By the equality $u_{q+1}^n = \mathrm{P}_{C(t_{q+1}^n)}\Big(u_q^n - \eta_q^n \zeta(\kappa_n(t_q^n), u_q^n) - \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s)\Big)$, the variation assumption on $C(\cdot)$ in (4.1) and the inclusion $u_q^n \in C(t_q^n)$, we have

$$\begin{aligned}
&\left\| u_{q+1}^n - u_q^n + \eta_q^n \zeta(\kappa_n(t_q^n), u_q^n) + \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s) \right\| \\
=& d_{C(t_{q+1}^n)}\Big(u_q^n - \eta_q^n \zeta(\kappa_n(t_q^n), u_q^n) - \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s) \Big) \\
\leq& \mu(]t_q^n, t_{q+1}^n]) + d_{C(t_q^n)}\Big(u_q^n - \eta_q^n \zeta(\kappa_n(t_q^n), u_q^n) - \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s) \Big) \\
\leq& \mu(]t_q^n, t_{q+1}^n]) + \eta_q^n \left\| \zeta(\kappa_n(t_q^n), u_q^n) \right\| + \left\| \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s) \right\|,
\end{aligned}$$

and then

$$\begin{aligned}
\left\| u_{q+1}^n \right\| \leq& \left\| u_q^n \right\| + \mu(]t_q^n, t_{q+1}^n]) \\
&+ 2\eta_q^n \left\| \zeta(\kappa_n(t_q^n), u_q^n) \right\| + 2 \left\| \int_{t_q^n}^{t_{q+1}^n} f(s, u_q^n) d\lambda(s) \right\|.
\end{aligned}$$

From this inequality, we deduce

$$\begin{aligned}
\left\| u_{q+1}^n \right\| \leq& \| u_0^n \| + \sum_{p=0}^{q} \mu(]t_p^n, t_{p+1}^n]) + 2 \sum_{p=0}^{q} \eta_p^n \left\| \zeta(\kappa_n(t_p^n), u_p^n) \right\| \\
&+ 2 \sum_{p=0}^{q} \left\| \int_{t_p^n}^{t_{p+1}^n} f(s, u_p^n) d\lambda(s) \right\|. \qquad (4.10)
\end{aligned}$$

For all $p \in \{0, \ldots, q\}$, we have by (ii)

$$\left\| \zeta(\kappa_n(t_p^n), u_p^n) \right\| \leq \beta(\kappa_n(t_p^n))(1 + \left\| u_p^n \right\|) \leq \beta(\kappa_n(t_p^n))(1 + \max_{0 \leq i \leq q} \| u_i^n \|), \qquad (4.11)$$

and by (iv)

$$\left\| f(s, u_p^n) \right\| \leq \alpha(s)(1 + \left\| u_p^n \right\|) \leq \alpha(s)(1 + \max_{0 \leq i \leq q} \| u_i^n \|) \quad \text{for all } s \in I. \qquad (4.12)$$

It follows from (4.10), (4.11) and (4.12) that

$$\begin{aligned}
\left\| u_{q+1}^n \right\| \leq& \| u_0^n \| + \sum_{p=0}^{q} \mu(]t_p^n, t_{p+1}^n]) \\
&+ 2(1 + \max_{0 \leq i \leq q} \| u_i^n \|) \Big(\sum_{p=0}^{q} \eta_p^n \beta(\kappa_n(t_p^n)) + \sum_{p=0}^{q} \int_{t_p^n}^{t_{p+1}^n} \alpha(s) d\lambda(s) \Big).
\end{aligned}$$

Since $q < k$ and $\sum_{p=0}^{q} \mu(]t_p^n, t_{p+1}^n]) \leq \mu(]T_0, T])$, we have

$$\begin{aligned}
\left\| u_{q+1}^n \right\| \leq& \| u_0^n \| + \mu(]T_0, T]) \\
&+ 2(1 + \max_{0 \leq i \leq k} \| u_i^n \|) \Big(\sum_{p=0}^{q} \eta_p^n \beta(\kappa_n(t_p^n)) + \sum_{p=0}^{q} \int_{t_p^n}^{t_{p+1}^n} \alpha(s) d\lambda(s) \Big).
\end{aligned}$$

As

$$\sum_{p=0}^{q} \eta_p^n \beta(\kappa_n(t_p^n)) = \sum_{p=0}^{q} \int_{t_p^n}^{t_{p+1}^n} \beta(\kappa_n(t_p^n)) ds \leq \int_{T_0}^{T} (\beta(s)+1) d\lambda(s),$$

and

$$\sum_{p=0}^{q} \int_{t_p^n}^{t_{p+1}^n} \alpha(s) d\lambda(s) \leq \int_{T_0}^{T} \alpha(s) d\lambda(s),$$

we get

$$\begin{aligned} \left\| u_{q+1}^n \right\| &\leq \|u_0^n\| + \mu(]T_0,T]) \\ &\quad + 2(1 + \max_{0\leq i\leq k} \|u_i^n\|) \left(\int_{T_0}^{T} (\beta(s)+1) d\lambda(s) + \int_{T_0}^{T} \alpha(s) d\lambda(s) \right). \end{aligned}$$

Combining this and (4.2), it follows

$$\max_{0\leq i\leq k} \|u_i^n\| \leq \|u_0^n\| + \mu(]T_0,T]) + 2(1 + \max_{0\leq i\leq k} \|u_i^n\|)\Big(\frac{1}{8} + \frac{1}{8}\Big).$$

Consequently,

$$\max_{0\leq i\leq k} \|u_i^n\| \leq 2\Big(\|u_0\| + \mu(]T_0,T]) + \frac{1}{2} \Big) = l - 2,$$

the equality being due to the definition of l in (4.3). In particular, we have

$$1 + \|u_k^n\| < l.$$

By (ii), we get

$$\begin{aligned} \eta_k^n \|\zeta(\kappa_n(t_k^n), u_k^n)\| &\leq \eta_k^n \beta(\kappa_n(t_k^n))(1 + \|u_k^n\|) \\ &\leq (1 + \|u_k^n\|) \int_{t_k^n}^{t_{k+1}^n} \beta(\kappa_n(t_k^n)) d\lambda(s) \\ &\leq l \int_{t_k^n}^{t_{k+1}^n} (\beta(s)+1) d\lambda(s) \\ &\leq l\beta_k^n, \end{aligned} \tag{4.13}$$

and by (iv)

$$\left\| \int_{t_k^n}^{t_{k+1}^n} f(s, u_k^n) d\lambda(s) \right\| \leq (1 + \|u_k^n\|) \int_{t_k^n}^{t_{k+1}^n} \alpha(s) d\lambda(s) \leq l\alpha_k^n. \tag{4.14}$$

According to the variation assumption on $C(\cdot)$ in (4.1), (4.5), the inclusion $u_k^n \in C(t_k^n)$, (4.13), (4.14), the definition of ξ_n, the inequality $l \geq 1$ and (4.7), we have

$$\begin{aligned}
&d_{C(t_{k+1}^n)}\Big(u_k^n - \eta_k^n \zeta(\kappa_n(t_k^n), u_k^n) - \int_{t_k^n}^{t_{k+1}^n} f(s, u_k^n) d\lambda(s)\Big) \\
\leq & \mu(]t_k^n, t_{k+1}^n]) + d_{C(t_k^n)}\Big(u_k^n - \eta_k^n \zeta(\kappa_n(t_k^n), u_k^n) - \int_{t_k^n}^{t_{k+1}^n} f(s, u_k^n) d\lambda(s)\Big) \\
\leq & \varepsilon_n + \eta_k^n \left\| \zeta(\kappa_n(t_k^n), u_k^n) \right\| + \left\| \int_{t_k^n}^{t_{k+1}^n} f(s, u_k^n) d\lambda(s) \right\| + \mu(\{t_{k+1}^n\}) \\
\leq & \varepsilon_n + l\beta_k^n + l\alpha_k^n + \frac{r}{2} \\
\leq & l(\varepsilon_n + \xi_n) + \frac{r}{2} \\
< & r.
\end{aligned}$$

