ON FIRST AND SECOND ORDER STATE-DEPENDENT PROX-REGULAR SWEEPING PROCESSES

FLORENT NACRY, JIMMY NOEL, AND LIONEL THIBAULT

Abstract. This paper is devoted to a new family of measure differential inclusions in Hilbert spaces. We show that it encompasses the first order BV prox-regular sweeping process and the second order one with outward normal at the velocity. Through a new suitable mixed catching-up algorithm coming from first and second order sweeping process theory, we provide sufficient conditions ensuring the existence of a trajectory solution for our evolution problem.

1. Introduction

R.A. Poliquin and R.T. Rockafellar ([46]) introduced *prox-regular sets* by requiring that their indicator functions be prox-regular. Geometric properties of those sets and their links with other previous concepts (positively reached, φ-convex, weakly convex, $O(2)$-convex, proximally smooth) were established by R.A. Poliquin, R.T. Rockafellar and L. Thibault ([47]). The present paper is a contribution to first and second order sweeping processes governed by prox-regular sets.

Let $C(t)$ be a time-dependent closed convex set of a Hilbert space $\mathcal{H}$ which moves in an absolutely continuous way, that is,

$$\operatorname{haus}\big(C(s), C(t)\big) \leq \varrho(t) - \varrho(s) \quad \text{for all } s, t \in I := [T_0, T], \tag{1.1}$$

for some nondecreasing absolutely continuous function $\varrho(\cdot) : I \to \mathbb{R}_+ := [0, +\infty[$. According to J.J. Moreau ([37]), for every initial condition $u_0 \in C(0)$, there is one and only one (absolutely continuous) trajectory solution of the following generalized Cauchy problem with $F \equiv 0$

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + F(t, u(t)) & \text{a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(T_0) = u_0 \in C(T_0). \end{cases} \tag{1.2}$$

The differential inclusion in (1.2) associated to the (outward) normal cone in the sense of convex analysis $N_{C(t)}(u(t))$ has also an interesting mechanical interpretation with $F \equiv 0$. Indeed, recalling that the latter set is reduced to $\{0\}$ if $u(t) \in \operatorname{int} C(t)$, (1.2) means that the velocity $\dot{u}(t)$ has to point inward $C(t)$ whenever $u(t)$ is caught-up by the boundary of $C(t)$. Such

---

2010 *Mathematics Subject Classification.* 34A60, 49J52, 49J53.

Key words and phrases. Variational analysis, first order Moreau sweeping process, second order sweeping process, bounded variation, Radon measure.

a kinematic point of view led J.J. Moreau to name *sweeping process* the evolution problem (1.2) with $F \equiv 0$.

Roughly speaking, there are three major ways to handle the so-called Moreau's sweeping process with $F \equiv 0$:
- the *catching-up algorithm* ([38]) which is some kind of Euler's explicit scheme associated to the iterates u_i^n defined through the metric projection as

$$u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \mathrm{proj}_{C(t_i^n)}(u_i^n),$$

where $T_0 =: t_0^n < \ldots < t_{p(n)}^n := T$ is a time discretization of I;
- the reduction to an unconstrained differential inclusion ([49, 6])

$$-\dot{u}(t) \in \dot{\varrho}(t)\partial_C d_{C(t)}(u(t))$$

with the absolutely continuous function $\varrho(\cdot)$ involved in (1.1) (here ∂_C denotes the Clarke subdifferential);
- the regularization of the normal cone ([37]) (see also the survey [41] and the references therein) through the family of ordinary differential equations (with a parameter $\nu > 0$)

$$u_\nu(T_0) = u_0 \quad \text{and} \quad -\dot{u}_\nu(t) = \frac{1}{2\nu}\nabla d^2_{C(t)}(u_\nu(t)) \quad \text{a.e. } t \in I.$$

The important role in many concrete problems of Moreau's sweeping process (see, e.g., [1, 32, 36, 34]) lies at the heart of the following extensions of (1.2): with bounded variation of the moving set ([38, 34, 27]), to the stochastic setting ([13]), to Banach spaces ([10]) and manifolds ([9]), to nonconvex moving sets ([52, 29, 30, 42]), with a perturbation $F \not\equiv 0$ ([33, 34]), with state-dependent moving sets ([17, 31, 28]), to the second-order case ([15, 2, 45]), to the delayed case ([43]), with outward normal at the velocity for first order ([3]), with truncated excess and Hausdorff distance ([2, 42]), in Wasserstein's space ([25]), in optimal control ([12]), etc.

Introduced by J.J. Moreau himself in [38], the bounded variation (or discontinuous) sweeping process depicts a situation where the moving set $C(t)$ is allowed to jump. This amounts to saying that the function $\varrho(\cdot)$ involved in (1.1) may have some discontinuities with respect to the time. Assuming that the function $\varrho(\cdot)$ is only right-continuous with bounded variation ([38, 34, 42]) leads to the following measure differential inclusion (see Section 3 for a precise meaning)

$$-du \in N_{C(t)}(u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I.$$

Another variant of a great interest is obtained by putting a state-dependence in the moving set, say

$$-\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I \tag{1.3}$$

or in its bounded variation form

$$-du \in N_{C(t,u(t))}(u(t)) + F(t, u(t)) \quad \text{a.e. } t \in I. \tag{1.4}$$

To the best of our knowledge, the dissertation thesis of K. Chraibi ([17]) contains the first work devoted to the (absolutely continuous) evolution problem (1.3) in the particular context of a closed convex $C(t,x) \subset \mathbb{R}^3$ and $F \equiv 0$. The existence result in [17] (with $F \equiv 0$) has been extended to Hilbert spaces by M. Kunze and M.D.P. Monteiro Marques ([31]). Their proof is strongly based on a generalized Schauder's fixed point theorem (see Section 4) which ensures the well-posedness of the implicit scheme

$$u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_{i+1}^n)}(u_i^n).$$

Let us point out here that (1.3) can also be handled without the use of any fixed point type result thanks to a semi-implicit algorithm (see, [17, 28, 6, 29, 40])

$$u_0^n := u_0 \quad \text{and} \quad u_{i+1}^n := \operatorname{proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n).$$

Unlike the original Moreau's sweeping process (1.2), any existence result in infinite dimensional setting for (1.3) makes a crucial use of a compactness type assumption. It remains an open question to establish their necessity or not.

One of the main area of research in sweeping process theory consists in going beyond the convexity of the involved moving set. It probably starts in 1988 with M. Valadier ([52]). With $\mathcal{H} = \mathbb{R}^n$, he established that (1.2) has a solution (with $N = N^C$ the Clarke normal cone) for a nonconvex set $C(t)$ provided that the multimapping $(t,x) \mapsto N^C_{C(t)}(x)$ has its graph closed. We also mention the work by C. Castaing ([14]) from which we can derive existence result for the case where $C(t)$ is a translation of an autonomous nonconvex set. Major developments have been done since the pioneer works of Castaing and Valadier. In the dimensional setting, it is known ([8, 21, 49]) that the closedness of $C(t)$ along with (1.1) ensure the existence of a solution for (1.2). What we know so far in a general Hilbert space are existence results for various classes of moving sets coming from variational analysis such as prox-regular ([21, 27, 42]), alpha-far ([30]) and subsmooth ([29]). In the present paper, the sets involved in the sweeping processes will be prox-regular.

Besides first order theory, C. Castaing introduced in [14] at the end of 80's the following second order evolution problem with outward normal at the velocity inside the set

$$-\ddot{u}(t) \in N_{C(u(t))}(\dot{u}(t)) \text{ with } u(T_0) = u_0 \text{ and } \dot{u}(T_0) = v_0 \in C(u_0). \tag{1.5}$$

For the other second order problem with outward normal at the state/position, that is, the differential inclusion $-\ddot{u}(t) \in N_{C(t)}(u(t))$, we refer to [45] and the references therein. As for the first order, in numerous ways and for various purposes (see, e.g., the monograph [34] and the references therein), many researchers studied (1.5) in the perturbed form with a possibly time dependence in the moving set

$$\begin{cases} -\ddot{u}(t) \in N_{C(t,u(t))}(\dot{u}(t)) + F(t,u(t),\dot{u}(t)), \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0). \end{cases}$$

Taking into account the problem (1.4), it is quite natural to consider the following measure differential inclusion

$$\begin{cases} -d\dot{u}(t) \in N_{C(t,u(t))}(\dot{u}(t)) + F(t,u(t),\dot{u}(t)), \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0). \end{cases} \tag{1.6}$$

The existence of solution has been developed in [15] for the first time in the absolutely continuous setting with a prox-regular moving set in a separable Hilbert space $\mathcal{H}$. The problem has also been investigated by S. Adly and B.K. Le ([2]) with a control on a convex set $C(t,x)$ involving only the truncated Hausdorff distance. The right-continuous bounded variation form appeared in [4], with a convex-valued multimapping F satisfying only a time-dependent growth condition

$$F(t,u,v) \subset \alpha(t)(1 + \|u\| + \|v\|)\mathbb{B},$$

for some fixed $\alpha(\cdot) \in L^1(I,\mathbb{R}_+)$, where $\mathbb{B}$ stands for the closed unit ball in $\mathcal{H}$. More recently, the evolution problem (1.6) has been examined in [5] with a subsmooth set in the context where the moving set is controlled in time by a continuous function with bounded variation. Using such a continuity assumption, the authors succeeded in adapting to the second-order setting the catching-up semi-implicit approach in [28, 29].

The papers [43, 53] brought to light a link beetween the above first and second order sweeping processes. Indeed, in an absolutely continuous setting with $\mathcal{H} = \mathbb{R}^n$, it is established in [43, 53] that the existence of solution for (1.6) can be obtained through a suitable first-order state-dependent sweeping process. The crucial assumption in those works seems to be the control of the minimal norm coming from the convex perturbation term, in the sense that

$$d\big(0, F(t,u,v)\big) \le \alpha$$

for some real $\alpha > 0$. The aim of the present paper is twofold. On one hand, we introduce a new mixed first order sweeping process (taking place in the product space $\mathcal{H}^2$)

$$(\mathcal{FMSP}) \begin{cases} -d\Phi \in N_{C(t,\Phi(t))\times Q}(\Phi(t)) + G(t,\Phi(t)) \times \{f(t,\Phi_1(t))\} \\ \Phi_1(T_0) = v_0, \Phi_2(T_0) = u_0, \end{cases}$$

for which we establish that it encompasses the BV evolution problems (1.4) and (1.6). On the other hand, we provide sufficient conditions ensuring the existence of solutions for $(\mathcal{FMSP})$. Here, our single-valued perturbation f is Lipschitz and G is convex-valued satisfying the following time-dependent minimal norm control

$$d\big(0, G(t,u,v)\big) \le \alpha(t)(1 + \|u\|),$$

for some $\alpha(\cdot) \in L^1(I, \mathbb{R}_+)$. The work is achieved thanks to a suitable mixed catching-up algorithm

$$\begin{cases} y_{p+1}^n = \operatorname{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s)\big), \\ x_{p+1}^n = \operatorname{proj}_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}\big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g(s, x_p^n, y_p^n) d\lambda(s)\big), \end{cases}$$

where $g(t,u,v)$ denotes the minimal norm element of $G(t,u,v)$. Then, from this existence result, we have been able to derive the existence of a trajectory solution for both above state-dependent first order and second order sweeping processes governed by a prox-regular moving set with bounded variation.

The paper is organized as follows. Section 2 is devoted to the introduction of notation and the necessary preliminaries. In Section 3, we develop the concept of solution for the discontinuous perturbed first and second order sweeping processes. In Sections 5 and 6, we focus on the existence of solution for such evolution problems.

2. Notation and preliminaries

Throughout, $\mathcal{H}$ is a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. The open (resp. closed) ball of $\mathcal{H}$ centered at $x \in \mathcal{H}$ with radius $r > 0$ is denoted by $B(x,r)$ (resp. $B[x,r]$). The letter $\mathbb{B}$ denotes the closed unit ball of $\mathcal{H}$, that is $\mathbb{B} := B[0,1]$. In all the paper, $I := [T_0, T]$ is an interval of $\mathbb{R}$ for some reals $T_0 < T$, λ stands for its Lebesgue measure. As usual, $\mathbb{N}$ denotes the set of integers starting from 1 and $\mathbb{R}_+ := [0, +\infty[$ the set of nonnegative reals.

Let S be nonempty subset of $\mathcal{H}$. The *distance function from* S is defined by

$$d_S(x) :=: d(x,S) := \inf_{y \in S} \|x - y\| \quad \text{for all } x \in \mathcal{H}.$$

For any $x \in \mathcal{H}$, the (possibly empty) set of *nearest points* of x in S is defined as

$$\operatorname{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

If $\operatorname{Proj}_S(x) = \{\overline{y}\}$ for some $\overline{y} \in S$, one says that $\operatorname{proj}_S(x)$ (or $P_S(x)$) is well-defined and in such a case one sets $\operatorname{proj}_S(x) := \overline{y}$ (or $P_S(x) := \overline{y}$).

2.1. **Normal cones and subdifferentials.**

Let S be a nonempty closed set of the Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is a *proximal normal vector to the set* $S \subset \mathcal{H}$ *at a point* $x \in S$ provided that (see, e.g., [35, 48, 51]) there exists a real $r > 0$ such that $x \in \operatorname{Proj}_S(x + rv)$. The set of all proximal normal vectors at x, denoted by $N_S^P(x)$ or $N^P(S;x)$, is a convex cone containing zero (not necessarily closed in $\mathcal{H}$). As usual, we set

$$N_S^P(x) := \emptyset \quad \text{for all } x \in \mathcal{H} \setminus S. \tag{2.1}$$

For each $v \in \mathcal{H}$ with $w \in \mathrm{Proj}_S(v) \neq \emptyset$, we may obviously write $w \in \mathrm{Proj}_S\big(w + (v - w)\big)$ and this ensures the crucial inclusion

$$v - w \in N_S^P(w). \tag{2.2}$$

The *Clarke tangent cone* $T_S(x)$ (denoted also by $T(S; x)$) of S at $x \in S$ is (see [19]) the set of $h \in \mathcal{H}$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ of S with $x_n \to x$, for every sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals with $t_n \to 0$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $h_n \to h$ satisfying

$$x_n + t_n h_n \in S \quad \text{for all } n \in \mathbb{N}.$$

It is an exercise to check that $T_S(x)$ is a closed convex cone containing 0. The *Clarke normal cone* of S at $x \in S$ is denoted by $N_S(x)$ or $N(S; x)$ and is defined as the polar cone of $T_S(x)$, i.e.,

$$N_S(x) := \{v \in \mathcal{H} : \langle v, h\rangle \leq 0, \forall h \in T_S(x)\}.$$

As in (2.1), one puts $T_S(x) := N_S(x) := \emptyset$ for every x outside S. It is routine to show that the proximal normal cone is always included in the Clarke one, i.e.,

$$N_S^P(x) \subset N_S(x) \quad \text{for all } x \in \mathcal{H}. \tag{2.3}$$

Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function defined on an open subset U of $\mathcal{H}$. Through the above concepts of normal cones, one defines the *proximal subdifferential* $\partial_P f(x)$ and the *Clarke subdifferential* $\partial f(x)$ of f at $x \in U$ by

$$\partial_P f(x) := \Big\{v \in \mathcal{H} : (v, -1) \in N_{E_f}^P\big(x, f(x)\big)\Big\} \tag{2.4}$$

and

$$\partial f(x) := \big\{v \in \mathcal{H} : (v, -1) \in N_{E_f}\big(x, f(x)\big)\big\}, \tag{2.5}$$

where $\mathcal{H} \times \mathbb{R}$ is endowed with the usual product structure and

$$E_f := \operatorname{epi} f := \{(u, r) \in U \times \mathbb{R} : f(u) \leq r\}.$$

It follows from the very definition of the latter subdifferentials that $\partial_P f(x) = \emptyset$ and $\partial f(x) = \emptyset$ whenever f is not finite at $x \in U$. From (2.4), (2.5) and (2.3), it is readily seen that

$$\partial_P f(x) \subset \partial f(x) \quad \text{for all } x \in U. \tag{2.6}$$

Of course, when U is convex and the function f is convex on U, the two latter subdifferentials coincide with the one in the sense of convex analysis, that is,

$$\partial_P f(x) = \partial f(x) = \big\{v \in \mathcal{H} : \big\langle v, x' - x\big\rangle \leq f(x') - f(x),\ \forall x' \in U\big\}. \tag{2.7}$$

If $f = d_S$ (here S is closed but possibly nonconvex) we have the following description of its proximal and Clarke subdifferential (see, e.g., [20, 11, 51])

$$\partial_P d_S(x) = N_S^P(x) \cap \mathbb{B} \quad \text{and} \quad \partial d_S(x) \subset N_S(x) \cap \mathbb{B} \quad \text{for all } x \in S. \tag{2.8}$$

For a function f which is Lipschitz near $x \in U$, it is known that (see [19, 51]) the Clarke subdifferential is nonempty, weakly compact and satisfies

$$\partial f(x) = \{v \in \mathcal{H} : \langle v, h\rangle \leq f^o(x; h)\, \forall h \in \mathcal{H}\},$$

where $f^o(x;h)$ is the *Clarke directional derivative* of f at x in the direction h defined by

$$f^o(x;h) := \limsup_{t\downarrow 0, x'\to x} t^{-1}\big(f(x'+th) - f(x')\big).$$

Under such a Lipschitz assumption, the Clarke directional derivative $f^o(x;h)$ is nothing but the support function of the closed convex set $\partial f(x)$. Recall that for any subset S of the Hilbert space $\mathcal{H}$, its *support function* $\sigma(\cdot, S)$ is defined by

$$\sigma(\xi, S) := \sup_{x\in S} \langle \xi, x\rangle \quad \text{for all } \xi \in \mathcal{H}.$$

As a direct consequence of the Hahn-Banach theorem, we see that the support function characterizes the closed convex sets of $\mathcal{H}$ since for every subsets S_1, S_2 of $\mathcal{H}$,

$$\overline{\mathrm{co}}\, S_1 \subset \overline{\mathrm{co}}\, S_2 \Leftrightarrow \sigma(\cdot, S_1) \leq \sigma(\cdot, S_2). \tag{2.9}$$

Here and below, co (resp. $\overline{\mathrm{co}}$) stands for the convex (resp. closed convex) hull of S. Through the support function, we define the concept of scalar upper semicontinuity as follows: a multimapping $F : \mathcal{T} \rightrightarrows \mathcal{H}$ from a Hausdorff topological space $\mathcal{T}$ to the Hilbert space $\mathcal{H}$ is said to be *scalarly upper semicontinuous* whenever, for any $\xi \in \mathcal{H}$, the extended real-valued function $\sigma(\xi, F(\cdot)) : \mathcal{T} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is upper semicontinuous.

2.2. **Prox-regularity in Hilbert spaces.** As mentionned above, we focus on evolution problems described through a prox-regular moving set, that is, a multimapping $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ with prox-regular values. Let us first recall the definition.

Definition 2.1. ([47]) Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0, +\infty]$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P_S(x) \cap \mathbb{B}$ and for all $t \in]0, r[$, one has $x \in \mathrm{Proj}_S(x + tv)$.

Now, let us provide some useful characterizations and properties of uniform prox-regular sets ([47]). The proofs as well as additional results, applications and historical comments can be found in the survey [22] and the book [51]. Before stating the next result, we need to recall that for any extended real $r > 0$, the *r-open enlargement* of a subset S of $\mathcal{H}$ is defined as the set

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}.$$

Theorem 2.2. *Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0, +\infty]$. Consider the following assertions.*

(a) The set S is r-prox-regular.
(b) For all $x_1, x_2 \in S$, for all $v \in N^P_S(x_1)$, one has

$$\langle v, x_2 - x_1 \rangle \leq \frac{1}{2r} \|v\| \, \|x_1 - x_2\|^2 .$$

(c) The mapping $\mathrm{proj}_S : U_r(S) \to S$ is well-defined and locally Lipschitz on $U_r(S)$.
(d) For all $u \in U_r(S) \setminus S$, one has

$$P_S(u) = P_S\Big(P_S(u) + t\frac{u - P_S(u)}{\|u - P_S(u)\|}\Big) \quad \textit{for all}\, t \in [0, r[.$$

(e) One has

$$N^P_S(x) = N_S(x) \quad \textit{for all}\, x \in S$$

and

$$\partial_P d_S(x) = \partial d_S(x) \quad \textit{for all}\, x \in U_r(S).$$

Then, the assertions (a), (b) and (c) are pairwise equivalent and each one implies both (d) and (e).

Let us end this paragraph with a crucial result established by M.V. Balashov and G.E. Ivanov [7] for projection onto prox-regular sets. We present it in a suitable form for the development of our analysis in the next section (see also [44]), and we sketch the proof for completeness.

Here and in the rest of the paper, $\mathrm{haus}(\cdot, \cdot)$ stands for the *Hausdorff-Pompeiu* distance on $\mathcal{H}$ which is defined for two nonempty subsets S and S' of $\mathcal{H}$ by

$$\mathrm{haus}(S, S') := \sup_{x \in S \cup S'} |d_S(x) - d_{S'}(x)| = \sup_{x \in \mathcal{H}} |d_S(x) - d_{S'}(x)| .$$

Theorem 2.3 ([7]). *Let S_1, S_2 be two r-prox-regular sets of $\mathcal{H}$ for some $r \in]0, +\infty]$, $s \in]0, r[$. If $\mathrm{haus}(S_1, S_2) < r$, then for every $x \in U_s(S_1) \cap U_s(S_2)$, one has*

$$\big\|\mathrm{proj}_{S_1}(x) - \mathrm{proj}_{S_2}(x)\big\| \leq \sqrt{2s\big(1 - \frac{s}{r}\big)^{-1} \mathrm{haus}(S_1, S_2)}.$$

Proof. Set $h := \mathrm{haus}(S_1, S_2)$ and assume that $h < r$. Fix any $x \in U_s(S_1) \cap U_s(S_2)$. For each $i \in \{1, 2\}$, set $x_i := \mathrm{proj}_{S_i}(x)$. Pick any $t \in [h, r[$. We claim that

$$2 \langle x - x_1, x_2 - x_1 \rangle \leq s\big(\frac{\|x_1 - x_2\|^2}{t} + 2h\big).$$

Without loss of generality, assume that $x \neq x_1$, in particular $x \in U_r(S_1) \setminus S_1$. By virtue of Theorem 2.2(d), we have

$$x_1 = \mathrm{proj}_{S_1}\Big(x_1 + \frac{t(x - x_1)}{\|x - x_1\|}\Big).$$

For all $z \in S_1$, we have

$$\left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - x_2\right\| \geq \left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - z\right\| - \|x_2 - z\| \geq t - \|x_2 - z\|.$$

Taking the supremum on both sides of the latter inequality ensures that

$$\left\|x_1 + \frac{t(x-x_1)}{\|x-x_1\|} - x_2\right\| \geq \sup_{z \in S_1} (t - \|x_2 - z\|) = t - d_{S_1}(x_2) \geq t - h.$$

We deduce from this (keeping in mind that $t \geq h$)

$$\|x_1 - x_2\|^2 + \frac{2t}{\|x-x_1\|} \langle x - x_1, x_1 - x_2\rangle + t^2 \geq t^2 - 2th,$$

or equivalently

$$2t \langle x - x_1, x_2 - x_1\rangle \leq \|x - x_1\| (\|x_1 - x_2\|^2 + 2th).$$

Using $d_{S_1}(x) = \|x - x_1\| < s$, we obtain

$$2 \langle x - x_1, x_2 - x_1\rangle \leq s\big(\frac{\|x_1 - x_2\|^2}{t} + 2h\big),$$

which is the inequality claimed. In a similar way, we see that

$$2 \langle x - x_2, x_1 - x_2\rangle \leq s\big(\frac{\|x_1 - x_2\|^2}{t} + 2h\big).$$

Adding the two latter inequalities yields

$$\|x_1 - x_2\|^2 \leq s\big(\frac{\|x_1 - x_2\|^2}{t} + 2h\big).$$

It remains to let $t \uparrow r$ to complete the proof. □

2.3. **BV multimappings and vector measure theory.** The present paper is devoted to the study of the following Moreau's sweeping process (see Section 3 for the detailed concept of solutions)

$$\begin{cases} -d\Phi \in N_{C(t,\Phi(t))\times Q}(\Phi(t)) + G(t,\Phi(t)) \times \{f(t,\Phi_1(t))\}, \\ \Phi(T_0) = (u_0, q_0). \end{cases} \tag{2.10}$$

Our existence result will require that there is a positive Radon measure μ on I such that for every $x, y \in \mathcal{H}$

$$\operatorname{haus}\big(C(s,x,y), C(t,x,y)\big) \leq \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \leq t. \tag{2.11}$$

Our first aim here is to show how the latter inequality (2.11) is strongly related to the notion of bounded variation for multimappings. Doing so, consider any multimapping $M : I = [T_0, T] \rightrightarrows \mathcal{H}$ and any real $\tau \in [T_0, T]$. Let σ be a *subdivision* of $[T_0, \tau]$, that is $\sigma = (t_0, \ldots, t_k)$ for some reals $T_0 = t_0 < \cdots < t_k = \tau$ with $k \in \mathbb{N}$. One associates to such a subdivision

σ, the real $h_\sigma := \sum_{i=0}^{k-1} \mathrm{haus}\big(M(t_i), M(t_{i+1})\big)$. The *variation* of M on $[T_0, \tau]$ is defined as the extended real

$$\mathrm{var}(M; [T_0, \tau]) := \sup_{\zeta \in \mathcal{S}_{[T_0,\tau]}} h_\zeta,$$

where $\mathcal{S}_{[T_0,\tau]}$ is the set of all subdivisions of $[T_0, \tau]$. The multimapping $M(\cdot)$ is said to be of *bounded variation* (BV for short) on $[T_0, \tau]$ if $\mathrm{var}(M; [T_0, \tau]) < +\infty$. Then, it is straightforward to check that the existence of a positive Radon measure μ on I satisfying

$$\mathrm{haus}\big(M(s), M(t)\big) \leq \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s < t \tag{2.12}$$

entails (keeping in mind that $\mathrm{haus}(\cdot,\cdot)$ satifies the triangle inequality) that

$$0 \leq \mathrm{var}(M; [T_0, t]) - \mathrm{var}(M; [T_0, \bar{t}]) \leq \mu(]\bar{t}, t]) \quad \text{for all } t \in]\bar{t}, T];$$

in particular $M(\cdot)$ has a bounded variation on I along with a variation function $\mathrm{var}(M; [T_0, \cdot])$ right-continuous on I (see below).

In order to develop the converse implication, we need to introduce the concept of differential measure. Let $u(\cdot) : I \to \mathcal{H}$ be a mapping. Assume for a moment that $M(t) = \{u(t)\}$ for every $t \in I$. From the equality

$$h_\sigma = \sum_{i=0}^{k-1} \mathrm{haus}\big(M(t_i), M(t_{i+1})\big) = \sum_{i=0}^{k-1} \|u(t_{i+1}) - u(t_i)\|,$$

it is clear that $M(\cdot)$ is of bounded variation if and only if $u(\cdot)$ is of bounded variation in the usual sense for mappings. In such a case, it is known (see, e.g., [24]) that $u(\cdot)$ has one-sided limits at each point of I denoted

$$u(\tau^-) := \lim_{t \uparrow \tau} u(t) \quad \text{for all } \tau \in]T_0, T],$$

and

$$u(\tau^+) := \lim_{t \downarrow \tau} u(t) \quad \text{for all } \tau \in [T_0, T[,$$

where in the whole paper, $t \uparrow \tau$ (resp. $t \downarrow \tau$) means $t \to \tau$ with $t < \tau$ (resp. with $t > \tau$). If in addition $u(\cdot)$ is right-continuous on I (i.e., $u(\tau) = u(\tau^+)$ for all $\tau \in [T_0, T[$) there exists a vector measure du on I called *differential measure* satisfying

$$du(]s,t]) = \int_{]s,t]} du = u(t) - u(s).$$

Now, let us come back to our problem with a general multimapping M by assuming that $M(\cdot)$ has a bounded variation on I along with a right-continuous variation function $\mathrm{var}(M; [T_0, \cdot])$ on I. Since the latter function is nondecreasing on I, it is of bounded variation on I, so if we denote by μ_M the differential measure associated with it, we have

$$\mathrm{var}(M; [T_0, t]) - \mathrm{var}(M; [T_0, s]) = \mu_M(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \leq t.$$

It follows that

$$\operatorname{haus}\big(M(s), M(t)\big) \le \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s < t,$$

that is, $M(\cdot)$ satisfies (2.12) with $\mu = \mu_M$.