Since $C(t_{k+1}^n)$ is r-prox-regular

$$u_{k+1}^n := \mathrm{P}_{C(t_{k+1}^n)}\Big(u_k^n - \eta_k^n \zeta(\kappa_n(t_k^n), u_k^n) - \int_{t_k^n}^{t_{k+1}^n} f(s, u_k^n) d\lambda(s)\Big),$$

is well-defined and this completes the induction. Let us define

$$z_i^n := \zeta(\kappa_n(t_i^n), u_i^n) \quad \text{for all } i \in \{0, \ldots, p(n) - 1\}.$$

With this definition and thanks to the latter induction, we have for all $i \in \{0, \ldots, p(n) - 1\}$,

$$z_i^n \in F(\kappa_n(t_i^n), u_i^n), \tag{4.15}$$

$$\begin{aligned}
d_{C(t_{i+1}^n)}\Big(u_i^n - \eta_i^n z_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) d\lambda(s)\Big) &\leq \mu(]t_i^n, t_{i+1}^n]) + l\beta_i^n + l\alpha_i^n \\
&< r,
\end{aligned} \tag{4.16}$$

$$u_{i+1}^n = \mathrm{P}_{C(t_{i+1}^n)}\Big(u_i^n - \eta_i^n z_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) d\lambda(s)\Big) \tag{4.17}$$

and

$$1 + \|u_i^n\| < l. \tag{4.18}$$

Note that by (4.17), (4.16) and (4.4) for any $i \in \{0, \ldots, p(n) - 1\}$,

$$\begin{aligned}
\left\| u_{i+1}^n - u_i^n + \eta_i^n z_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) d\lambda(s) \right\| &\leq \mu(]t_i^n, t_{i+1}^n]) + l\beta_i^n + l\alpha_i^n \\
&\leq \nu(]t_i^n, t_{i+1}^n]).
\end{aligned} \tag{4.19}$$

Fix any $i \in \{0, \ldots, p(n) - 1\}$. By (4.15) and (ii), we have

$$z_i^n \in \beta(\kappa_n(t_i^n))(1 + \|u_i^n\|)K$$

and this entails

$$\frac{l}{(1+\|u_i^n\|)} z_i^n \in l\beta(\kappa_n(t_i^n))K.$$

Using the fact that K is a convex set containing 0 and the inequality $(1+\|u_i^n\|) < l$, we see that

$$\frac{1+\|u_i^n\|}{l}\Big(\frac{l}{1+\|u_i^n\|} z_i^n\Big) + \Big(1 - \frac{1+\|u_i^n\|}{l}\Big)0 \in l\beta(\kappa_n(t_i^n))K,$$

that is,

$$z_i^n \in l\beta(\kappa_n(t_i^n))K. \tag{4.20}$$

Hence,

$$\|z_i^n\| \le l\beta(\kappa_n(t_i^n)) \quad \text{for all } i \in \{0,\dots,p(n)-1\}, \tag{4.21}$$

according to the inclusion $K \subset \mathbb{B}$.

Step 2: Definition of the sequence $(u_n(\cdot))_{n\ge n_0}$.

Fix any integer $n \ge n_0$. Let us define $z_n(\cdot), u_n(\cdot) : I \to \mathcal{H}$ by

$$z_n(t) = \begin{cases} z_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in E_n, \\ z_{p(n)-1}^n & \text{if } t = T \end{cases}$$

and

$$u_n(t) = u_i^n + \frac{\nu(]t_i^n, t])}{\nu(]t_i^n, t_{i+1}^n])}\Big(u_{i+1}^n - u_i^n + \eta_i^n z_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n)d\lambda(s)\Big) - (t - t_i^n)z_i^n - \int_{t_i^n}^{t} f(s, u_i^n)d\lambda(s)$$

where $i \in \{0,\dots,p(n)-1\}$ such that $t \in [t_i^n, t_{i+1}^n]$. Observe that $u_n(\cdot)$ is right continuous and of bounded variation on each $[t_i^n, t_{i+1}^n]$. Hence it is right continuous and of bounded variation on the whole interval I. Set for all $t \in I$,

$$\Pi_n(t) = \sum_{i=0}^{p(n)-1} \frac{u_{i+1}^n - u_i^n + \eta_i^n z_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n)d\lambda(s)}{\nu(]t_i^n, t_{i+1}^n])} \mathbf{1}_{]t_i^n, t_{i+1}^n]}(t).$$

Define $\delta_n : I \to \mathcal{H}$ by

$$\delta_n(t) = \begin{cases} t_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in E_n, \\ t_{p(n)-1}^n & \text{if } t = T. \end{cases}$$

Using the definition of $u_n(\cdot)$, $\Pi_n(\cdot)$ and $\delta_n(\cdot)$, we get for all $t \in I$

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \Pi_n(s)d\nu(s) - \int_{]T_0,t]} \Big(z_n(s) + f\big(s, u_n(\delta_n(s))\big)\Big)d\lambda(s).$$

Since λ is absolutely continuous with respect to ν, it has $\frac{d\lambda}{d\nu}$ as a density in $L^\infty(I, \mathbb{R}_+, \nu)$ relative to ν and then by (2.9), we have for all $t \in I$

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \Big(\Pi_n(s) - z_n(s)\frac{d\lambda}{d\nu}(s) - f\big(s, u_n(\delta_n(s))\big)\frac{d\lambda}{d\nu}(s)\Big)d\nu(s).$$

This tells us that the vector measure du_n has $\Pi_n(\cdot) - z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot) - f(\cdot, u_n(\delta_n(\cdot)))\frac{d\lambda}{d\nu}(\cdot)$ (that is, the latter integrand) as a density in $L^\infty(I, \mathcal{H}, \nu)$ relative to ν. Consequently, the derivative $\frac{du_n}{d\nu}(\cdot)$ is a density of du_n relative to ν and

$$\frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) + f(t, u_n(\delta_n(t)))\frac{d\lambda}{d\nu}(t) = \Pi_n(t) \quad \nu\text{-a.e. } t \in I. \tag{4.22}$$

By (4.19), for ν-almost every $t \in I$, we have

$$\begin{aligned} \left\| \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) + f(t, u_n(\delta_n(t)))\frac{d\lambda}{d\nu}(t) \right\| &= \|\Pi_n(t)\| \\ &\leq 1. \end{aligned} \tag{4.23}$$

On the other hand, by (4.4), the measure $l(\beta(\cdot)+1+\alpha(\cdot))\lambda$ is absolutely continuous with respect to ν, thus it has $\frac{d(l(\beta(\cdot)+1+\alpha(\cdot))}{d\nu}$ as a density relative to ν. Hence, for ν-almost every $t \in I$, we have

$$0 \leq l(\beta(t) + 1 + \alpha(t))\frac{d\lambda}{d\nu}(t) = \frac{d(l(\beta(\cdot) + \alpha(\cdot) + 1)\lambda)}{d\nu}(t) \leq 1. \tag{4.24}$$

Using (4.21), we get

$$\|z_n(t)\| \leq l(\beta(t) + 1) \quad \text{for all } t \in I. \tag{4.25}$$

From the assumption (iv) and (4.18), we deduce

$$\|f(t, u_n(\delta_n(t)))\| \leq l\alpha(t) \quad \text{for all } t \in I. \tag{4.26}$$

Thanks to (4.25) and (4.24), we have

$$\left\| z_n(t)\frac{d\lambda}{d\nu}(t) \right\| \leq l(\beta(t) + 1)\frac{d\lambda}{d\nu}(t) \leq 1 \quad \nu\text{-a.e. } t \in I. \tag{4.27}$$

Using (4.25), (4.26) and (4.24), we obtain

$$\left\| z_n(t)\frac{d\lambda}{d\nu}(t) + f(t, u_n(\delta_n(t)))\frac{d\lambda}{d\nu}(t) \right\| \leq 1 \quad \nu\text{-a.e. } t \in I. \tag{4.28}$$

Taking the latter inequality into account, it results from (4.23) that

$$\left\| \frac{du_n}{d\nu}(t) \right\| \leq 2 \quad \nu\text{-a.e. } t \in I. \tag{4.29}$$

Combining this inequality with the fact that $\frac{du_n}{d\nu}$ is a density of du_n relative to ν, we obtain

$$\|u_n(\tau_1) - u_n(\tau_2)\| \leq 2\nu(]\tau_1, \tau_2]) \quad \text{for all } \tau_1, \tau_2 \in I \text{ with } \tau_1 \leq \tau_2. \tag{4.30}$$

By (4.15), we have

$$z_n(t) \in F(\kappa_n(\delta_n(t)), u_n(\delta_n(t))) \quad \text{for all } t \in I.$$

Let us define $\theta_n : I \to \mathcal{H}$ by

$$\theta_n(t) = \begin{cases} t_{i+1}^n & \text{if } t \in]t_i^n, t_{i+1}^n] \text{ with } i \in E_n, \\ t_1^n & \text{if } t = T_0. \end{cases}$$

Using the definition of Π_n, (4.17) and (2.1), we have

$$\Pi_n(t) \in -N^P\big(C(\theta_n(t)); u_n(\theta_n(t))\big) \quad \nu\text{-a.e. } t \in I. \tag{4.31}$$

According to (4.31), (4.23) and (2.5), we have

$$\Pi_n(t) \in -\partial_P d_{C(\theta_n(t))}\big(u_n(\theta_n(t))\big) \quad \nu\text{-a.e. } t \in I. \tag{4.32}$$

Step 3: Convergence of $(u_n(\cdot))_n$ up to a subsequence.
As in [14], we are going to prove that $(u_n(\cdot))_n$ has a subsequence that converges pointwise to a mapping $u(\cdot)$ which is a solution of $(\mathcal{P})$.
According to (4.27), the sequence $(z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot))_n$ is bounded in $L^2([T_0,T],\mathcal{H},\nu)$. Without loss of generality, we can suppose that $(z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot))_n$ converges weakly in $L^2([T_0,T],\mathcal{H},\nu)$ to some mapping $\tilde{z} : I \to \mathcal{H}$ with $\tilde{z} \in L^2([T_0,T],\mathcal{H},\nu)$. Fix any integer $n \geq n_0$. Let us define $Z_n : I \to \mathcal{H}$ by

$$Z_n(t) = \int_{]T_0,t]} z_n(s)\frac{d\lambda}{d\nu}(s)d\nu(s) \quad \text{for all } t \in I.$$

For all $t \in I$, we have

$$Z_n(t) \to \int_{]T_0,t]} \tilde{z}(s)d\nu(s) \quad \text{weakly in } \mathcal{H}. \tag{4.33}$$

By (4.20), the definition of $z_n(\cdot)$, the fact that K is a convex set containing 0, and the choice of s_i^n $(i \in \{0,\dots,p(n)-1\})$, we get

$$z_n(t) \in l(\beta(t)+1)K \quad \text{for all } t \in I. \tag{4.34}$$

According to (4.34), (4.24) and the fact that K is a closed convex set containing 0, we have

$$Z_n(t) \in \nu(]T_0,t])K \quad \text{for all } t \in I,$$

hence,

$$Z_n(t) \in \nu(]T_0,T])K \quad \text{for all } t \in I.$$

Since K is strongly compact, the convergence in (4.33) holds with respect to the strong topology of $\mathcal{H}$. Observe that the mapping $Z : I \to \mathcal{H}$ defined by

$$Z(t) = \int_{]T_0,T]} \tilde{z}(s)d\nu(s) \quad \text{for all } t \in I,$$

is right continuous on I, of bounded variation on I and satisfies for all $t \in I$

$$Z_n(t) \to Z(t).$$

Let us define the mapping $w_n : I \to \mathcal{H}$ by

$$w_n(t) = u_n(t) + Z_n(t) \quad \text{for all } t \in I,$$

which is right continuous on I and of bounded variation on I. We are going to prove that for any $t \in I$, the sequence $(w_n(t))_n$ is a Cauchy sequence of $\mathcal{H}$. Fix any $m,n \in \mathbb{N}$ with $m,n \geq n_0$. According to the definitions of u_n and θ_n, we have

$$u_n(\theta_n(t)) \in C(\theta_n(t)) \quad \text{for all } t \in I. \tag{4.35}$$

From the latter inclusion and the variation assumption on $C(\cdot)$, it results for any $t \in I$

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\le d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(t)) + \|u_m(\theta_m(t)) - u_m(t)\| \\ &\le \max\{\mu(]t,\theta_n(t)],\mu(]t,\theta_m(t)])\} + \|u_m(\theta_m(t)) - u_m(t)\|. \end{aligned} \tag{4.36}$$

Fix any $s \in [T_0, T[$. Choose any $i_s \in \{0, \dots, p(m)-1\}$ such that $s \in [t^m_{i_s}, t^m_{i_s+1}[$. According to the definitions of $\theta_m(\cdot)$ and $u_m(\cdot)$, we have

$$\begin{aligned} &u_m(\theta_m(s)) - u_m(s) \\ &= u^m_{i_s+1} - u^m_{i_s} - \frac{\nu(]t^m_{i_s}, s])}{\nu(]t^m_{i_s}, t^m_{i_s+1}])}\left(u^m_{i_s+1} - u^m_{i_s} + \eta^m_{i_s} z^m_{i_s} + \int_{t^m_{i_s}}^{t} f(w, u^m_{i_s})d\lambda(w)\right) \\ &\quad - (t - t^m_{i_s})z^m_{i_s} - \int_{t^m_{i_s}}^{t} f(w, u^m_{i_s})d\lambda(w). \end{aligned}$$

Combining the latter equality with (4.19) and with the assumptions (ii) and (iv), we get

$$\begin{aligned} &\|u_m(\theta_m(s)) - u_m(s)\| \\ &\le \nu(]t^m_{i_s}, s]) + \left\| u^m_{i_s+1} - u^m_{i_s} + (t - t^m_{i_s})z^m_{i_s} + \int_{t^m_{i_s}}^{t} f(w, u^m_{i_s})d\lambda(w)\right\| \\ &\le \varepsilon_m + \left\| u^m_{i_s+1} - u^m_{i_s} + \eta^m_{i_s} z^m_{i_s} + \int_{t_{i^m_s}}^{t^m_{i_s+1}} f(w, u^m_{i_s})d\lambda(w)\right\| \\ &\quad + \int_{t_{i^m_s}}^{t^m_{i_s+1}} \left\|f(w, u^m_{i_s})\right\| d\lambda(w) + \eta^m_{i_s}\left\|z^m_{i_s}\right\| \\ &\le \varepsilon_m + \nu(]t^m_{i_s}, t^m_{i_s+1}]) + (1+l)\alpha^m_{i_s} + l\beta^m_{i_s} \\ &\le 2\varepsilon_m + (1+l)\alpha^m_{i_s} + l\beta^m_{i_s} + \sup_{\tau\in]T_0,T]} \mu(\{\tau\}). \end{aligned}$$

This inequality with (4.36) and (4.5) give

$$\begin{aligned} d_{C(\theta_n(s))}(u_m(s)) &\le \max\{\mu(]s,\theta_n(s)],\mu(]s,\theta_m(s)])\} + \|u_m(\theta_m(s)) - u_m(s)\| \\ &\le \max\{\mu(]s,\theta_n(s)[,\mu(]s,\theta_m(s)[)\} + \|u_m(\theta_m(s)) - u_m(s)\| \\ &\quad + \sup_{\tau\in]T_0,T]} \mu(\{\tau\}) \\ &\le \max\{\varepsilon_n, \varepsilon_m\} + 2\varepsilon_m + (1+l)\alpha^m_{i_s} + l\beta^m_{i_s} + 2\sup_{\tau\in]T_0,T]} \mu(\{\tau\}) \\ &\le \max\{\varepsilon_n, \varepsilon_m\} + 2\varepsilon_m + (1+l)\xi_m + 2\sup_{\tau\in]T_0,T]} \mu(\{\tau\}). \end{aligned}$$