Besides BV multimappings and differential mesures, some additional preliminaries on vector measure theory are also needed. Let us end this section by developing it.

Take any positive Radon measure ν on I and any real $p \ge 1$. We denote by $L^p(I, \mathcal{H}, \nu)$ the real space of (classes of) mappings from I to $\mathcal{H}$ which are ν-Bochner integrable on I. Recall that a (class of) mapping $f : I \to \mathcal{H}$ belongs to $L^p(I, \mathcal{H}, \nu)$ whenever if it is ν-Bochner (or strongly) measurable on I (see, e.g., [26, Chapter 2, Definition 1]) and $\int_I \|f\|^p \, d\nu < \infty$. For more details on Bochner integral, we refer the reader to [26] and the references therein.

Let $\nu, \hat{\nu}$ be two positive Radon measures on I. We recall (see, e.g., [24]) that, with $I(t,r) := I \cap [t-r, t+r]$ ($r > 0$ and $t \in I$) the limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\hat{\nu}(I(t,r))}{\nu(I(t,r))} \tag{2.13}$$

(with the convention $\frac{0}{0} = 0$) exists and is finite for ν-almost every $t \in I$. The (nonnegative Borel) function $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is called the *derivative of the measure* $\hat{\nu}$ *with respect to* ν. Moreover, the measure $\hat{\nu}$ is *absolutely continuous with respect to* ν if and only if $\hat{\nu} = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$ (i.e., $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a density of $\hat{\nu}$ relative to ν). If the latter equality holds, a mapping $u(\cdot) : I \to \mathcal{H}$ is $\hat{\nu}$-integrable on I if and only if $u(\cdot)\frac{d\hat{\nu}}{d\nu}(\cdot)$ is ν-integrable on I. In such a case, one has

$$\int_I u(t) d\hat{\nu}(t) = \int_I u(t) \frac{d\hat{\nu}}{d\nu}(t) d\nu(t).$$

If the two Radon measures ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other one, one says that ν and $\hat{\nu}$ are *absolutely continuously equivalent.*

It is worth pointing out that, taking $\hat{\nu}$ equal to the Lebesgue measure λ, the relation (2.13) gives

$$\frac{d\lambda}{d\nu}(t) = \frac{\lambda(\{t\})}{\nu(\{t\})} = 0 \quad \text{for all } t \in I \text{ with } \nu(\{t\}) > 0,$$

hence

$$\frac{d\lambda}{d\nu}(t)\nu(\{t\}) = 0 \quad \nu\text{-a.e. } t \in I.$$

Now, consider ν a positive Radon measure on I, $u(\cdot) : I \to \mathcal{H}$ a mapping and $\tilde{u}(\cdot) \in L^1(I, \mathcal{H}, \nu)$. If, for any $t \in I$,

$$u(t) = u(T_0) + \int_{]T_0, t]} \tilde{u} \, d\nu,$$

then $u(\cdot)$ is of bounded variation, right-continuous on I and

$$du = \tilde{u}\, d\nu.$$

In such a case, the mapping $\tilde{u}(\cdot)$ is said to be a *density of the measure du relative to ν*. According to J.J. Moreau and M.Valadier ([39]), for ν-almost every $t \in I$,

$$\tilde{u}(t) = \frac{du}{d\nu}(t) := \lim_{r\downarrow 0} \frac{du(I(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{du(I^+(t,r))}{\nu(I^+(t,r))} = \lim_{r\downarrow 0} \frac{du(I^-(t,r))}{\nu(I^-(t,r))},$$

where $I^-(t,r) = [t-r,t] \cap I$ and $I^+(t,r) = [t,t+r] \cap I$ for each $t \in I$ and each real $r > 0$.

Given a positive Radon measure ν on I we will also use the property that

$$\text{the set } \{t \in I : \nu(\{t\}) > 0\} \text{ is countable.} \tag{2.14}$$

This known property can be seen from the fact that the latter set coincides with $\bigcup_{k\in\mathbb{N}} A_k$, where $A_k := \{t \in I : \nu(\{t\}) > 1/k\}$ is a finite set for each $k \in \mathbb{N}$ since $\nu(I) < +\infty$.

3. Concept of solutions

This section is devoted to the concept of solutions for first and second order discontinuous sweeping processes. For more details on bounded variation solution of sweeping process, we refer the reader to [34, 50] and the references therein.

We start with the following definition which is a slight extension of [27, Definition 2.1] to the context of bounded variation state-dependent sweeping processes. Before giving it, we need to associate to a multimapping $M : I \times X \rightrightarrows Y$ with closed values, where X and Y are two normed vector spaces, the real ϱ_M defined by

$$\begin{cases} \varrho_M = 0 & \text{if } 0 \in M(t,x) \text{ for all } (t,x) \in I \times X, \\ \varrho_M = 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

Definition 3.1. Let $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping and let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping such that there exists some positive Radon measure μ on I satisfying for every $x \in \mathcal{H}$,

$$\operatorname{haus}\big(C(s,x), C(t,x)\big) \le \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \le t.$$

One says that a mapping $u : I \to \mathcal{H}$ is a solution of the $\mathcal{F}$irst order bounded variation state-dependent Moreau $\mathcal{S}$weeping $\mathcal{P}$rocess associated to μ for the initial condition $u_0 \in \mathcal{H}$ with $u_0 \in C(T_0, u_0)$

$$(\mathcal{FSP}) \begin{cases} -du \in N_{C(t,u(t))}(u(t)) + G(t,u(t)) \\ u(T_0) = u_0, \end{cases}$$

provided:
(a) the mapping $u(\cdot)$ is of bounded variation on I, right-continuous on I and

satisfies $u(T_0) = u_0$ and $u(t) \in C(t, u(t))$ for all $t \in I$;
(b) there exist a λ-Bochner integrable mapping $z(\cdot) : I \to \mathcal{H}$ with $z(t) \in G(t, u(t))$ for λ-almost every $t \in I$ and a positive Radon measure ν on I, absolutely continuously equivalent to $\mu + \varrho_G\lambda$ and with respect to which the differential measure du of u is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1(I, \mathcal{H}, \nu)$-density and such that

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N_{C(t,u(t))}(u(t)) \quad \nu\text{-a.e. } t \in I.$$

Let us mention that such a concept *does not depend* on the involved Radon measure ν, that is, a mapping $u(\cdot) : I \to \mathcal{H}$ satisfying (a) above is a solution of $(\mathcal{FSP})$ if and only if (b) holds for any positive Radon measure ν which is absolutely continuously equivalent to $\mu + \varrho_G\lambda$.

Now, let us focus on second order evolution problems. The following definition has been introduced in [4] as a careful adaptation of Definition 3.1 to the context of second order sweeping processes with outward normal at the velocity.

Definition 3.2. Let $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be a multimapping and let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping such that there exists a positive Radon measure μ on I satisfying for every $x \in \mathcal{H}$,

$$\text{haus}\big(C(s, x), C(t, x)\big) \le \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \le t.$$

One says that a mapping $u : I \to \mathcal{H}$ satisfies the $\mathcal{S}$econd order bounded variation $\mathcal{S}$weeping $\mathcal{P}$rocess associated to μ for the initial conditions $u_0, v_0 \in \mathcal{H}$ with $v_0 \in C(T_0, u_0)$

$$(\mathcal{SSP}) \begin{cases} -d\dot{u} \in N_{C(t,u(t))}(\dot{u}(t)) + G(t, u(t), \dot{u}(t)) \\ u(T_0) = u_0, \dot{u}(T_0) = v_0, \end{cases}$$

whenever:
(a) the mapping $u(\cdot)$ is absolutely continuous on I and $u(T_0) = u_0$;
(b) there exists a mapping $v : I \to \mathcal{H}$ (called derivative for $u(\cdot)$ relative to $(\mathcal{SSP})$) right-continuous with bounded variation such that $v(T_0) = v_0$, $v(t) \in C(t, u(t))$ for all $t \in I$ and $v(t) = \dot{u}(t)$ for λ-almost every $t \in I$;
(c) there exist a λ-Bochner integrable mapping $z : I \to \mathcal{H}$ with $z(t) \in G(t, u(t), v(t))$ for λ-almost every $t \in I$ and a positive Radon measure ν on I absolutely continuously equivalent to $\mu + \varrho_G\lambda$ with respect to which dv admits a density in $L^1(I, \mathcal{H}, \nu)$ such that

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N_{C(t,u(t))}(v(t)) \quad \nu\text{-a.e. } t \in I.$$

It is worth pointing out (as above) that such a concept of solution is independent of the involved Radon measure ν.

An important link between problems $(\mathcal{FSP})$ and $(\mathcal{SSP})$ is given through the following result. It asserts that the derivative of a solution of second order sweeping process is nothing but a solution of a first order Moreau sweeping process (independent of state). More precisely:

Proposition 3.3 ([4]). *Let $C : I \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be two multimappings. Let μ be a positive Radon measure on I such that*

$$\operatorname{haus}\big(C(s), C(t)\big) \leq \mu(]s,t]) \quad \text{for all}\, s,t \in I \text{ with } s \leq t.$$

If $u(\cdot) : I \to \mathcal{H}$ is a solution of the second order sweeping process associated to μ for the initial conditions $u_0, v_0 \in \mathcal{H}$ with $v_0 \in C(T_0)$

$$\begin{cases} -d\dot{u} \in N_{C(t)}(\dot{u}(t)) + G(t, \dot{u}(t)) \\ u(T_0) = u_0, \dot{u}(T_0) = v_0, \end{cases}$$

then there exists a solution $v : I \to \mathcal{H}$ of the first order sweeping process associated to μ for the initial condition $v_0 \in C(T_0)$

$$\begin{cases} -dv \in N_{C(t)}(v(t)) + G(t, v(t)) \\ v(T_0) = v_0 \end{cases}$$

such that

$$u(t) = u_0 + \int_{T_0}^{t} v(s) d\lambda(s) \quad \text{for all}\, t \in I.$$

The next definition gives the exact meaning of (2.10). Here and below, for a prescribed mapping $\Phi : I \to \mathcal{H}^2$, it is convenient to set $\Phi_i := \pi_i \circ \Phi$ for each $i \in \{1, 2\}$, where $\pi_i : \mathcal{H}^2 \to \mathcal{H}$ is defined by

$$\pi_i(x_1, x_2) := x_i \quad \text{for all } (x_1, x_2) \in \mathcal{H}^2.$$

Definition 3.4. Let $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping and let $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $D : I \rightrightarrows \mathcal{H}$ be two multimappings. Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be a multimapping such that there exists a positive Radon measure μ on I satisfying for every $(x, y) \in \mathcal{H}^2$,

$$\operatorname{haus}\big(C(s, x, y), C(t, x, y)\big) \leq \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

One says that a mapping $\Phi = (\Phi_1, \Phi_2) : I \to \mathcal{H}^2$ is a solution of the $\mathcal{F}$irst order $\mathcal{M}$ixed partially BV $\mathcal{S}$weeping $\mathcal{P}$rocess associated to μ for the initial conditions $u_0, v_0 \in \mathcal{H}$ with $(u_0, v_0) \in D(T_0) \times C(T_0, v_0, u_0)$

$$(\mathcal{FMSP}) \begin{cases} -d\Phi \in N_{C(t,\Phi(t)) \times D(t)}(\Phi(t)) + G(t, \Phi(t)) \times \{f(t, \Phi_1(t))\} \\ \Phi_1(T_0) = v_0, \Phi_2(T_0) = u_0, \end{cases}$$

whenever:
(a) the mapping $\Phi_1(\cdot)$ (resp. $\Phi_2(\cdot)$) is right-continuous with bounded variation (resp. absolutely continuous) on I, $\Phi(T_0) = (v_0, u_0)$, and $\Phi(t) \in C(t, \Phi(t)) \times D(t)$ for all $t \in I$;
(b) for λ-almost every $t \in I$,

$$\dot{\Phi}_2(t) + f(t, \Phi_1(t)) \in -N_{D(t)}(\Phi_2(t));$$

(c) there exist a λ-Bochner integrable mapping $z_1(\cdot) : I \to \mathcal{H}$ with $z_1(t) \in G(t, \Phi(t))$ for λ-almost all $t \in I$ and a positive Radon measure ν absolutely continuously equivalent to $\mu + \varrho_P \lambda$ where $P : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ is the multimapping defined by

$$P(t,x,y) := G(t,x,y) \times \{f(t,x)\} \quad \text{for all } (t,x,y) \in I \times \mathcal{H}^2$$

such that

$$\frac{d\Phi_1}{d\nu}(t) + z_1(t)\frac{d\lambda}{d\nu}(t) \in -N_{C(t,\Phi(t))}(\Phi_1(t)) \quad \nu\text{-a.e. } t \in I.$$

Assume that the positive Radon measure μ on I involved above is absolutely continuously equivalent to the Lebesgue measure λ on I. Then, a mapping $\Phi = (\Phi_1, \Phi_2) : I \to \mathcal{H}^2$ satisfies $(\mathcal{FSMP})$ if and only if

$$\begin{cases} -\dot{\Phi}(t) \in N_{C(t,\Phi(t)) \times D(t)}(\Phi(t)) + G(t, \Phi(t)) \times \{f(t, \Phi_1(t))\}, \\ \Phi_1(T_0) = v_0, \Phi_2(T_0) = u_0 \end{cases}$$

i.e., the following conditions hold:
(a') the mappings Φ_1, Φ_2 are absolutely continuous on I;
(b') for λ-almost every $t \in I$,

$$\dot{\Phi}_2(t) + f(t, \Phi_1(t)) \in -N_{D(t)}(\Phi_2(t)).$$

(c') there exist a λ-Bochner integrable mapping $z_1 : I \to \mathcal{H}$ with $z_1(t) \in G(t, \Phi(t))$ for λ-almost every $t \in I$ such that

$$\dot{\Phi}_1(t) + z_1(t) \in -N_{C(t,\Phi(t))}(\Phi_1(t)) \quad \lambda\text{-a.e. } t \in I.$$

4. Preparatory results

In the present section, we list for sake of completeness the technical results which will be necessary in order to establish our main existence theorem.

Let us start with the following classical Gronwall lemma and its discrete version.