Since the right-hand side of the latter inequality (which is independent of s) goes to $2\sup_{\tau\in]T_0,T]}\mu(\{\tau\}) < r$ as $n, m \to +\infty$, there exists some integer $n_1 \ge n_0$ such that, for all integers $k_1, k_2 \ge n_1$, for all $t \in I$

$$d_{C(\theta_{k_1}(t))}(u_{k_2}(t)) < r. \tag{4.37}$$

Fix now any integers $m, n \geq n_1$ and $t \in I$. Thanks to (4.37), (4.32) and (4.22), we can apply Proposition 2.4 to obtain

$$\begin{aligned}
&\left\langle \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\
\leq &\frac{1}{2r}\left\|u_n(\theta_n(t)) - u_m(t)\right\|^2 + \frac{1}{2r}d^2_{C(\theta_n(t))}(u_m(t)) \\
&+ \left[\frac{1}{r}\left\|u_n(\theta_n(t)) - u_m(t)\right\| + 1\right] d_{C(\theta_n(t))}(u_m(t)) \\
\leq &\frac{1}{2r}\big(\, \|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\,\big)^2 + \frac{1}{2r}d^2_{C(\theta_n(t))}(u_m(t)) \\
&+ \left[\frac{1}{r}(\|u_n(\theta_n(t)) - u_n(t)\| + \|u_n(t) - u_m(t)\|) + 1\right] d_{C(\theta_n(t))}(u_m(t)).
\end{aligned} \tag{4.38}$$

Set

$$\gamma_k(t) := \nu(]t, \theta_k(t)]) + \mu(]t, \theta_k(t)]) \quad \text{for all } k \in \mathbb{N} \text{ with } k \geq n_1.$$

By (4.36) and (4.29), we have

$$d_{C(\theta_n(t))}(u_m(t)) \leq \mu(]t, \theta_n(t)]) + \mu(]t, \theta_m(t)]) + 2\nu(]t, \theta_m(t)]) \leq \gamma_n(t) + 2\gamma_m(t). \tag{4.39}$$

Note that by (4.30)

$$\|u_k(\theta_k(\tau)) - u_k(\tau)\| \leq 2\nu(]\tau, \theta_k(\tau)]) \leq 2\gamma_k(\tau) \tag{4.40}$$

for all $\tau \in I$, all integers $k \geq n_1$. Referring to (4.38), (4.39) and (4.40), we have

$$\begin{aligned}
&\left\langle \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\
\leq &\frac{1}{2r}\big(\, \|u_n(t) - u_m(t)\| + 2\gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\
&+ \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big).
\end{aligned} \tag{4.41}$$

Put

$$\psi_k(t) := \gamma_k(t) + \|Z_k(t) - Z(t)\| \quad \text{for all } k \in \mathbb{N} \text{ with } k \geq n_1.$$

Using the definition of $Z_n(\cdot)$ and (4.27), we obtain

$$\begin{aligned}
\|Z_n(\theta_n(t)) - Z_n(t)\| &= \left\| \int_{]T_0, \theta_n(t)]} z_n(s)\frac{d\lambda}{d\nu}(s)d\nu(s) - \int_{]T_0, t]} z_n(s)\frac{d\lambda}{d\nu}(s)d\nu(s) \right\| \\
&= \left\| \int_{]t, \theta_n(t)]} z_n(s)\frac{d\lambda}{d\nu}(s)d\nu(s) \right\| \\
&\leq \int_{]t, \theta_n(t)]} \left\| z_n(s)\frac{d\lambda}{d\nu}(s) \right\| d\nu(s) \\
&\leq \int_{]t, \theta_n(t)]} d\nu(s) \\
&= \nu(]t, \theta_n(t)]) \\
&\leq \gamma_n(t).
\end{aligned} \tag{4.42}$$

It follows

$$\begin{aligned}\|Z_n(\theta_n(t)) - Z_m(t)\| &\leq \|Z_n(\theta_n(t)) - Z_n(t)\| + \|Z_n(t) - Z(t)\| + \|Z(t) - Z_m(t)\| \\ &\leq \gamma_n(t) + \|Z_n(t) - Z(t)\| + \|Z(t) - Z_m(t)\| \\ &\leq \psi_n(t) + \psi_m(t). \qquad (4.43)\end{aligned}$$

Observe that, according to the definition of $w_n(\cdot)$, the differential measure dw_n of $w_n(\cdot)$ has $\frac{dw_n}{d\nu} \in L^\infty(I, \mathcal{H}, \nu)$ as a density relative to ν such that

$$\frac{dw_n}{d\nu}(t) = \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) \quad \nu\text{-a.e. } t \in I. \qquad (4.44)$$

According to (4.41), (4.44) and (4.23), we have

$$\begin{aligned}&\left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(\theta_n(t)) - w_m(t) \right\rangle \\ =& \left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\ &+ \left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), Z_n(\theta_n(t)) - Z_m(t) \right\rangle \\ \leq& \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + 2\gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\ &+ \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) \\ &+ \|Z_n(\theta_n(t)) - Z_m(t)\|. \qquad (4.45)\end{aligned}$$

for ν-almost every $t \in I$. Keeping in mind the definition of $w_n(\cdot)$ and $w_m(\cdot)$, it is readily seen that for all $t \in I$,

$$\|u_n(t) - u_m(t)\| \leq \|w_n(t) - w_m(t)\| + \|Z_m(t) - Z_n(t)\|. \qquad (4.46)$$

Using (4.45) and (4.46), we obtain

$$\begin{aligned}&\left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(\theta_n(t)) - w_m(t) \right\rangle \\ \leq& \frac{1}{2r}\big(\|w_n(t) - w_m(t)\| + \|Z_m(t) - Z_n(t)\| + 2\gamma_n(t)\big)^2 \\ &+ \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) \\ &+ \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 + \|Z_n(\theta_n(t)) - Z_m(t)\|. \qquad (4.47)\end{aligned}$$

It is straightforward that for all $t \in I$,

$$\begin{aligned}\|Z_n(t) - Z_m(t)\| + 2\gamma_n(t) &\leq \|Z_n(t) - Z(t)\| + \|Z_m(t) - Z(t)\| + 2\gamma_n(t) \\ &\leq 2\psi_n(t) + \psi_m(t). \qquad (4.48)\end{aligned}$$

Combining (4.47), (4.48) and (4.43), we get

$$\begin{aligned}
&\left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(\theta_n(t)) - w_m(t) \right\rangle \\
\leq &\frac{1}{2r}\big(\|w_n(t) - w_m(t)\| + \psi_m(t) + 2\psi_n(t)\big)^2 + \frac{1}{2r}\big(\psi_n(t) + 2\psi_m(t)\big)^2 \\
&+ \left[\frac{1}{r}\big(2\psi_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right] \big(\psi_n(t) + 2\psi_m(t)\big) \\
&+ \big(\psi_n(t) + \psi_m(t)\big). \qquad (4.49)
\end{aligned}$$

for ν-almost every $t \in I$. According to (4.40) and (4.42), we have

$$\begin{aligned}
\|w_n(\theta_n(t)) - w_n(t)\| &\leq \|u_n(\theta_n(t)) - u_n(t)\| + \|Z_n(\theta_n(t)) - Z_n(t)\| \\
&\leq 2\gamma_n(t) + \gamma_n(t) \\
&= 3\gamma_n(t), \qquad (4.50)
\end{aligned}$$

for all $t \in I$. Using (4.50), (4.49), (4.23) and the inequality $\gamma_n \leq \psi_n$, it follows

$$\begin{aligned}
&\left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(t) - w_m(t) \right\rangle \\
= &\left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(t) - w_n(\theta_n(t)) \right\rangle \\
&+ \left\langle \frac{dw_n}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t), w_n(\theta_n(t)) - w_m(t) \right\rangle \\
\leq &\left\| \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) + f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t) \right\| 3\gamma_n(t) \\
&+ \frac{1}{2r}\big(\|w_n(t) - w_m(t)\| + \psi_m(t) + 2\psi_n(t)\big)^2 + \frac{1}{2r}(\psi_n(t) + 2\psi_m(t))^2 \\
&+ \left[\frac{1}{r}\big(2\psi_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right] \big(\psi_n(t) + 2\psi_m(t)\big) + \big(\psi_n(t) + \psi_m(t)\big) \\
\leq &\frac{1}{2r}\big(\|w_n(t) - w_m(t)\| + 2(\psi_m(t) + \psi_n(t))\big)^2 + \frac{2}{r}\big(\psi_n(t) + \psi_m(t)\big)^2 \\
&+ 2\left[\frac{1}{r}\big(2\psi_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right] \big(\psi_n(t) + \psi_m(t)\big) + 4\big(\psi_n(t) + \psi_m(t)\big),
\end{aligned}$$