Lemma 4.1 (Gronwall's inequality)**.** *Let $\varphi : [T_0, T] \to \mathbb{R}$ be an absolutely continuous function on $[T_0, T]$, $a : [T_0, T] \to \mathbb{R}$ and $b : [T_0, T] \to \mathbb{R}$ be Lebesgue integrable functions on $[T_0, T]$. If for λ-almost every $t \in [T_0, T]$,*

$$\dot{\varphi}(t) \le b(t) + a(t)\varphi(t),$$

then for all $t \in [T_0, T]$,

$$\varphi(t) \le \varphi(T_0) \exp\left(\int_{T_0}^t a(s) d\lambda(s)\right) + \int_{T_0}^t b(\tau) \exp\left(\int_\tau^t a(s) d\lambda(s)\right) d\lambda(\tau).$$

Lemma 4.2 (Discrete version of Gronwall's inequality)**.** *Let $A \geq 0$ be a real and let $(v_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ be two sequences of nonnegative reals such that*

$$v_n \leq A + \sum_{k=0}^{n-1} B_k v_k \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Then, one has

$$v_n \leq A \exp\left(\sum_{k=0}^{n-1} B_k\right) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

The celebrated Schauder's fixed point theorem (see, e.g., [23, Theorem 8.8]) will also be used in the proof of our main result (Theorem 5.1).

Theorem 4.3 (Schauder's fixed point)**.** *Let C be a nonempty closed bounded convex subset of $\mathcal{H}$ and $f : C \to C$ be a continuous mapping. If $f(C)$ is relatively compact, then f has a fixed point.*

The scalar upper semicontinuity provided by the next proposition will be fundamental. It can be seen as an adaptation of [4, Proposition 2.2]. Before giving it, the following lemma is needed.

Lemma 4.4 ([27])**.** *Let S a subset of the real Hilbert space $\mathcal{H}$ which is r-prox-regular, with $r \in]0, +\infty]$. Let $x \in S$ and $\zeta \in \partial_P d_S(x)$. Then, for all $z \in \mathcal{H}$ such that $d_S(z) < r$, one has*

$$\langle \zeta, z - x\rangle \leq \frac{1}{2r}\|z - x\|^2 + \frac{1}{2r} d_S^2(z) + \left(\frac{1}{r}\|z - x\| + 1\right) d_S(z),$$

and

$$\langle \zeta, z - x\rangle \leq \frac{2}{r}\|z - x\|^2 + d_S(z).$$

Now, we can state and prove:

Proposition 4.5. *Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be a multimapping satisfying:*
(i) there exists an extended real $r \in]0, +\infty]$ such that for all $(t, x) \in I \times \mathcal{H}$, $C(t, x, y)$ is r-prox-regular;
(ii) there exist a positive measure μ on I, a norm $\|\cdot\|_{\mathcal{H}^2}$ on $\mathcal{H}^2$ and a function $\varphi : \mathcal{H}^2 \times \mathcal{H}^2 \to [0, +\infty[$ with $\lim\limits_{\|X-Y\|_{\mathcal{H}^2}\to 0} \varphi(X, Y) = 0$ such that for all $s, t \in I$ with $s \leq t$, all $u \in \mathcal{H}$ and all $X, Y \in \mathcal{H}^2$,

$$d\big(u, C(t, X)\big) - d\big(u, C(s, Y)\big) \leq \mu(]s, t]) + \varphi(X, Y).$$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of I converging to some $t \in I$ with $t_n \geq t$ for all $n \in \mathbb{N}$ and $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}^2$ converging to some $x \in C(t, x, y)$ and such that $x_n \in C(t_n, x_n, y_n)$ for all $n \in \mathbb{N}$.

If there exists $N \in \mathbb{N}$ with $\mu(]t, t_N]) < +\infty$, then for any $z \in \mathcal{H}$, one has

$$\limsup_{n\to\infty} \sigma\big(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)\big) \leq \sigma\big(z, \partial_P d_{C(t,x,y)}(x)\big).$$

Proof. Assume there exists $N \in \mathbb{N}$ with $\mu(]t, t_N]) < +\infty$. Fix any $z \in \mathcal{H}$. We may assume that the sequence $\big(\sigma(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)\big)_{n\in\mathbb{N}}$ converges. Doing so, we have

$$\limsup_{n\to\infty} \sigma(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)) = \lim_{n\to\infty} \sigma(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)).$$

From assumption (i) and Theorem 2.2(e), we get

$$\partial_P d_{C(t_n,x_n,y_n)}(x_n) = \partial d_{C(t_n,x_n,y_n)}(x_n) \quad \text{for all } n \in \mathbb{N}.$$

In particular, for every $n \in \mathbb{N}$, the set $\partial_P d_{C(t_n,x_n,y_n)}(x_n)$ is weakly compact, hence there is $\xi_n \in \partial_P d_{C(t_n,x_n,y_n)}(x_n)$ such that $\sigma(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)) = \langle \xi_n, z\rangle$. Thanks to the inequality $\|\xi_n\| \le 1$ for all $n \in \mathbb{N}$ (see (2.8)), we may assume that $(\xi_n)_{n\in\mathbb{N}}$ converges weakly to some $\xi \in \mathcal{H}$. Let us establish that $\xi \in \partial d_{C(t,x,y)}(x)$. Fix any $u \in \mathcal{H}$. As $x_n \in C(t_n, x_n, y_n)$ for all $n \in \mathbb{N}$, there exists a real $\alpha_0 > 0$ such that for all $\alpha \in]0, \alpha_0[$ and all $n \in \mathbb{N}$,

$$d_{C(t_n,x_n,y_n)}(x_n + \alpha u) \le \|\alpha u\| < r.$$

This allows us to apply Lemma 4.4 to obtain for all $\alpha \in]0, \alpha_0[$ and for all $n \in \mathbb{N}$,

$$\langle \xi_n, \alpha u\rangle \le \frac{2}{r}\alpha^2 \|u\|^2 + d_{C(t_n,x_n,y_n)}(x_n + \alpha u). \tag{4.1}$$

On the other hand, the assumption (ii) gives for all $\alpha \in]0, \alpha_0[$ and all $n \in \mathbb{N}$,

$$d_{C(t_n,x_n,y_n)}(x_n + \alpha u) \le d_{C(t,x,y)}(x_n + \alpha u) + \mu(]t, t_n]) + \varphi(x_n, y_n, x, y).$$

Extracting a subsequence if necessary, we may suppose that $(t_n)_{n\in\mathbb{N}}$ is non-increasing, so

$$\lim_{n\to\infty} \mu(]t, t_n]) = \mu\Big(\bigcap_{k\in\mathbb{N}}]t, t_k]\Big) = 0.$$

It follows that for all $\alpha \in]0, \alpha_0[$, $\limsup_{n\to\infty} d_{C(t_n,x_n,y_n)}(x_n + \alpha u) \le d_{C(t,x,y)}(x + \alpha u)$. Using (4.1), we obtain for all $\alpha \in]0, \alpha_0[$

$$\langle \xi, \alpha u\rangle \le \frac{2}{r}\alpha^2 \|u\|^2 + d_{C(t,x,y)}(x + \alpha u).$$

Combining the latter inequality with $d_{C(t,x,y)}(x) = 0$, we arrive to

$$\langle \xi, u\rangle \le \liminf_{\alpha\downarrow 0} \frac{1}{\alpha}\big(d_{C(t,x,y)}(x + \alpha u) - d_{C(t,x,y)}(x)\big) \le d^\circ_{C(t,x,y)}(x; u).$$

This being true for any $u \in \mathcal{H}$, it results that

$$\xi \in \partial d_{C(t,x,y)}(x) = \partial_P d_{C(t,x,y)}(x),$$

hence

$$\lim_{n\to\infty} \sigma(z, \partial_P d_{C(t_n,x_n,y_n)}(x_n)) = \lim_{n\to\infty} \langle \xi_n, z\rangle = \langle \xi, z\rangle \le \sigma\big(z, \partial_P d_{C(t,x,y)}(x)\big).$$

The proof is then complete. □

The following proposition is a slight variant of a previous result stated and proved in [41, Remark 7.4]. The closed unit ball of a real normed space $(Z, \|\cdot\|_Z)$ is denoted $\mathbb{B}_Z$.

Proposition 4.6. *Let X, Y be two real normed spaces and $C : I \times X \rightrightarrows Y$ be a multimapping. Assume that there exists a positive measure μ on I such that*

$$\operatorname{haus}\big(C(s,x), C(t,x)\big) \le \mu(]s,t]),$$

for all $s, t \in I$ with $s \le t$ and $x \in X$. Consider the following assertions.
(a) For every real $\tau > 0$, every bounded set $A \subset X$ and every $t \in I$, $C(t, A) \cap \tau\mathbb{B}_Y$ is relatively compact.
(b) For every real $\tau > 0$ and every bounded set $A \subset X$, $C(I \times A) \cap \tau\mathbb{B}_Y$ is relatively compact.
(c) Given any real $\tau > 0$, any sequence $(t_n)_{n\in\mathbb{N}}$ in I tending to t with $t_n \ge t$ for some $t \in I$ and any bounded sequence $(x_n)_{n\in\mathbb{N}}$ of X, then every sequence $(y_n)_{n\in\mathbb{N}}$ of Y with $y_n \in C(t_n, x_n) \cap \tau\mathbb{B}_Y$ for all $n \in \mathbb{N}$ has a convergent subsequence in Y.

Then, the implications $(b) \Rightarrow (a) \Leftrightarrow (c)$ hold. Further, if $\mu(\{t\}) = 0$ for every $t \in I$, then the first implication is an equivalence.

Proof. The implication $(b) \Rightarrow (a)$ is obvious. Assume (c) and fix any real $\tau > 0$, any $t \in I$ and any nonempty bounded set $A \subset X$. Let any sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in C(t, A) \cap \tau\mathbb{B}_Y$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ choosing $x_n \in A$ with $y_n \in C(t, x_n) \cap \tau\mathbb{B}_Y$, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, so by (c) the sequence $(y_n)_{n\in\mathbb{N}}$ admits a convergent subsequence. Then, $C(t, A) \cap \tau\mathbb{B}_Y$ is relatively compact, so the implication $(c) \Rightarrow (a)$ is proved. To prove the converse implication, assume that (a) holds and take any real $\tau > 0$, any sequence $(t_n)_{n\in\mathbb{N}}$ in I tending to t with $t_n \ge t$ for some $t \in I$, any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X and any sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in C(t_n, x_n)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ the inequality

$$\operatorname{haus}\big(C(t, x_n), C(t_n, x_n)\big) \le \mu(]t, t_n])$$

furnishes some $z_n \in C(t, x_n)$ such that

$$\|y_n - z_n\|_Y \le \mu(]t, t_n]) + 2^{-n} =: \varepsilon_n$$

and such an inequality entails that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded. Here and below, $\|\cdot\|_Y$ stands for the norm on Y. Consequently, there is a real $\kappa > 0$ such that for all $n \in \mathbb{N}$,

$$y_n \in C(t, x_n) \cap \kappa\mathbb{B}_Y + \varepsilon_n\mathbb{B}_Y.$$

Using this and the bounded set $A := \{x_n : n \in \mathbb{N}\}$ we obtain some sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{B}_Y$ such that

$$y_n + \varepsilon_n b_n \in C(t, A) \cap \kappa\mathbb{B}_Y \quad \text{for all } n \in \mathbb{N}.$$

Then the relative compactness of $C(t,A)\cap\kappa\mathbb{B}_Y$ by (a) combined with the (strong) convergence $\varepsilon_n b_n\to 0$ (due to $\mu(]t,t_n]\to 0)$ ensures that $(y_n)_{n\in\mathbb{N}}$ admits a convergent subsequence. This justifies the implication $(a)\Rightarrow(c)$.

Finally, assume that the μ-measures of singletons of I are null and let us prove in this case that $(a)\Rightarrow(b)$. Fix any real $\tau>0$ and any nonempty bounded set $A\subset X$. Take any sequence $(y_n)_{n\in\mathbb{N}}$ in $C(I\times A)\cap\tau\mathbb{B}_Y$, so for each $n\in\mathbb{N}$ there is some $t_n\in I$ and $x_n\in A$ with $y_n\in C(t_n,x_n)\cap\tau\mathbb{B}_Y$. Let $t\in I$ be a cluster point of $(t_n)_{n\in\mathbb{N}}$. There exists a subsequence $(t_{s(n)})_{n\in\mathbb{N}}$ tending to t with either $t_{s(n)}\geq t$ for all $n\in\mathbb{N}$ or $t_{s(n)}<t$ for all $n\in\mathbb{N}$. In the first situation where $t_{s(n)}\geq t$ for all $n\in\mathbb{N}$, (keeping in mind $(a)\Leftrightarrow(c)$) the assertion (c) says that $(y_{s(n)})_{n\in\mathbb{N}}$ has a convergent subsequence. Suppose that $t_{s(n)}<t$ for all $n\in\mathbb{N}$. As above, for each $n\in\mathbb{N}$ by the inequality

$$\mathrm{haus}\big(C(t_{s(n)},x_{s(n)}),C(t,x_{s(n)})\big)\leq\mu(]t_{s(n)},t])$$

we can choose some $z_n\in C(t,x_{s(n)})$ such that

$$\big\|y_{s(n)}-z_n\big\|_Y\leq\mu(]t_{s(n)},t])+2^{-n}=:\varepsilon_n,$$

and clearly $\varepsilon_n\to 0$ since $\mu(\{t\})=0$. The latter inequality tells us that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded, hence there is a real $\kappa>0$ such that for all $n\in\mathbb{N}$,

$$y_{s(n)}\in C(t,x_{s(n)})\cap\kappa\mathbb{B}_Y+\varepsilon_n\mathbb{B}_Y.$$

Using this we obtain some sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{B}_Y$ such that

$$y_{s(n)}+\varepsilon_n b_n\in C(t,A)\cap\kappa\mathbb{B}_Y\quad\text{for all } n\in\mathbb{N}.$$

Since $\varepsilon_n b_n\to 0$, the relative compactness of $C(t,A)\cap\kappa\mathbb{B}_Y$ entails that $(y_{s(n)})_{n\in\mathbb{N}}$ has a convergent subsequence. Consequently, in any case the sequence $(y_n)_{n\in\mathbb{N}}$ admits a convergent subquence, so $C(I\times A)\cap\tau\mathbb{B}_Y$ is relatively compact. This confirms the implication $(a)\Rightarrow(b)$ and finishes the proof. $\square$

5. Existence of solution for mixed partially BV state-dependent sweeping process

This section is devoted to the development of sufficient conditions ensuring existence of solutions for the first order mixed partially BV sweeping process

$$-d\Phi\in N_{C(t,\Phi(t))\times Q}(\Phi(t))+G(t,\Phi(t))\times\{f(t,\Phi_1(t))\}.$$

Before giving the main result in that direction, let us introduce for a multimapping $P:I\times\mathcal{H}^2\rightrightarrows\mathcal{H}$ its associated mapping of minimal norm $m_P:I\times\mathcal{H}^2\rightrightarrows\mathcal{H}$ defined by

$$m_P(t,x,y):=\mathrm{proj}_{P(t,x,y)}(0)\quad\text{for all } (t,x,y)\in I\times\mathcal{H}^2.$$

Now, we are in position to prove the following existence result.

Theorem 5.1. *Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings and $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Let Q be a closed convex subset of $\mathcal{H}$, $(u_0, q_0) \in \mathcal{H} \times Q$ with $u_0 \in C(T_0, u_0, q_0)$.*

Assume that:

(i) there exists $r \in]0, +\infty]$ such that for every $t \in I$ and every $x, y \in \mathcal{H}$, the set $C(t, x, y)$ is r-prox-regular;

(ii) there exist two reals $L \in]0, 1[$, $L' \geq 0$ and a positive Radon measure μ on I with $\sup\limits_{s \in]T_0, T]} \mu(\{s\}) < (1 - L)r$ such that

$$\operatorname{haus}\big(C(t_1, x_1, y_1), C(t_2, x_2, y_2)\big) \leq \mu(]t_1, t_2]) + L \|x_1 - x_2\| + L' \|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $x_1, x_2, y_1, y_2 \in \mathcal{H}$;

(iii) the mapping $f(\cdot, x)$ is λ-Bochner measurable on I for each $x \in \mathcal{H}$, there exists a real $\beta \geq 0$ such that

$$\|f(t, x)\| \leq \beta(1 + \|x\|) \quad \textit{for all } t \in I, x \in \mathcal{H},$$

and for each bounded subset B of $\mathcal{H}$ there exists a real $l_B \geq 0$ satisfying

$$\left\|f(t, x) - f(t, x')\right\| \leq l_B \left\|x - x'\right\| \quad \textit{for all } t \in I, x, x' \in B;$$

(iv) the multimapping G is nonempty closed convex valued, $G(t, \cdot, \cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $(x, y) \in \mathcal{H}^2$, the mapping $m_G(\cdot, x, y) : I \to \mathcal{H}$ is λ-Bochner measurable on I and there exists a nonnegative function $\alpha(\cdot) \in L^1(I, \lambda, \mathbb{R}_+)$ such that

$$\|m_G(t, x, y)\| = d(0, G(t, x, y)) \leq \alpha(t)(1 + \|x\|),$$

for all $t \in I$, $x, y \in \mathcal{H}$ with $(x, y) \in C(t, x, y) \cap \kappa\mathbb{B} \times Q \cap \kappa'\mathbb{B}$, where

$$\kappa' := \|q_0\| + 2\beta(1 + \kappa)(T - T_0)$$

and

$$\kappa := \Big(\|u_0\| + \frac{\mu(]T_0, T])}{1 - L} + s\Big)e^s \quad \textit{with} \quad s := \frac{2}{1 - L} \int_{T_0}^{T} \big(\alpha(s) + L'\beta\big) d\lambda(s);$$

(v) for every bounded subset B of $\mathcal{H}$, the set $C(I, B \times \kappa'\mathbb{B}) \cap B$ is relatively compact.

Then, there exists a solution $\Phi(\cdot) : I \to \mathcal{H}^2$ of the mixed partially BV differential inclusion

$$(\mathcal{P}) \quad \begin{cases} -d\Phi \in N_{C(t, \Phi(t)) \times Q}(\Phi(t)) + G(t, \Phi(t)) \times \{f(t, \Phi_1(t))\}, \\ \Phi(T_0) = (u_0, q_0). \end{cases}$$

Further, Φ_2 is $2\beta(1 + \kappa)$-Lipschitz continuous on I.

Proof. Let us first define the multimapping $P : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ by

$$P(t, x, y) := G(t, x, y) \times \{f(t, x)\} \quad \text{for all } (t, x, y) \in I \times \mathcal{H}^2.$$

For each $(t, x, y) \in I \times \mathcal{H}^2$ denote by $g(t, x, y)$ the element of minimal norm of the nonempty closed convex set $G(t, x, y)$ of $\mathcal{H}$, that is,

$$g(t, x, y) := m_G(t, x, y) = \operatorname{proj}_{G(t, x, y)}(0).$$

Observe that the inequality in assumption (ii) can be rewritten as

$$
\begin{aligned}
&\big|d\big(z_1, C(t_1,x_1,y_1)\big) - d\big(z_2, C(t_2,x_2,y_2)\big)\big| \\
&\le \|z_1 - z_2\| + \mu\,(]t_1,t_2]) + L\|x_1 - x_2\| + L'\|y_1 - y_2\|, \qquad (5.1)
\end{aligned}
$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathcal{H}$. On one hand, the assumption (iii) gives (see (3.1))

$$\|f(t,x)\| \le \beta\varrho_P(1 + \|x\|) \quad \text{for all } (t,x) \in I \times \mathcal{H}. \tag{5.2}$$

On the other hand, the assumption (iv) ensures for any $t \in I$ and any $x, y \in \mathcal{H}$ with $x \in C(t,x,y) \cap \kappa\mathbb{B}$ and $y \in Q \times \kappa'\mathbb{B}$ that

$$\|g(t,x,y)\| = d(0, G(t,x,y)) \le \varrho_P\alpha(t)(1 + \|x\|). \tag{5.3}$$

Hence in particular $g(\cdot,x,y)$ is λ-Bochner integrable according to the λ-Bochner measurability of $m_G(\cdot,x,y) = g(\cdot,x,y)$ on I. Let ν be the positive Radon measure on I absolutely continuously equivalent to the measure $\mu + \varrho_P\lambda$ defined by

$$\nu := \frac{\mu + 2\varrho_P(1+\kappa)\big(\alpha(\cdot) + (L'+1)(\beta+1)\big)\lambda}{1 - L}. \tag{5.4}$$

Now, consider the function $v(\cdot) : I \to \mathbb{R}$ defined by

$$v(t) := \nu(]T_0, t]) \quad \text{for all } t \in I$$

and set

$$V := v(T) = \nu(]T_0, T]).$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive reals with $\varepsilon_n \downarrow 0$ and a real $0 < r' < r$ such that

$$\varepsilon_n + (1-L)^{-1} \sup_{s\in]T_0,T]} \mu(\{s\}) < r' \quad \text{for all } n \in \mathbb{N}. \tag{5.5}$$

Step 1. Time discretization.

As in J.J. Moreau [38], choose for each $n \in \mathbb{N}$, $0 = V_0^n < V_1^n < \ldots < V_{q_n}^n = V$ (with $q_n \in \mathbb{N}$) such that

(a) for all $j \in \{0, \ldots, q_n - 1\}$, $V_{j+1}^n - V_j^n \le \varepsilon_n$;

(b) for all $k \in \mathbb{N}$, $\{V_0^k, \ldots, V_{q_k}^k\} \subset \Big\{V_0^{k+1}, \ldots, V_{q_{k+1}}^{k+1}\Big\}$.

For each $n \in \mathbb{N}$, set $V_{1+q_n}^n := V + \varepsilon_n$ and consider the partition $(J_j^n)_{j\in\{0,\ldots,q_n-1\}}$ of I where for each $j \in \{0, \ldots, q_n\}$

$$J_j^n := v^{-1}\big(\big[V_j^n, V_{j+1}^n\big[\big) = \{t \in I : V_j^n \le \nu(]T_0,t]) < V_{j+1}^n\}.$$

Observe that $(J_j^m)_{0\le j\le q_m}$ is a refinement of $(J_j^n)_{0\le j\le q_n}$ for all $m, n \in \mathbb{N}$ with $m \ge n$. Using the fact that $v(\cdot)$ is nondecreasing and right-continuous on I, it is not difficult to see that, for each $n \in \mathbb{N}$, $j \in \{0, \ldots, q_n - 1\}$, the set J_j^n is either empty or an interval of the form $[a, b[$ with $a < b$. Furthermore, for each $n \in \mathbb{N}$ we have $J_{q_n}^n$ of the form $J_{q_n}^n = [a, b[\cap I$. This gives for each $n \in \mathbb{N}$ an integer $k_n \in \mathbb{N}$ and a finite sequence

$$T_0 = t_0^n < \ldots < t_{k_n}^n = T$$

such that for each $i \in \{0, \ldots, k_n - 2\}$, there is some $j \in \{0, \ldots, q_n - 1\}$ satisfying $J_j^n = \left[t_i^n, t_{i+1}^n\right[$ and such that for $i = k_n - 1$ the interval $[t_{k_n-1}^n, t_{k_n}^n[$ is either $J_{q_n}^n \setminus \{T\}$ (if $J_{q_n}^n \neq \{T\}$) or J_k^n for some $k \in \{0, \ldots, q_n - 1\}$. For each integer n, put

$$\eta_i^n := \int_{t_i^n}^{t_{i+1}^n} (\alpha(s) + L'\beta) d\lambda(s). \tag{5.6}$$

Without loss of generality (including new points if necessary) we may and do suppose that for every $n \in \mathbb{N}$,

$$\max_{i \in \{0, \cdots, k_n - 1\}} (t_{i+1}^n - t_i^n) \leq \frac{1}{n} \text{ and } \{t_0^{n+1}, \cdots, t_{k_{n+1}}^{n+1}\} \supset \{t_0^n, \cdots, t_{k_n}^n\}.$$

Note that $(k_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence and that for each $n \in \mathbb{N}$, for all $i \in \{0, \ldots, k_n - 1\}$ and all $t \in \left[t_i^n, t_{i+1}^n\right[$, we have

$$\nu(]t_i^n, t]) = v(t) - v(t_i^n) \leq \varepsilon_n,$$

hence

$$\nu(\left]t_i^n, t_{i+1}^n\right[) \leq \varepsilon_n \quad \text{for all } i \in \{0, \ldots, k_n - 1\}. \tag{5.7}$$

Step 2. Construction of sequences $(x_p^n)_{0 \leq p \leq k_n}$ and $(y_p^n)_{0 \leq p \leq k_n}$.

Fix for a moment any integer $n \in \mathbb{N}$. Put $x_0^n := u_0 \in C(T_0, u_0, q_0) \cap \kappa\mathbb{B}$ and $y_0^n := q_0 \in Q \cap \kappa'\mathbb{B}$. We are going to construct by (finite) induction $x_0^n, x_1^n, \ldots, x_{k_n}^n, y_0^n, \ldots, y_{k_n}^n \in \mathcal{H}$ satisfying for each $p \in \{0, \ldots, k_n - 1\}$ the following relations with $\sigma_p^n := \int_{t_p^n}^{t_{p+1}^n} g(s, x_p^n, y_p^n) d\lambda(s)$

$$\begin{cases} y_{p+1}^n = \operatorname{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s)\big), \\ x_{p+1}^n = \operatorname{proj}_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}\big(x_p^n - \sigma_p^n\big) \\ \left\|y_{p+1}^n - y_p^n\right\| \leq 2\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_p^n), \\ (x_{p+1}^n, y_{p+1}^n) \in \kappa\mathbb{B} \times \kappa'\mathbb{B}, \end{cases} \tag{5.8}$$

along with

$$\left\|x_{p+1}^n\right\| \leq \left\|x_p^n\right\| + \frac{\mu(]t_p^n, t_{p+1}^n]) + 2\varrho_P \eta_p^n}{1 - L} + \frac{2}{1 - L} \eta_p^n \left\|x_p^n\right\|. \tag{5.9}$$

Set $y_1^n := \operatorname{proj}_Q\big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) d\lambda(s)\big)$ and observe that

$$\begin{aligned} \|y_1^n - y_0^n\| &\leq \left\|y_1^n - \big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) d\lambda(s)\big)\right\| + \left\|\int_{t_0^n}^{t_1^n} f(s, x_0^n) d\lambda(s)\right\| \\ &\leq d_Q\big(y_0^n - \int_{t_0^n}^{t_1^n} f(s, x_0^n) d\lambda(s)\big) + \int_{t_0^n}^{t_1^n} \|f(s, x_0^n)\| \, d\lambda(s) \\ &\leq 2 \int_{t_0^n}^{t_1^n} \|f(s, x_0^n)\| \, d\lambda(s) \\ &\leq 2\beta\varrho_P(1 + \|x_0^n\|)(t_1^n - t_0^n) \leq 2\beta\varrho_P(1+\kappa)(t_1^n - t_0^n), \end{aligned} \tag{5.10}$$

where the third inequality is due to $y_0^n \in Q$, the fourth to (5.2) and the last one to the inclusion $x_0^n \in \kappa\mathbb{B}$. Then, we see that

$$\|y_1^n\| \leq \|y_0^n\| + 2\beta\varrho_P(1+\kappa)(t_1^n - t_0^n) \leq \kappa'. \tag{5.11}$$

Now, for the construction of x_1^n we will use a method inspired by the ones used in [15] and [44]. Putting together (5.1), the inclusion $(x_0^n, y_0^n) \in C(t_0^n, x_0^n, y_0^n) \cap \kappa\mathbb{B} \times Q \cap \kappa'\mathbb{B}$, (5.10), (5.3), (5.4), (5.7) and (5.5), we see that for every vector $v \in B\left[u_0, \nu\left(]t_0^n, t_1^n]\right)\right]$,

$$\begin{aligned}
&d\big(x_0^n - \sigma_0^n, C(t_1^n, v, y_1^n)\big)\\
\leq & d\big(x_0^n - \sigma_0^n, C(t_0^n, x_0^n, y_0^n)\big) + \mu(]t_0^n, t_1^n]) + L\|v - x_0^n\| + L'\|y_1^n - y_0^n\|\\
\leq & \int_{t_0^n}^{t_1^n} \|g\left(s, x_0^n, y_0^n\right)\| \, d\lambda(s) + \mu\left(]t_0^n, t_1^n]\right) + L\nu\left(]t_0^n, t_1^n]\right)\\
&+ 2L'\beta\varrho_P(1+\kappa)(t_1^n - t_0^n)\\
\leq & \varrho_P(1+\kappa)\int_{t_0^n}^{t_1^n} (\alpha(s) + 2L'\beta)\, d\lambda(s) + \mu\left(]t_0^n, t_1^n[\right)\\
&+ L\nu\left(]t_0^n, t_1^n[\right) + \mu(\{t_1^n\}) + L\nu(\{t_1^n\})\\
\leq & (1-L)\frac{\varrho_P(1+\kappa)\int_{t_0^n}^{t_1^n}(\alpha(s) + 2L'\beta) d\lambda(s) + \mu\left(]t_0^n, t_1^n[\right)}{1-L}\\
&+ L\nu\left(]t_0^n, t_1^n[\right) + (1 + \frac{L}{1-L})\mu(\{t_1^n\})\\
\leq & (1 - L + L)\nu\left(]t_0^n, t_1^n[\right) + (1-L)^{-1}\mu(\{t_1^n\})\\
\leq & \varepsilon_n + (1-L)^{-1} \sup_{s \in]T_0, T]} \mu(\{s\}) < r' < r. \qquad (5.12)
\end{aligned}$$