for ν-almost every $t \in I$. By interchanging m and n, we get

$$\begin{aligned}
&\left\langle \frac{dw_m}{d\nu}(t) + f\big(t, u_m(\delta_m(t))\big)\frac{d\lambda}{d\nu}(t), w_m(t) - w_n(t) \right\rangle \\
\leq &\frac{1}{2r}\big(\|w_n(t) - w_m(t)\| + 2(\psi_n(t) + \psi_m(t))\big)^2 \\
&+ 2\left[\frac{1}{r}\big(2\psi_m(t) + \|u_n(t) - u_m(t)\|\big) + 1\right] \big(\psi_n(t) + \psi_m(t)\big) \\
&+ \frac{2}{r}\big(\psi_n(t) + \psi_m(t)\big)^2 + 4\big(\psi_n(t) + \psi_m(t)\big),
\end{aligned}$$

for ν-almost every $t \in I$. For ν-almost every $t \in I$, for all integers $k \geq n_1$, put

$$B_k(t) := \frac{dw_k}{d\nu}(t) + f(t, u_k(\delta_k(t)))\frac{d\lambda}{d\nu}(t).$$

Since the sequences $(u_k(\cdot))_{k\geq n_1}$, $(w_k(\cdot))_{k\geq n_1}$ and $(\psi_k(\cdot))_{k\geq n_1}$ are uniformly bounded on I, adding the two latter inequalities, there exists some real $A > 0$ (depending on r) such that, for ν-almost every $t \in I$, for all integers $k_1, k_2 \geq n_1$,

$$\begin{aligned}\langle B_{k_1}(t) - B_{k_2}(t), w_{k_1}(t) - w_{k_2}(t)\rangle \leq A\Big(\big(\psi_{k_1}(t) + \psi_{k_2}(t)\big)^2 + \big(\psi_{k_1}(t) + \psi_{k_2}(t)\big)\Big)\\ + \frac{1}{r}\left\|w_{k_1}(t) - w_{k_2}(t)\right\|^2. \end{aligned} \tag{4.51}$$

Fix any real $c > 0$ such that, for all integers $k \geq n_1$, for all $t \in I$

$$\|u_k(t)\| \leq c \quad \text{and} \quad \|w_k(t)\| \leq c.$$

Applying assumption (iii), we get for all $t \in I$,

$$\begin{aligned}&\|f(t, u_n(\delta_n(t))) - f(t, u_m(\delta_m(t)))\|\\ \leq &L_c(t)\left\|u_n(\delta_n(t)) - u_m(\delta_m(t))\right\|\end{aligned} \tag{4.52}$$

From (4.51) and (4.52), it results

$$\begin{aligned}&\left\langle \frac{dw_n}{d\nu}(t) - \frac{dw_m}{d\nu}(t), w_n(t) - w_m(t)\right\rangle\\ \leq&\frac{d\lambda}{d\nu}(t)\left\langle f\big(t, u_n(\delta_n(t))\big) - f\big(t, u_m(\delta_m(t))\big), w_m(t) - w_n(t)\right\rangle\\ &+ \frac{1}{r}\left\|w_n(t) - w_m(t)\right\|^2 + A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big)\\ \leq&\frac{d\lambda}{d\nu}(t)\left\|f\big(t, u_n(\delta_n(t))\big) - f\big(t, u_m(\delta_m(t))\big)\right\|\left\|w_m(t) - w_n(t)\right\|\\ &+ \frac{1}{r}\left\|w_n(t) - w_m(t)\right\|^2 + A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big)\\ \leq&\frac{d\lambda}{d\nu}(t)L_c(t)\left\|u_n(\delta_n(t)) - u_m(\delta_m(t))\right\|\left\|w_m(t) - w_n(t)\right\|\\ &+ \frac{1}{r}\left\|w_n(t) - w_m(t)\right\|^2 + A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big).\end{aligned} \tag{4.53}$$

for ν-almost every $t \in I$. According to the definition of $(w_k(\cdot))_k$, we have for all $t \in I$

$$\begin{aligned}&\|u_n(\delta_n(t)) - u_m(\delta_m(t))\|\\ \leq&\|u_n(\delta_n(t)) - u_n(t)\| + \|u_n(t) - u_m(t)\| + \|u_m(t) - u_m(\delta_m(t))\|\\ \leq&\|u_n(\delta_n(t)) - u_n(t)\| + \|u_m(t) - u_m(\delta_m(t))\|\\ &+ \|w_n(t) - w_m(t)\| + \|Z_n(t) - Z_m(t)\|.\end{aligned} \tag{4.54}$$

Note that, for all $t \in I$

$$\begin{aligned}\|Z_n(t) - Z_m(t)\| &\leq \|Z_n(t) - Z(t)\| + \|Z(t) - Z_m(t)\|\\ &\leq \psi_n(t) + \psi_m(t).\end{aligned} \tag{4.55}$$

Keeping in mind that $\|w_n(t) - w_m(t)\| \leq 2c$ for all $t \in I$ and using (4.53), (4.54), (4.55) and (4.40), we get

$$\begin{aligned}
&\left\langle \frac{dw_n}{d\nu}(t) - \frac{dw_m}{d\nu}(t), w_n(t) - w_m(t) \right\rangle \\
\leq &\Big(\frac{d\lambda}{d\nu}(t)L_c(t) + \frac{1}{r}\Big) \|w_n(t) - w_m(t)\|^2 + A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big) \\
&+ 2c\frac{d\lambda}{d\nu}(t)L_c(t)\Big(\|u_n(\delta_n(t)) - u_n(t)\| + \|u_m(t) - u_m(\delta_m(t))\| + \|Z_n(t) - Z_m(t)\| \Big) \\
\leq &\Big(\frac{d\lambda}{d\nu}(t)L_c(t) + \frac{1}{r}\Big) \|w_n(t) - w_m(t)\|^2 + A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big) \\
&+ 2c\frac{d\lambda}{d\nu}(t)L_c(t)\big(2\nu(]\delta_n(t), t]) + 2\nu(]\delta_m(t), t]) + \psi_n(t) + \psi_m(t)\big),
\end{aligned}$$

for ν-almost every $t \in I$. According to Proposition 2.7, we have

$$d(\|w_n(\cdot) - w_m(\cdot)\|^2) \leq 2 \left\langle \frac{dw_n}{d\nu}(\cdot) - \frac{dw_m}{d\nu}(\cdot), w_n(\cdot) - w_m(\cdot) \right\rangle d\nu.$$

Let us define, for all $t \in I$

$$\Phi_{n,m}(t) := \|w_n(t) - w_m(t)\|^2 ,$$

and

$$\begin{aligned}
A_{n,m}(t) :=&A\Big(\big(\psi_n(t) + \psi_m(t)\big)^2 + \big(\psi_n(t) + \psi_m(t)\big)\Big) \\
&+ 2c\frac{d\lambda}{d\nu}(t)L_c(t)\big(2\nu(]\delta_n(t), t]) + 2\nu(]\delta_m(t), t]) + \psi_n(t) + \psi_m(t)\big).
\end{aligned}$$

Set

$$\alpha_{n,m} := \int_{]T_0,T]} A_{n,m}(s)d\nu(s).$$

Since $w_n(T_0) = w_m(T_0)$, we obtain, for all $t \in]T_0, t]$

$$\Phi_{n,m}(t) \leq \int_{]T_0,t]} 2\Big(\frac{d\lambda}{d\nu}(s)L_c(s) + \frac{1}{r}\Big)\Phi_{n,m}(s)d\nu(s) + \alpha_{n,m}.$$