Then, the uniform prox-regularity of constant r of the set $C\big(t_1^n, v, y_1^n\big)$ for each $v \in B[u_0, \nu\left(]t_0^n, t_1^n]\right)]$ and Theorem 2.2 allow us to define the mapping $\varphi_1^n : B[u_0, \nu\left(]t_0^n, t_1^n]\right)] \to \mathcal{H}$ by

$$\varphi_1^n(v) := \mathrm{proj}_{C(t_1^n, v, y_1^n)}\big(x_0^n - \sigma_0^n\big) \text{ for all } v \in B\left[u_0, \nu\left(]t_0^n, t_1^n]\right)\right].$$

Now, we show that $\varphi_1^n(\cdot)$ is continuous on $B[u_0, \nu(]t_0^n, t_1^n])]$. Fix any $\overline{x} \in B[u_0, \nu(]t_0^n, t_1^n])]$. According to (ii) we have for all $x \in B[u_0, \nu(]t_0^n, t_1^n])]$ with $\|x - \overline{x}\| < L^{-1}r$

$$\mathrm{haus}\big(C(t_1^n, x, y_1^n), C(t_1^n, \overline{x}, y_1^n)\big) \leq L\,\|x - \overline{x}\| < r.$$

Thanks to (5.12), we also have the inclusion

$$x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) d\lambda(s) \in U_{r'}\big(C(t_1^n, \overline{x}, y_1^n)\big) \cap U_{r'}\big(C(t_1^n, x, y_1^n)\big),$$

for all $x \in B[u_0, \nu(]t_0^n, t_1^n])]$. Then, we can apply Theorem 2.3 to get for all $x \in B[u_0, \nu(]t_0^n, t_1^n])]$ with $\|x - \overline{x}\| < L^{-1}r$

$$\|\varphi_1^n(x) - \varphi_1^n(\overline{x})\| \leq \sqrt{2Lr'(1 - \frac{r'}{r})^{-1}}\, \|x - \overline{x}\|^{1/2}.$$

Consequently, the mapping $\varphi_1^n(\cdot)$ is continuous as claimed above. From the definition of $\varphi_1^n(\cdot)$, (5.1), (5.3), the inclusion $x_0^n \in C(t_0^n, x_0^n, y_0^n) \cap \kappa\mathbb{B}$, (5.10) and the definition of ν, observe that for all $v \in B[u_0, \nu(]t_0^n, t_1^n])]$,

$$\begin{aligned}
&\|\varphi_1^n(v) - u_0\| \\
&\leq \big\|\varphi_1^n(v) - \big(u_0 - \sigma_0^n\big)\big\| + \|\sigma_0^n\| \\
&\leq d\big(x_0^n - \sigma_0^n, C(t_1^n, v, y_1^n)\big) + \int_{t_0^n}^{t_1^n} \|g\,(s, x_0^n, y_0^n)\|\, d\lambda(s) \\
&\leq d\big(x_0^n - \sigma_0^n, C(t_0^n, x_0^n, y_0^n)\big) + \mu\,(]t_0^n, t_1^n]) + L\,\|v - x_0^n\| \\
&\quad + L'\|y_1^n - y_0^n\| + \varrho_P(1 + \|x_0^n\|) \int_{t_0^n}^{t_1^n} \alpha(s)\, d\lambda(s) \\
&\leq 2\varrho_P(1 + \|x_0^n\|) \int_{t_0^n}^{t_1^n} (\alpha(s) + L'\beta)\, d\lambda(s) + \mu\,(]t_0^n, t_1^n]) + L\,\|v - x_0^n\| \qquad (5.13) \\
&\leq 2\varrho_P(1 + \kappa) \int_{t_0^n}^{t_1^n} (\alpha(s) + L'\beta)\, d\lambda(s) + \mu\,(]t_0^n, t_1^n]) + L\nu\,(]t_0^n, t_1^n]) \\
&\leq (1 - L)\nu\,(]t_0^n, t_1^n]) + L\nu\,(]t_0^n, t_1^n]) = \nu\,(]t_0^n, t_1^n])\,,
\end{aligned}$$

hence for all $v \in B[u_0, \nu(]t_0^n, t_1^n])]$,

$$\varphi_1^n(v) \in C\Big(t_1^n, B[u_0, \nu(]t_0^n, t_1^n])], y_1^n\Big) \cap B[u_0, \nu(]t_0^n, t_1^n])].$$

We derive from the latter inclusion, (5.11) and the assumption (v) that the set $\varphi_1^n\Big(B[u_0, \nu(]t_0^n, t_1^n])]\Big)$ is relatively compact. By virtue of Schauder's fixed point theorem recalled above (see Section 4), we know that $\varphi_1^n(\cdot)$ has a fixed point x_1^n in $B[u_0, \nu(]t_0^n, t_1^n])]$, i.e.,

$$\begin{cases} x_1^n = \mathrm{proj}_{C(t_1^n, x_1^n, y_1^n)}\big(x_0^n - \int_{t_0^n}^{t_1^n} g(s, x_0^n, y_0^n) d\lambda(s)\big) = \varphi_1^n(x_1^n), \\ \|x_1^n - x_0^n\| \leq \nu\,(]t_0^n, t_1^n])\,. \end{cases}$$

Applying (5.13) with $v = x_1^n$ yields

$$(1 - L)\,\|x_1^n - x_0^n\| \leq \mu(]t_0^n, t_1^n]) + 2\varrho_P(1 + \|x_0^n\|) \int_{t_0^n}^{t_1^n} (\alpha(s) + L'\beta)\, d\lambda(s).$$

Thus, we have (keeping in mind the definition of η_0^n in (5.6))

$$\begin{aligned}
\|x_1^n\| &\leq \|x_0^n\| + \frac{\mu(]t_0^n, t_1^n]) + 2\varrho_P\eta_0^n}{1 - L} + \frac{2}{1 - L}\eta_0^n\|x_0^n\| \\
&\leq \left(\|x_0^n\| + \frac{\mu(]t_0^n, t_1^n]) + 2\eta_0^n}{1 - L}\right)\left(1 + \frac{2}{1 - L}\eta_0^n\right) \\
&\leq \left(\|x_0^n\| + \frac{\mu(]t_0^n, t_1^n]) + 2\eta_0^n}{1 - L}\right)\exp\left(\frac{2}{1 - L}\eta_0^n\right) \leq \kappa.
\end{aligned}$$

Now, let $p \in \{1, \ldots, k_n - 1\}$. Assume that $x_0^n, \ldots, x_p^n$ and $y_0^n, \ldots, y_p^n$ have been constructed, so that properties in (5.8) and (5.9) hold true. Set $y_{p+1}^n :=$

$\mathrm{proj}_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s,x_p^n)d\lambda(s)\big)$ and note that

$$\begin{aligned}
\left\|y_{p+1}^n - y_p^n\right\| &\leq \left\|y_{p+1}^n - \big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s,x_p^n)d\lambda(s)\big)\right\| + \left\|\int_{t_p^n}^{t_{p+1}^n} f(s,x_p^n)d\lambda(s)\right\| \\
&\leq d_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s,x_p^n)d\lambda(s)\big) + \int_{t_p^n}^{t_{p+1}^n} \left\|f(s,x_p^n)\right\| d\lambda(s) \\
&\leq 2\int_{t_p^n}^{t_{p+1}^n} \left\|f(s,x_p^n)\right\| d\lambda(s) \leq 2\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_p^n).
\end{aligned}$$

The latter inequality with the second inequality of (5.8) ensure that

$$\begin{aligned}
\left\|y_{p+1}^n\right\| &\leq \left\|y_p^n\right\| + 2\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_p^n) \\
&\leq \left\|y_{p-1}^n\right\| + 2\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_{p-1}^n) \\
&\vdots \\
&\leq \|y_0^n\| + 2\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_0^n) \leq \kappa'.
\end{aligned}$$

Let us focus on x_{p+1}^n. Taking into account the inclusion $x_p^n \in C(t_p^n, x_p^n, y_p^n) \cap \kappa\mathbb{B}$, we may proceed as above to get for any $v \in B\left[x_p^n, \nu\left(\left]t_p^n, t_{p+1}^n\right]\right)\right]$,

$$\begin{aligned}
&d\left(x_p^n - \sigma_p^n, C\left(t_{p+1}^n, v, y_{p+1}^n\right)\right) \\
\leq& d\left(x_p^n - \sigma_p^n, C\left(t_p^n, x_p^n, y_p^n\right)\right) + \mu\left(\left]t_p^n, t_{p+1}^n\right]\right) \\
&+ L\left\|v - x_p^n\right\| + L'\|y_{p+1}^n - y_p^n\| \\
\leq& \int_{t_p^n}^{t_{p+1}^n} \left\|g\left(s, x_p^n, y_p^n\right)\right\| d\lambda(s) + \mu\left(\left]t_p^n, t_{p+1}^n\right]\right) \\
&+ L\nu\left(\left]t_p^n, t_{p+1}^n\right]\right) + 2L'\beta\varrho_P(1+\kappa)(t_{p+1}^n - t_p^n) \\
\leq& \varrho_P(1+\kappa)\int_{t_p^n}^{t_{p+1}^n} \alpha(s)\, d\lambda(s) + \mu(\{t_{p+1}^n\}) + \mu\left(\left]t_p^n, t_{p+1}^n\right[\right) \\
&+ L\nu(\{t_{p+1}^n\}) + L\nu\left(\left]t_p^n, t_{p+1}^n\right[\right) + 2L'\varrho_P(1+\kappa)\int_{t_p^n}^{t_{p+1}^n} \beta\, d\lambda(s) \\
\leq& (1-L)\frac{2\varrho_P(1+\kappa)\int_{t_p^n}^{t_{p+1}^n}\left(\alpha(s) + L'\beta\right)d\lambda(s) + \mu\left(\left]t_p^n, t_{p+1}^n\right[\right)}{1-L} \\
&+ L\nu\left(\left]t_p^n, t_{p+1}^n\right[\right) + (1-L)^{-1}\mu(\{t_{p+1}^n\}) \\
\leq& \nu\left(\left]t_p^n, t_{p+1}^n\right[\right) + (1-L)^{-1}\mu(\{t_{p+1}^n\}) \\
\leq& \varepsilon_n + (1-L)^{-1}\sup_{s\in]T_0,T]}\mu(\{s\}) < r' < r.
\end{aligned}$$

Using the prox-regularity of $C\big(t_{p+1}^n, v, y_{p+1}^n\big)$ for each $v \in \mathcal{H}$ and Theorem 2.2, we may define the mapping $\varphi_{p+1}^n : B[x_p^n, \nu(]t_p^n, t_{p+1}^n])] \to \mathcal{H}$

$$\varphi_{p+1}^n(v) := \mathrm{proj}_{C(t_{p+1}^n, v, y_{p+1}^n)}\big(x_p^n - \sigma_p^n\big),$$

for all $v \in B[x_p^n, \nu(]t_p^n, t_{p+1}^n])]$. In the same way as above for φ_1^n, we establish that the mapping φ_{p+1}^n is continuous on $B[x_p^n, \nu(]t_p^n, t_{p+1}^n])]$. Further, for all $v \in B[x_p^n, \nu(]t_p^n, t_{p+1}^n])]$, we have

$$\begin{aligned}
&\left\|\varphi_{p+1}^n(v) - x_p^n\right\| \\
\le &\left\|\varphi_{p+1}^n(v) - \left(x_p^n - \sigma_p^n\right)\right\| + \left\|\sigma_p^n\right\| \\
\le &d\Big(x_p^n - \sigma_p^n, C\big(t_{p+1}^n, v, y_{p+1}^n\big)\Big) + \int_{t_p^n}^{t_{p+1}^n} \left\|g\big(s, x_p^n, y_p^n\big)\right\| d\lambda(s) \\
\le &d\Big(x_p^n - \sigma_p^n, C\big(t_p^n, x_p^n, y_p^n\big)\Big) + \mu\left(]t_p^n, t_{p+1}^n]\right) + L\left\|v - x_p^n\right\| \\
&+ L'\|y_{p+1}^n - y_p^n\| + \varrho_P(1 + \|x_p^n\|) \int_{t_p^n}^{t_{p+1}^n} \alpha(s)\, d\lambda(s) \\
\le &2\varrho_P(1 + \|x_p^n\|) \int_{t_p^n}^{t_{p+1}^n} (\alpha(s) + L'\beta) d\lambda(s) \\
&+ \mu(]t_p^n, t_{p+1}^n]) + L\left\|v - x_p^n\right\| \\
\le &2\varrho_P(1 + \kappa) \int_{t_p^n}^{t_{p+1}^n} \big(\alpha(s) + L'\beta\big)\, d\lambda(s) \\
&+ \mu\left(]t_p^n, t_{p+1}^n]\right) + L\nu\left(]t_p^n, t_{p+1}^n]\right) \\
\le &(1 - L)\nu\left(]t_p^n, t_{p+1}^n]\right) + L\nu\left(]t_p^n, t_{p+1}^n]\right) = \nu\left(]t_p^n, t_{p+1}^n]\right),
\end{aligned} \tag{5.14}$$

hence

$$\varphi_{p+1}^n(v) \in C\Big(t_{p+1}^n, B[x_p^n, \nu\big(]t_p^n, t_{p+1}^n]\big)], y_{p+1}^n\Big) \cap B[x_p^n, \nu\big(]t_p^n, t_{p+1}^n]\big)].$$