According to $\mu(\{t\}) = \nu(\{t\})$ for all $t \in I$ and $\sup_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$, we have

$$\sup_{s\in]T_0,T]} \nu(\{s\}) < \frac{r}{2}.$$

Let us set

$$a := \frac{2}{r} \sup_{s\in]T_0,T]} \nu(\{s\}) < 1.$$

Since $L_c(\cdot)$ is λ-integrable on $[T_0,T]$ and λ is absolutely continuous with respect to ν, we know that $L_c(\cdot)\frac{d\lambda}{d\nu}(\cdot)$ is ν-integrable on $[T_0,T]$ and

$$\int_{]T_0,t]} L_c(s)d\lambda(s) = \int_{]T_0,t]} L_c(s)\frac{d\lambda}{d\nu}(s)d\nu(s) \quad \text{for all } t \in I.$$

Applying Lemma 2.8, it follows

$$\Phi_{n,m}(t) \le \alpha_{n,m} \exp\left(\frac{2}{1-a}\int_{]T_0,t]}\Big(\frac{d\lambda}{d\nu}(s)L_c(s)+\frac{1}{r}\Big)d\nu(s)\right) \quad \text{for all } t\in I.$$

So, we have

$$\Phi_{n,m}(t) \le \alpha_{n,m} \exp\left(\frac{2}{1-a}\Big(\int_{]T_0,t]} L_c(s)d\lambda(s)+\frac{1}{r}\nu(]T_0,t])\Big)\right) \quad \text{for all } t\in I.$$

As a consequence, we get

$$\sup_{t\in[T_0,T]} \Phi_{n,m}(t) \le \alpha_{n,m} \exp\left(\frac{2}{1-a}\Big(\int_{]T_0,T]} L_c(s)d\lambda(s)+\frac{1}{r}\nu(]T_0,T])\Big)\right).$$

Observe that, $\lim_{n,m\to+\infty}\alpha_{n,m}=0$ via Lebesgue dominated convergence theorem, since the sequence $(\psi_n(\cdot))_n$ is uniformly bounded and $\lim_{n\to+\infty}\psi_n(t)=0$ for all $t\in]T_0,T]$ and thanks to the fact that $\nu(]\delta_n(t),t])\le\varepsilon_n$ for all integers $n\ge n_1$, all $t\in I$. This proves that $(w_n(t))_n$ is a Cauchy sequence for each $t\in[T_0,T]$. Then, we get some mapping $w(\cdot): I\to\mathcal{H}$ such that, for all $t\in[T_0,T]$,

$$u_n(t)\to u(t):=w(t)-Z(t).$$

On the other hand, according to (4.29), extracting a subsequence if necessary, we may suppose that $(\frac{du_n}{d\nu}(\cdot))_n$ converges weakly in $L^2(I,\mathcal{H},\nu)$ to some mapping $g(\cdot)\in L^2(I,\mathcal{H},\nu)$. So, for any $t\in I$

$$\int_{]T_0,t]}\frac{du_n}{d\nu}(s)d\nu(s)\to\int_{]T_0,t]}g(s)d\nu(s) \quad \text{weakly in } \mathcal{H}.$$

As $u_n(t)\to u(t)$, it results that

$$u(t)=u_0+\int_{]T_0,t]}g(s)d\nu(s),$$

hence $u(\cdot)$ is right continuous on I and of bounded variation on I, and du has $\frac{du}{d\nu}(\cdot)=g(\cdot)\in L^2(I,\mathcal{H},\nu)$ as a density relative to ν. As a result,

$$\frac{du_n}{d\nu}(\cdot)\to\frac{du}{d\nu}(\cdot) \quad \text{weakly in } L^2(I,\mathcal{H},\nu),$$

and this yields

$$\frac{du_n}{d\nu}(\cdot)\to\frac{du}{d\nu}(\cdot) \quad \text{weakly in } L^1(I,\mathcal{H},\nu).$$

Step 4: Let us prove that, $u(\cdot)$ is a solution of $(\mathcal{P})$.
Fix for a moment any $t\in I$. Using (4.6), we get

$$\lim_{n\to+\infty}\delta_n(t)=t \quad \text{and} \quad \lim_{n\to+\infty}\theta_n(t)=t. \tag{4.56}$$

By the continuity of $f(t,\cdot)$ on $c\mathbb{B}$ and the inequality $\|u_n(t)\|\le c$ for all $n\ge n_1$, we have

$$\lim_{n\to+\infty} f\big(t,u_n(\delta_n(t))\big)=f(t,u(t)).$$

From (4.30), we obtain

$$\begin{aligned}\|u_n(\theta_n(t)) - u(t)\| &\le \|u_n(\theta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\| \\ &\le 2\nu(]t, \theta_n(t)]) + \|u_n(t) - u(t)\|.\end{aligned}$$

By (4.56), we have

$$\lim_{n \to +\infty} u_n(\theta_n(t)) = u(t).$$

Using the variation assumption on $C(\cdot)$ in (4.1) and (4.35), it follows

$$\begin{aligned}d_{C(t)}(u_n(\theta_n(t))) &= d_{C(t)}(u_n(\theta_n(t))) - d_{C(\theta_n(t))}(u_n(\theta_n(t))) \\ &\le \mu(]t, \theta_n(t)]).\end{aligned}$$

Combining the latter inequality, (4.56) and the fact that $C(t)$ is closed, we have

$$u(t) \in C(t).$$

According to (4.25), we may suppose that, $(z_n(\cdot))_n$ converges weakly in $L^1(I, \mathcal{H}, \lambda)$ to some mapping $z(\cdot) \in L^1(I, \mathcal{H}, \lambda)$. Since $\frac{d\lambda}{d\nu}(\cdot) \in L^\infty(I, \mathbb{R}_+, \nu)$, it entails that

$$z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot) \to z(\cdot)\frac{d\lambda}{d\nu}(\cdot) \quad \text{weakly in } L^1(I, \mathcal{H}, \nu).$$

By Lebesgue dominated convergence, we have for ν-almost every $t \in I$

$$f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t) \to f(t, u(t))\frac{d\lambda}{d\nu}(t) \quad \text{strongly in } L^1(I, \mathcal{H}, \nu).$$

Now, we apply a classical technique due to C. Castaing ([7]). Thanks to Mazur's lemma, there exists a sequence $(\zeta_n(\cdot))_n$ which converges strongly in $L^1(I, \mathcal{H}, \nu)$ to $\frac{du}{d\nu}(\cdot) + z(\cdot)\frac{d\lambda}{d\nu}(\cdot) + f(\cdot, u(\cdot))\frac{d\lambda}{d\nu}$ with

$$\zeta_n(\cdot) \in \mathrm{co}\left\{\frac{du_k}{d\nu}(\cdot) + z_k(\cdot)\frac{d\lambda}{d\nu}(\cdot) + f(\cdot, u_k(\cdot)) : k \ge n\right\}$$

for each $n \ge n_1$. Extracting a subsequence if necessary, we may suppose that

$$\zeta_n(t) \to \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) \quad \nu\text{-a.e. } t \in I.$$

Then, we have

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu} \in \bigcap_{n \ge n_1} \overline{\mathrm{co}}\left\{\frac{du_k}{d\nu}(\cdot) + z_k(\cdot)\frac{d\lambda}{d\nu}(\cdot) + f(\cdot, u_k(\cdot)) : k \ge n\right\},$$

for ν-almost every $t \in I$. This inclusion yields for ν-almost every $t \in I$ that

$$\begin{aligned}&\left\langle \xi, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t)\right\rangle \\ &\le \inf_{n \ge n_1} \sup_{k \ge n} \left\langle \xi, \frac{du_k}{d\nu}(t) + z_k(t)\frac{d\lambda}{d\nu}(t) + f(t, u_k(t))\right\rangle,\end{aligned}$$

for all $\xi \in \mathcal{H}$. It follows that, for ν-almost every $t \in I$, for all $\xi \in \mathcal{H}$,

$$\left\langle \xi, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t)\right\rangle \le \limsup_{n \to +\infty} \sigma(\xi, -\partial_P d_{C(\theta_n(t))}(u_n(\theta_n(t)))).$$

Hence, for ν-almost every $t \in I$, according to Proposition 2.6

$$\left\langle \xi, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t,u(t))\frac{d\lambda}{d\nu}(t) \right\rangle \leq \sigma(\xi, -\partial_C d_{C(t)}(u(t))),$$

for all $\xi \in \mathcal{H}$. Thanks to (2.7), we have

$$\left\{ \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t,u(t))\frac{d\lambda}{d\nu}(t) \right\} \subset \overline{\text{co}}\left(-\partial_C d_{C(t)}(u(t))\right) \quad \nu\text{-a.e. } t \in I$$

Since the Clarke subdifferential is always closed and convex, this last inclusion gives us

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t,u(t))\frac{d\lambda}{d\nu}(t) \in -\partial_C d_{C(t)}(u(t)) \quad \nu\text{-a.e. } t \in I.$$

Combining this inclusion, (2.6) and (4.23), we have

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) + f(t,u(t))\frac{d\lambda}{d\nu}(t) \in -N(C(t);u(t)) \quad \nu\text{-a.e. } t \in I.$$

Let us show that, $z(t) \in F(t,u(t))$ for λ-almost every $t \in I$. Fix any $t \in I$ and $n \geq n_1$ an integer. Note that, by (4.6)

$$|\kappa_n(\delta_n(t)) - t| \leq \varepsilon_n.$$

It results

$$\lim_{k \to +\infty} \kappa_k(\delta_k(t)) = t.$$

Thanks to the fact that $z_k(\cdot)$ converges to $z(\cdot)$ weakly in $L^1(I, \mathcal{H}, \nu)$, via Mazur's lemma again, extracting a subsequence if necessary, we may write

$$z(t) \in \bigcap_{n \geq n_1} \overline{\text{co}}\{z_k(t) : k \geq n\} \quad \nu\text{-a.e. } t \in I.$$

Thus, for ν-almost every $t \in I$, we have

$$\langle \xi, z(t) \rangle \leq \limsup_{n \to +\infty} \sigma\Big(\xi, F\big(\kappa_n(\delta_n(t)), u_n(\delta_n(t))\big)\Big).$$

for all $\xi \in \mathcal{H}$. Applying assumption (i), we get for ν-almost every $t \in I$

$$\langle \xi, z(t) \rangle \leq \sigma(\xi, F(t,u(t))).$$

for all $\xi \in \mathcal{H}$. Since $F(t,u(t))$ is closed and convex for all $t \in I$, we have (thanks to (2.7))

$$z(t) \in F(t,u(t)) \quad \nu\text{-a.e. } t \in I.$$

As $u(T_0) = \lim_{n \to +\infty} u_n(T_0) = u_0$, $u(\cdot)$ is a solution of $(\mathcal{P})$. On the other hand, by passing to the limit in (4.30), we have

$$\|u(\tau_1) - u(\tau_2)\| \leq 2\nu(]\tau_1, \tau_2]) \quad \text{for all } \tau_1, \tau_2 \in I \text{ with } \tau_1 \leq \tau_2.$$

It results

$$\left\|u(t) - u(t^-)\right\| \leq 2\nu(\{t\}) = 2\mu(\{t\}) \quad \text{for all } t \in]T_0, T].$$

This completes the first case.

Case 2: Now, we assume

$$\int_{T_0}^{T} (\beta(s)+1)d\lambda(s) > \frac{1}{8} \quad \text{or} \quad \int_{T_0}^{T} \alpha(s)d\lambda(s) > \frac{1}{8}.$$

Consider a subdivision $(T_0,\ldots,T_k)$ (with $k\geq 2$) such that,

$$T_0<\cdots<T_k=T,$$

and satisfying for each $i\in\{1,\ldots,k\}$

$$\int_{T_{i-1}}^{T_i}(\beta(s)+1)d\lambda(s)\leq\frac{1}{8}\quad\text{and}\quad\int_{T_{i-1}}^{T_i}\alpha(s)d\lambda(s)\leq\frac{1}{8}.$$

For each $i\in\{1,\ldots,k\}$, denote by μ_i the Radon measure induced on $[T_{i-1},T_i]$ by μ and set $\nu_i:=\mu_i+\lambda$. Then, the case 1 provides a mapping $u_1:[T_0,T_1]\to\mathcal{H}$, a λ-integrable mapping $z_1:[T_0,T_1]\to\mathcal{H}$ such that, $u_1(\cdot)$ is right continuous on $[T_0,T_1]$ and of bounded variation on $[T_0,T_1]$, $u_1(T_0)=u_0$, $u_1(t)\in C(t)$ for all $t\in[T_0,T_1]$, du_1 has $\frac{du_1}{d\nu_1}$ as a density in $L^1([T_0,T],\mathcal{H},\nu_1)$ relative to ν_1,

$$\left\|u_1(t)-u_1(t^-)\right\|\leq 2\mu_1(\{t\})\quad\text{for all } t\in]T_0,T],$$

$$z_1(t)\in F(t,u_1(t))\quad\lambda\text{-a.e.}t\in[T_0,T_1].$$

and

$$\frac{du_1}{d\nu_1}(t)+z_1(t)\frac{d\lambda}{d\nu_1}(t)+f(t,u_1(t))\frac{d\lambda}{d\nu_1}(t)\in -N(C(t);u_1(t))\quad\lambda\text{-a.e.}t\in[T_0,T_1].$$

By finite induction, we obtain a finite sequence of right continuous mappings of bounded variations $u_i(\cdot):[T_{i-1},T_i]\to\mathcal{H}$ $(2\leq i\leq k)$ and a finite sequence of λ-integrable mappings $z_i(\cdot):[T_{i-1},T_i]\to\mathcal{H}$ such that, for each $i\in\{2,\ldots,k\}$, $u_i(T_{i-1})=u_{i-1}(T_{i-1})$, $u_i(t)\in C(t)$ for all $t\in[T_{i-1},T_i]$, du_i has $\frac{du_i}{d\nu_i}$ as a density in $L^1([T_{i-1},T_i],\mathcal{H},\nu_i)$ relative to ν_i,

$$\left\|u_i(t)-u_i(t^-)\right\|\leq 2\mu_i(\{t\})\quad\text{for all } t\in]T_{i-1},T_i],$$

$$z_i(t)\in F(t,u_i(t))\quad\lambda\text{-a.e.}t\in[T_{i-1},T_i].$$

and

$$\frac{du_i}{d\nu_i}(t)+z_i(t)\frac{d\lambda}{d\nu_i}(t)+f(t,u_i(t))\frac{d\lambda}{d\nu_i}(t)\in -N(C(t);u_i(t))\quad\lambda\text{-a.e.}t\in[T_{i-1},T_i].$$

Now, let us define $u(\cdot),z(\cdot),g(\cdot):[T_0,T]\to\mathcal{H}$ by

$$u(t)=u_i(t)\quad\text{if } t\in[T_{i-1},T_i]\text{ for some } i\in\{1,\ldots,k\},$$

$$\begin{cases} z_1(t) & \text{if } t\in[T_0,T_1],\\ z_i(t) & \text{if } t\in]T_{i-1},T_i]\text{ for some } i\in\{2,\ldots,k\},\end{cases}$$

and

$$g(t)=\mathbf{1}_{[T_0,T_1]}(t)\frac{du_1}{d\nu_1}(t)+\sum_{i=2}^{k}\mathbf{1}_{]T_{i-1},T_i]}(t)\frac{du_i}{d\nu_i}(t).$$

It is clear that $z(\cdot)$ is λ-integrable on $[T_0,T]$ and $u(\cdot)$ is right continuous and of bounded variation on $[T_0,T]$ satisfying $u(T_0)=u_0$ and for $\tilde{\nu}:=\mu+\lambda$, one has

$$u(t)\in C(t)\quad\text{and}\quad u(t)=u(T_0)+\int_{]T_0,t]}g(s)d\tilde{\nu}(s).$$

Note that, the differential measure du of $u(\cdot)$ has $\frac{du}{d\tilde{\nu}}(\cdot) = g(\cdot) \in L^1([T_0, T], \mathcal{H}, \tilde{\nu})$ as a density relative to $\tilde{\nu}$. Moreover,

$$\left\| u(t) - u(t^-) \right\| \leq 2\mu(\{t\}) \quad \text{for all } t \in]T_0, T],$$

$$z(t) \in F(t, u(t)) \quad \lambda\text{-a.e. } t \in I.$$

and

$$\frac{du}{d\tilde{\nu}}(t) + z(t)\frac{d\lambda}{d\tilde{\nu}}(t) + f(t, u(t))\frac{d\lambda}{d\tilde{\nu}}(t) \in -N(C(t); u(t)) \quad \tilde{\nu}\text{-a.e. } t \in I.$$

This finishes the proof. □

Remark 4.2. It is straightforward that Theorem 4.1 encompasses [2, Theorem 4.1] (with $F \equiv \{0\}$) and [14, Theorem 3.1] (with $f \equiv 0$).