Keeping in mind that the set in the right-hand side of the latter inclusion is relatively compact, we can apply Schauder's fixed point theorem to obtain a fixed point $x_{p+1}^n \in B[x_p^n, \nu\left(]t_p^n, t_{p+1}^n]\right)]$ of $\varphi_{p+1}^n(\cdot)$, otherwise stated

$$\begin{cases} x_{p+1}^n = \mathrm{proj}_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}\Big(x_p^n - \int_{t_p^n}^{t_{p+1}^n} g\left(s, x_p^n, y_p^n\right) d\lambda(s)\Big) = \varphi_{p+1}^n(x_{p+1}^n), \\ \left\|x_{p+1}^n - x_p^n\right\| \le \nu\left(]t_p^n, t_{p+1}^n]\right). \end{cases}$$

Further, it follows from (5.14), taking $v = x_{p+1}^n$, that

$$(1-L)\|x_{p+1}^n - x_p^n\| \le 2\varrho_P(1+\|x_p^n\|) \int_{t_p^n}^{t_{p+1}^n} (\alpha(s) + L'\beta)\, d\lambda(s) + \mu\left(]t_p^n, t_{p+1}^n]\right).$$

Thus, we have

$$
\begin{aligned}
\left\|x_{p+1}^n\right\| &\leq \left\|x_p^n\right\| + \frac{\mu(]t_p^n, t_{p+1}^n]) + 2\eta_p^n}{1-L} + \frac{2}{1-L}\eta_p^n\|x_p^n\| \\
&\leq \left\|x_{p-1}^n\right\| + \frac{\mu(]t_p^n, t_{p+1}^n]) + \mu(]t_{p-1}^n, t_p^n]) + 2(\eta_p^n + \eta_{p-1}^n)}{1-L} \\
&\quad + \frac{2}{1-L}(\eta_p^n\|x_p^n\| + \eta_{p-1}^n\|x_{p-1}^n\|) \\
&\vdots \\
&\leq \|x_0^n\| + \frac{\mu(]t_0^n, t_{p+1}^n]) + 2\sum_{i=0}^p \eta_i^n}{1-L} + \frac{2}{1-L}\sum_{i=0}^p \|x_i^n\|\eta_i^n \\
&\leq A + \frac{2}{1-L}\sum_{i=0}^p \|x_i^n\|\eta_i^n,
\end{aligned}
$$

where

$$
A := \|u_0\| + \frac{\mu(]T_0, T]) + 2\int_{T_0}^T \left(\alpha(s) + L'\beta\right) d\lambda(s)}{1-L}.
$$

It remains to apply Lemma 4.2 (see also [18]) to get that

$$
\|x_{p+1}^n\| \leq A \exp\left(\frac{2}{1-L}\sum_{i=0}^p \eta_i^n\right) \leq \kappa,
$$

to complete the induction.

Now, let $n \in \mathbb{N}$. Coming back to (5.8), we get (thanks to the inclusion (2.2)) for all $p \in \{0, \ldots, k_n - 1\}$,

$$
-x_{p+1}^n + x_p^n - \int_{t_p^n}^{t_{p+1}^n} g\left(s, x_p^n, y_p^n\right) d\lambda(s) \in N_{C\left(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n\right)}\left(x_{p+1}^n\right).
$$

On the other hand, using the second equality in (5.8), assumption (ii), the inclusion $x_p^n \in C(t_p^n, x_p^n, y_p^n) \cap \kappa\mathbb{B}$, (5.3), the inequalities in (5.8) and the definition of ν, it results that for all $p \in \{0, \ldots, k_n - 1\}$,

$$\begin{aligned}
&\left\|x_{p+1}^n - x_p^n + \sigma_p^n\right\| \\
=& d\Big(x_p^n - \sigma_p^n, C\big(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n\big)\Big) \\
\leq& d\Big(x_p^n - \sigma_p^n, C\big(t_p^n, x_p^n, y_p^n\big)\Big) + \mu\left(\left]t_p^n, t_{p+1}^n\right]\right) \\
&+ L\left\|x_{p+1}^n - x_p^n\right\| + L'\|y_{p+1}^n - y_p^n\| \\
\leq& \varrho_P(1+\kappa)\int_{t_p^n}^{t_{p+1}^n} \alpha(s)\, d\lambda(s) + \mu\left(\left]t_p^n, t_{p+1}^n\right]\right) \\
&+ L\nu\left(\left]t_p^n, t_{p+1}^n\right]\right) + 2\varrho_P L'(1+\kappa)\int_{t_p^n}^{t_{p+1}^n} \beta\, d\lambda(s) \\
\leq& 2\varrho_P(1+\kappa)\int_{t_p^n}^{t_{p+1}^n} \big(\alpha(s) + L'\beta\big)\, d\lambda(s) \\
&+ \mu\left(\left]t_p^n, t_{p+1}^n\right]\right) + L\nu\left(\left]t_p^n, t_{p+1}^n\right]\right) \\
\leq& (1-L)\nu\left(\left]t_p^n, t_{p+1}^n\right]\right) + L\nu\left(\left]t_p^n, t_{p+1}^n\right]\right) = \nu\left(\left]t_p^n, t_{p+1}^n\right]\right). \qquad (5.15)
\end{aligned}$$

Step 3. Construction of sequences $(u_n(\cdot))_{n\in\mathbb{N}}$ and $(q_n(\cdot))_{n\in\mathbb{N}}$.

Fix any integer $n \in \mathbb{N}$. Let us define the mapping $u_n(\cdot) : I \to \mathcal{H}$ by $u_n(T) := x_{k_n}^n$ and for each $t \in [t_p^n, t_{p+1}^n[$ with $p \in \{0, \ldots, k_n - 1\}$,

$$u_n(t) := x_p^n + \frac{\nu(]t_p^n, t])}{\nu(]t_p^n, t_{p+1}^n])}\left(x_{p+1}^n - x_p^n + \sigma_p^n\right) - \int_{t_p^n}^{t} g\left(s, x_p^n, y_p^n\right) d\lambda(s)$$

if $\nu(]t_p^n, t_{p+1}^n]) > 0$ and

$$u_n(t) := x_p^n \quad \text{if } \nu(]t_p^n, t_{p+1}^n]) = 0.$$

Fix for a moment any $p \in \{0, \ldots, k_n - 1\}$. Assume first that $\nu(]t_p^n, t_{p+1}^n]) = 0$. In view of the definitions of ν and ϱ_P (see (5.4) and (3.1)), we must have $f \equiv 0$ and $g \equiv 0$. Combining (2.3) and the equalities in (5.8), we see that $y_p^n = y_{p+1}^n$ and since $\mu(]t_p^n, t_{p+1}^n]) = 0$ (by definition of ν)

$$\begin{aligned}
\left\|x_{p+1}^n - x_p^n\right\| &= d_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}(x_p^n) \\
&\leq \operatorname{haus}(C(t_p^n, x_p^n, y_p^n), C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)) \leq L\left\|x_p^n - x_{p+1}^n\right\|.
\end{aligned}$$

Since $L < 1$, the latter inequality $\|x_{p+1}^n - x_p^n\| \leq L\|x_p^n - x_{p+1}^n\|$ entails that $x_p^n = x_{p+1}^n$. Hence, if $\nu(]t_p^n, t_{p+1}^n]) = 0$, we have $g \equiv 0$, $\sigma_p^n = 0$ and

$$u_n(t) = x_p^n = x_{p+1}^n \quad \text{for all } t \in [t_p^n, t_{p+1}^n].$$

If $\nu(]t_p^n, t_{p+1}^n]) > 0$, it is not difficult to see, for all $t \in [t_p^n, t_{p+1}^n]$,

$$u_n(t) = x_p^n + \frac{\nu(]t_p^n, t])}{\nu(]t_p^n, t_{p+1}^n])}\left(x_{p+1}^n - x_p^n + \sigma_p^n\right) - \int_{t_p^n}^{t} g\left(s, x_p^n, y_p^n\right) d\lambda(s).$$

Now, let us consider the mapping $z_n : I \to \mathcal{H}$ defined for all $t \in I$ by

$$z_n(t) = \begin{cases} g(t, x_p^n, y_p^n) & \text{if } t \in [t_p^n, t_{p+1}^n[, \\ g(T, x_{k_n}^n, y_{k_n}^n) & \text{if } t = T. \end{cases} \tag{5.16}$$

Let us also consider the mapping $\Lambda_n : I \to \mathcal{H}$ with $\Lambda_n(T_0) := 0$, defined on $]t_p^n, t_{p+1}^n]$ with $\nu(]t_p^n, t_{p+1}^n]) = 0$ by $\Lambda_n(t) := 0$ for all $t \in]t_p^n, t_{p+1}^n]$, and on $]t_p^n, t_{p+1}^n]$ with $\nu(]t_p^n, t_{p+1}^n]) > 0$ by

$$\Lambda_n(t) := \frac{x_{p+1}^n - x_p^n + \sigma_p^n}{\nu\big(]t_p^n, t_{p+1}^n]\big)} \quad \text{for all } t \in]t_p^n, t_{p+1}^n].$$

It can be checked that $\Lambda_n(\cdot)$ (resp. $z_n(\cdot)$) is ν-Bochner measurable on I and such a property combined with (5.15) (resp. assumption (iv)) gives the ν-Bochner integrability of $\Lambda_n(\cdot)$ (resp. $z_n(\cdot)$) on I. Consequently, we may write for all $t \in I$,

$$u_n(t) = u_0 + \int_{]T_0,t]} \Lambda_n(s) d\nu(s) - \int_{]T_0,t]} z_n(s) d\lambda(s), \tag{5.17}$$

and this says in particular (see Section 2) that the mapping $u_n(\cdot)$ is right-continuous with bounded variation on I. Assume for a moment that $\varrho_P \neq 0$. From (5.4), we see that λ is absolutely continuous relative to ν, which ensures (see again Section 2) that $\frac{d\lambda}{d\nu} \in L^\infty(I, \mathbb{R}_+, \nu)$ is a density of λ relative to ν, hence the equality (5.17) yields

$$u_n(t) = u_0 + \int_{]T_0,t]} \left(\Lambda_n(s) - z_n(s) \frac{d\lambda}{d\nu}(s) \right) d\nu(s) \quad \text{for all } t \in I.$$

Note that such an equality still holds if $\varrho_P = 0$ (thanks to the equality $z_n(t) = 0$ for each $t \in I$ and to (5.17)). Then, du_n has $\Lambda_n(\cdot) - z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot)$ as a density in $L^1(I, \mathcal{H}, \nu)$ relative to ν, which allows us to write

$$\frac{du_n}{d\nu}(t) = \Lambda_n(t) - z_n(t) \frac{d\lambda}{d\nu}(t) \quad \nu\text{-a.e. } t \in I. \tag{5.18}$$

On the other hand, from the definition of $\Lambda_n(\cdot)$, (5.8), (2.2) and the fact that the proximal normal cone always contains zero, we have for every $t \in]t_p^n, t_{p+1}^n]$ with $p \in \{0, \ldots, k_n - 1\}$,

$$\Lambda_n(t) \in -N^P_{C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n)}(x_{p+1}^n) \tag{5.19}$$

and by (5.15), (5.18) and the definition of $\Lambda_n(\cdot)$ again

$$\|\Lambda_n(t)\| = \left\| \frac{du_n}{d\nu}(t) + z_n(t) \frac{d\lambda}{d\nu}(t) \right\| \leq 1 \quad \nu\text{-a.e. } t \in I. \tag{5.20}$$

Now, let us define the mapping $q_n : I \to \mathcal{H}$ by setting

$$q_n(t) := y_p^n + \frac{t - t_p^n}{t_{p+1}^n - t_p^n} \big(y_{p+1}^n - y_p^n + \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s) \big) - \int_{t_p^n}^{t} f(s, x_p^n) d\lambda(s)$$

for all $t \in [t_p^n, t_{p+1}^n]$ with $p \in \{0, \ldots, k_n - 1\}$. Fix for a moment any $p \in \{0, \ldots, k_{n-1}\}$. It is clear that $q_n(t_p^n) = y_p^n$ and that $q_n(\cdot)$ is absolutely continuous on I and that for each $p \in \{0, \ldots, k_n - 1\}$ we have

$$\dot{q}_n(t) = \frac{1}{t_{p+1}^n - t_p^n}\Big(y_{p+1}^n - y_p^n + \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s)\Big) - f(t, x_p^n) \tag{5.21}$$

for λ-almost every $t \in]t_p^n, t_{p+1}^n[$. On the other hand, for λ-almost every $t \in]t_p^n, t_{p+1}^n[$ we have the estimate (see (5.8) and (5.2))

$$\begin{aligned}
\|\dot{q}_n(t)\| &\leq \frac{1}{t_{p+1}^n - t_p^n}\left\| y_{p+1}^n - \big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s)\big)\right\| + \left\|f(t, x_p^n)\right\| \\
&\leq \frac{1}{t_{p+1}^n - t_p^n} d_Q\big(y_p^n - \int_{t_p^n}^{t_{p+1}^n} f(s, x_p^n) d\lambda(s)\big) + \beta(1 + \left\|x_p^n\right\|) \\
&\leq \frac{1}{t_{p+1}^n - t_p^n} \int_{t_p^n}^{t_{p+1}^n} \left\|f(s, x_p^n)\right\| d\lambda(s) + \beta(1 + \kappa) \\
&\leq 2\beta(1 + \kappa), \qquad (5.22)
\end{aligned}$$

and this guarantees the $2\beta(1 + \kappa)$-Lipschitz property of the mapping $q_n(\cdot)$ on I. In particular, we must have

$$\|q_n(t)\| \leq \|q_n(T_0)\| + 2\beta(1 + \kappa)(t - T_0) \leq \kappa' \quad \text{for all } t \in I.$$

It follows from (5.22), (5.2) and the inclusion $x_p^n \in \kappa\mathbb{B}$ that, for λ-almost every $t \in I$

$$\left\|\dot{q}_n(t) + f(t, x_p^n)\right\| \leq \|\dot{q}_n(t)\| + \left\|f(t, x_p^n)\right\| \leq 2\beta(1+\kappa) + \beta(1+\kappa) =: c. \tag{5.23}$$

Using (5.8) and (5.21), we observe that

$$\dot{q}_n(t) + f(t, x_p^n) \in -N_Q(y_{p+1}^n) \quad \lambda\text{-a.e. } t \in [t_p^n, t_{p+1}^n]. \tag{5.24}$$