5. Consequences

In this section, we deal with consequences of our existence theorem.

Profiting from an idea of [2], we have the following result.

Corollary 5.1. *Under the assumption of Theorem 4.1, for each $u_0 \in C(T_0)$, the perturbed sweeping process*

$$\begin{cases} -du \in N(C(t); u(t)) + F(t, u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has a solution satisfying

$$u(t) = \mathrm{P}_{C(t)}(u(t^-)) \quad \textit{for all } t \in]T_0, T].$$

Proof. By Theorem 4.1, there exists a mapping $u(\cdot) : I \to \mathcal{H}$ satisfying

$$\begin{cases} -du \in N(C(t); u(t)) + F(t, u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

and

$$\left\| u(t) - u(t^-) \right\| \leq 2\mu(\{t\}) \quad \text{for all } t \in]T_0, T]. \tag{5.1}$$

Fix any $t \in]T_0, T]$.
Case 1: $\mu(\{t\}) = 0$.
The inequality (5.1) gives us

$$u(t^-) = u(t) \in C(t).$$

So, it is readily seen that, $u(t) = P_{C(t)}(u(t^-))$.
Case 2: $\mu(\{t\}) > 0$.
In this second case, we have

$$\left\| u(t) - u(t^-) \right\| \leq 2\mu(\{t\}) \leq 2 \sup_{s \in]T_0, T]} \mu(\{s\}) < r. \tag{5.2}$$

Set $\nu := \mu + \lambda$. The inequality $\mu(\{t\}) > 0$ entails straightforwardly $\nu(\{t\}) > 0$. Combining the definition of a solution and the equality $\frac{d\lambda}{d\nu}(t) = 0$ (thanks to (2.10)), we get

$$\frac{du}{d\nu}(t) \in -N(C(t); u(t)).$$

This inclusion with (2.11) give us

$$\frac{du}{d\nu}(t) = \lim_{s \uparrow t} \frac{du(]s,t])}{\nu(]s,t])} = \lim_{s \uparrow t} \frac{u(t) - u(s)}{\nu(]s,t])} = \frac{u(t) - u(t^-)}{\nu(\{t\})} \in -N(C(t); u(t)),$$

Since $N(C(t); u(t))$ is a cone, the latter inclusion is equivalent to

$$u(t^-) - u(t) \in N(C(t); u(t)). \tag{5.3}$$

It follows,

$$u(t) \in C(t). \tag{5.4}$$

Using (5.2), (5.3), (5.4) and Proposition 2.5, we get

$$u(t) = \mathrm{P}_{C(t)}(u(t^-)).$$

□

Now, in the same way as [2], we deal with the case where the measure μ is absolutely continuous relative to λ.

Proposition 5.2. *Let $C : I \rightrightarrows \mathcal{H}$ be a multimapping such that for some extended real $r \in]0, +\infty]$, $C(t)$ is r-prox-regular for every $t \in I$. Assume that there exists a nondecreasing absolutely continuous function $v(\cdot) : I \to \mathbb{R}$ on I such that*

$$|d(y, C(s)) - d(y, C(t))| \leq v(t) - v(s) \quad \text{for all } y \in \mathcal{H}, \text{ for all } s, t \in I \text{ with } s \leq t.$$

Let $F : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ (resp., $f : I \times \mathcal{H} \to \mathcal{H}$) be a multimapping with nonempty convex compact values satisfying (i) and (ii) (resp., be a mapping satisfying (iii) and (iv)) in Theorem 4.1.

Then, with μ the Radon measure on I satisfying $\mu(]s,t]) = v(t) - v(s)$ for all $s, t \in I$ with $s < t$, any solution of the measure differential sweeping process

$$(\mathcal{P}) \begin{cases} -du \in N(C(t); u(t)) + F(t, u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \in C(T_0) \end{cases}$$

is a solution in the classical sense, that is,

(*a*) *u is absolutely continuous on I;*

(*b*) *there is a λ-integrable mapping $z(\cdot) : I \to \mathcal{H}$ with $z(t) \in F(t, u(t))$ λ-a.e. $t \in I$ such that*

$$-\frac{du}{dt}(t) \in N(C(t); u(t)) + z(t) + f(t, u(t)) \quad \lambda - a.e.\, t \in I;$$

(*c*) *$u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$.*

So, $(\mathcal{P})$ admits at least one absolutely continuous solution $u(\cdot)$ on I.

Proof. Let $u(\cdot) : I \to \mathcal{H}$ be a solution of $(\mathcal{P})$ in the measure differential sense. Set $\nu = \mu + \lambda$ and observe that ν is absolutely continuously equivalent to the Lebesgue measure λ. Then, there exists a mapping $h : I \to [0, +\infty[$ λ-integrable on I such that

$$\nu = h(\cdot)\lambda.$$

Thanks to the equalities (λ-a.e.) $h(\cdot) = \frac{d\nu}{d\lambda}(\cdot)$ and $\frac{d\nu}{d\lambda}(\cdot)\frac{du}{d\nu}(\cdot) = \frac{du}{d\lambda}(\cdot)$, we have

$$u(t) = u_0 + \int_{]T_0,t]} h(s)\frac{du}{d\nu}(s)d\lambda(s) \quad \text{for all } t \in I.$$

As a consequence, the mapping $u(\cdot)$ is absolutely continuous on I and there exists a Borel set B_1 of I with $\lambda(B_1) = 0$ such that

$$\frac{du}{dt}(t) = h(t)\frac{du}{d\nu}(t) \quad \text{for all } t \in I \setminus B_1.$$

Since $u(\cdot)$ is a solution of $(\mathcal{P})$ in the measure differential sense, there exist a λ-integrable mapping $z : I \to \mathcal{H}$ with $z(t) \in F(t, u(t))$ for λ-almost every $t \in I$ and a Borel set B_2 in I with $\nu(B_2) = 0$ such that

$$\frac{du}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \text{for all } t \in I \setminus B_2.$$

Setting $B = B_1 \cup B_2$, we see that $\lambda(B) = 0$ and, for all $t \in I \setminus B$,

$$h(t)\frac{du}{d\nu}(t) + f(t, u(t))h(t)\frac{d\lambda}{d\nu}(t) + z(t)h(t)\frac{d\lambda}{d\nu} \in -N(C(t); u(t)).$$

On the other hand, for all $s, t \in I$ with $s < t$,

$$\int_{]s,t]} h(\theta)\frac{d\lambda}{d\nu}(\theta)d\lambda(\theta) = \int_{]s,t]} \frac{d\lambda}{d\nu}(\theta)h(\theta)d\lambda(\theta) = \int_{]s,t]} \frac{d\lambda}{d\nu}(\theta)d\nu(\theta) = \int_{]s,t]} d\lambda(\theta).$$

It follows that

$$\frac{du}{dt}(t) + f(t, u(t)) + z(t) \in -N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I,$$

and this finishes the proof. □

Remark 5.3. As a direct consequence of Proposition 5.2, we get [3, Theorem 3.1].

6. Concluding remarks

In this paper, we proved that the perturbed discontinuous Moreau's sweeping process with a prox-regular moving set

$$\begin{cases} -du \in N(C(t); u(t)) + f(t, u(t)) + F(t, u(t)) & \lambda\text{-a.e.} t \in [T_0, T] \\ u(T_0) \in C(T_0), \end{cases}$$

has at least one solution satisfying $u(t) = \mathrm{P}_{C(t)}(u(t^-))$ for all $t \in]T_0, T]$. It is of interest to deal with a discontinuous second order sweeping process with a perturbation in the form $f + F$ as above. Such a study is out of the scope of this manuscript and will be the subject of a future work.

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