Now, let us consider the mappings $\theta_n, \delta_n : I \to I$ defined by $\delta_n(T) := t_{k_n-1}^n$, $\theta_n(T) := T$ and for all $t \in \big[t_p^n, t_{p+1}^n\big[$ with $p \in \{0, \ldots, k_n - 1\}$,

$$\delta_n(t) := t_p^n \quad \text{and} \quad \theta_n(t) := t_{p+1}^n.$$

It is routine to check that

$$\delta_n(t) \uparrow t \quad \text{and} \quad \theta_n(t) \downarrow t \quad \text{for all } t \in I.$$

By the definition of $u_n(\cdot)$ and (5.8), we see that for all $p \in \{0, \ldots, k_n - 1\}$

$$u_n(t_{p+1}^n) = x_{p+1}^n \in C(t_{p+1}^n, x_{p+1}^n, y_{p+1}^n) = C\big(t_{p+1}^n, u_n(t_{p+1}^n), q_n(t_{p+1}^n)\big),$$

and this can be rewritten as

$$u_n(\theta_n(t)) \in C\big(\theta_n(t), u_n(\theta_n(t)), q_n(\theta_n(t))\big) \quad \text{for all } t \in I. \tag{5.25}$$

According to (5.18), the inclusion (5.19) can also be rewritten as

$$\Lambda_n(t) = \frac{du_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) \in -N_{C\big(\theta_n(t), u_n(\theta_n(t)), q_n(\theta_n(t))\big)}\big(u_n(\theta_n(t))\big) \quad \nu\text{-a.e.} t \in I.$$

Putting together the latter inclusion with (5.20) and (2.8), we arrive to

$$\Lambda_n(t) \in -\partial_P d_{C\left(\theta_n(t),u_n(\theta_n(t)),q_n(\theta_n(t))\right)}\big(u_n(\theta_n(t))\big) \quad \nu\text{-a.e. } t \in I. \tag{5.26}$$

On the other hand, the inclusion (5.24) yields

$$\dot{q_n}(t) + f\big(t, u_n(\delta_n(t))\big) \in -N_Q\big(q_n(\theta_n(t))\big) \quad \lambda\text{-a.e.} t \in I. \tag{5.27}$$

Concerning the set Q, from (5.23) and (5.27), it is easily seen that

$$c^{-1}\Big(\dot{q_n}(t) + f\big(t, u_n(\delta_n(t))\big)\Big) \in -N_Q\big(q_n(\theta_n(t))\big) \cap \mathbb{B} \quad \lambda\text{-a.e. } t \in I,$$

hence (see (2.8))

$$c^{-1}\Big(\dot{q_n}(t) + f\big(t, u_n(\delta_n(t))\big)\Big) \in -\partial_P d_Q\Big(q_n\big(\theta_n(t)\big)\Big) \quad \lambda\text{-a.e. } t \in I. \tag{5.28}$$

Assume for a moment that $\varrho_P \neq 0$. From the definition of ν (see (5.4)), it is straightforward to check that the measure $\varrho_P\alpha(\cdot)(1+\kappa)\lambda = \alpha(\cdot)(1+\kappa)\lambda$ is absolutely continuous with respect to ν, then $\frac{d(\alpha(\cdot)(1+\kappa)\lambda)}{d\nu}(\cdot)$ exists as a density and

$$0 \leq \alpha(t)(1+\kappa)\frac{d\lambda}{d\nu}(t) = \frac{d\left(\alpha(\cdot)(1+\kappa)\lambda\right)}{d\nu}(t) \leq 1 \quad \nu\text{-a.e. } t \in I.$$

By (5.16) and (5.3), it is not difficult to check that

$$\|z_n(t)\| \leq \alpha(t)(1+\kappa) \quad \text{for all } t \in I. \tag{5.29}$$

The two latter inequalities guarantee that

$$\left\|\frac{d\lambda}{d\nu}(t) z_n(t)\right\| \leq \alpha(t)(1+\kappa)\frac{d\lambda}{d\nu}(t) \leq 1 \quad \nu\text{-a.e. } t \in I.$$

Clearly, such an inequality is still valid if $\varrho_P = 0$. Then, we derive (see (5.20)) in both cases $\varrho_P = 0$ and $\varrho_P \neq 0$ that

$$\left\|\frac{du_n}{d\nu}(t)\right\| \leq \left\|\frac{d\lambda}{d\nu}(t) z_n(t)\right\| + 1 \leq 2 \quad \nu\text{-a.e. } t \in I. \tag{5.30}$$

Step 4. Convergence of $(u_n(\cdot))_{n\in\mathbb{N}}$ up to a subsequence.

By the inequality (5.30) there is a subsequence of $\left(\frac{du_n}{d\nu}(\cdot)\right)_{n\in\mathbb{N}}$ (that we do not relabel) which weakly converges in $L^2(I, \mathcal{H}, \nu)$ to a (class of) mapping $v(\cdot) \in L^2(I, \mathcal{H}, \nu)$. Defining the mapping $u : [T_0, T] \to \mathcal{H}$ by

$$u(t) := u_0 + \int_{]T_0,t]} v(s) d\nu(s) \quad \text{ for all } t \in I,$$

the latter weak convergence yields that for each $t \in I$

$$u_n(t) = u_0 + \int_{]T_0,t]} \frac{du_n}{d\nu}(s)\, d\nu(s) \xrightarrow{w} u_0 + \int_{]T_0,t]} v(s) d\nu(s) = u(t).$$

Then, the mapping $u(\cdot)$ is right-continuous with bounded variation on I (see Section 2) and du has $\frac{du}{d\nu}(\cdot) = v(\cdot) \in L^2(I, \mathcal{H}, \nu)$ as a density relative to ν. Further, note that

$$\frac{du_n}{d\nu} \to \frac{du}{d\nu} \quad \text{weakly in } L^2(I, \mathcal{H}, \nu),$$

which entails in particular that

$$\frac{du_n}{d\nu} \to \frac{du}{d\nu} \quad \text{weakly in } L^1(I, \mathcal{H}, \nu). \tag{5.31}$$

On the other hand, by (5.30) we have for every $n \in \mathbb{N}$ and every $t \in I$,

$$\|u_n(\theta_n(t)) - u_n(t)\| \le \int_{]t,\,\theta_n(t)]} \left\| \frac{du_n}{d\nu}(s) \right\| d\nu(s) \le 2\nu\left(]t,\,\theta_n(t)]\right). \tag{5.32}$$

Then for each $t \in I$ since $\theta_n(t) \downarrow t$ and $(u_n(t))_{n\in\mathbb{N}}$ weakly converges to $u(t)$, we deduce that $(u_n(\theta_n(t))_{n\in\mathbb{N}}$ weakly converges also to $u(t)$. Further, for every $t \in I$ and every $n \in \mathbb{N}$, the very definition of u_n and θ_n along with (5.8) furnish

$$\|u_n(\theta_n(t))\| \le \sup_{p\in\{0,\ldots,k_n\}} \left\|x_p^n\right\| \le \kappa,$$

hence by (5.25)

$$u_n(\theta_n(t)) \in C(I \times \kappa\mathbb{B} \times \kappa'\mathbb{B}) \cap \kappa\mathbb{B} \quad \text{for all } n \in \mathbb{N}. \tag{5.33}$$

Since the set $C(I\times\kappa\mathbb{B}\times\kappa'\mathbb{B})\cap\kappa\mathbb{B}$ is compact according to assumption (v), the inclusion (5.33) assures us that for each $t \in I$, the sequence $(u_n(\theta_n(t)))_{n\in\mathbb{N}}$ strongly converges to $u(t)$, hence $(u_n(t))_{n\in\mathbb{N}}$ also converges to $u(t)$, i.e.,

$$u_n(\theta_n(t)) \to u(t) \quad \text{and} \quad u_n(t) \to u(t) \quad \text{for all } t \in I.$$

Step 5. Cauchy property of $(q_n(\cdot))_{n\in\mathbb{N}}$.

Fix any integers $m, n \ge 1$. Thanks to the convexity of Q, (2.7), (5.28) and the inclusion $q_n(\theta_n(t)) \in Q$, we get

$$\begin{aligned}&\Big\langle c^{-1}\Big(\dot q_n(t) + f\big(t, u_n(\delta_n(t))\big)\Big), q_n(\theta_n(t)) - q_m(t)\Big\rangle\\ &\le d_Q(q_m(t)) - d_Q\big(q_n(\theta_n(t))\big) = d_Q(q_m(t)),\end{aligned}$$

for λ-almost every $t \in I$. From the latter inequality, the inequality $\|\dot q_n(t)\| \le c$ valid for λ-almost every $t \in I$ (see (5.23)) and from $q_m(\theta_m(t)) \in Q$ we deduce that

$$\begin{aligned}
&\left\langle c^{-1}\Big(\dot{q}_n(t)+f\big(t,u_n(\delta_n(t))\big)\Big),q_n(t)-q_m(t)\right\rangle \\
=&\left\langle c^{-1}\Big(\dot{q}_n(t)+f\big(t,u_n(\delta_n(t))\big)\Big),q_n(t)-q_n(\theta_n(t))\right\rangle \\
&+\left\langle c^{-1}\Big(\dot{q}_n(t)+f\big(t,u_n(\delta_n(t))\big)\Big),q_n(\theta_n(t))-q_m(t)\right\rangle \\
\le&\left\|q_n(t)-q_n(\theta_n(t))\right\|+d_Q(q_m(t)) \\
\le&\left\|q_n(t)-q_n(\theta_n(t))\right\|+\left\|q_m(\theta_m(t))-q_m(t)\right\| \\
\le& c(\theta_n(t)-t)+c(\theta_m(t)-t) \\
=& c(\theta_m(t)+\theta_n(t)-2t), \qquad (5.34)
\end{aligned}$$

for λ-almost every $t\in I$. Since m,n have been arbitrarily choosen, we also have the following inequality

$$\begin{aligned}
&\left\langle c^{-1}\Big(\dot{q}_m(t)+f\big(t,u_m(\delta_m(t))\big)\Big),q_m(t)-q_n(t)\right\rangle \\
\le& c(\theta_m(t)+\theta_n(t)-2t), \qquad (5.35)
\end{aligned}$$

for λ-almost every $t\in I$. Adding the inequalities (5.34) and (5.35) yield

$$\begin{aligned}
&\left\langle c^{-1}\big(\dot{q}_n(t)-\dot{q}_m(t)\big),q_n(t)-q_m(t)\right\rangle \\
\le&\left\langle -c^{-1}f\big(t,u_n(\delta_n(t))\big),q_n(t)-q_m(t)\right\rangle \\
&+\left\langle -c^{-1}f\big(t,u_m(\delta_m(t))\big),q_m(t)-q_n(t)\right\rangle+2c\big(\theta_m(t)+\theta_n(t)-2t\big) \\
\le& c^{-1}\left\|q_n(t)-q_m(t)\right\|\left\|f\big(t,u_n(\delta_n(t))\big)-f\big(t,u_m(\delta_m(t))\big)\right\| \\
&+2c\big(\theta_m(t)+\theta_n(t)-2t\big), \qquad (5.36)
\end{aligned}$$

for λ-almost every $t\in I$. From the Lipschitz assumption in (iii) and the inclusion $\{u_k(\delta_k(t)):t\in I,k\in\mathbb{N}\}\subset\kappa\mathbb{B}$, there is a real $l>0$ such that for all $t\in I$, all $k,k'\in\mathbb{N}$,

$$\left\|f\big(t,u_k(\delta_k(t))\big)-f\big(t,u_{k'}(\delta_{k'}(t))\big)\right\|\le l\left\|u_k(\delta_k(t))-u_{k'}(\delta_{k'}(t))\right\|. \qquad (5.37)$$

Putting together (5.36) and (5.37) and applying the elementary inequality $ab\le 2^{-1}(a^2+b^2)$ valid for every $(a,b)\in\mathbb{R}^2$ yield

$$\begin{aligned}
&\left\langle c^{-1}\big(\dot{q}_n(t)-\dot{q}_m(t)\big),q_n(t)-q_m(t)\right\rangle \\
\le&(2c)^{-1}\big(\left\|q_m(t)-q_n(t)\right\|^2+l^2\left\|u_n(\delta_n(t))-u_m(\delta_m(t))\right\|^2\big) \\
&+2c\big(\theta_m(t)+\theta_n(t)-2t\big). \qquad (5.38)
\end{aligned}$$

Now, define $\psi_{m,n}:I\to\mathbb{R}$ by

$$\psi_{m,n}(t):=\frac{1}{2c}\left\|q_m(t)-q_n(t)\right\|^2 \quad \text{for all } t\in I$$

and observe that (5.38) ensures

$$\dot{\psi}_{m,n}(t)\le\psi_{m,n}(t)+A_{m,n}(t) \quad \lambda\text{-a.e. } t\in I,$$

where

$$A_{m,n}(t) := \frac{l^2}{2c}\|u_n(\delta_n(t)) - u_m(\delta_m(t))\|^2 + 2c(\theta_m(t) + \theta_n(t) - 2t) \quad \text{for all} t \in I.$$

A direct application of Gronwall lemma (see Lemma 4.1) gives

$$\psi_{m,n}(t) \le e^{T-T_0}\int_{T_0}^{T} A_{m,n}(s)d\lambda(s) = e^{T-T_0}\int_{T_0}^{T} A_{m,n}(s)\frac{d\lambda}{d\nu}(s)d\nu(s).$$

Fix for a moment any $t \in I$ for which $\frac{d\lambda}{d\nu}(t)$ is well-defined. If $\nu(\{t\}) > 0$, we have (see Section 2) $A_{m,n}(t)\frac{d\lambda}{d\nu}(t) = 0$. If $\nu(\{t\}) = 0$, we observe through

$$\begin{aligned}\|u_n(t) - u_n(\delta_n(t))\| &= \| \int_{]\delta_n(t),t]} \frac{du_n}{d\nu}(s)d\nu(s)\| \\ &\le \int_{]\delta_n(t),t]} \left\|\frac{du_n}{d\nu}(s)\right\| d\nu(s) \le 2\nu(]\delta_n(t),t])\end{aligned}$$

that $A_{m,n}(t)\frac{d\lambda}{d\nu}(t) \to 0$ as $m,n \to \infty$ as well as

$$|A_{m,n}(t)| \le \frac{l^2}{2c}(2\kappa)^2 + 2c\big(2(T-T_0)\big) \quad \text{for all } m,n \in \mathbb{N}.$$

In such a case, we may apply the Lebesgue dominated convergence theorem to obtain for every $t \in I$,

$$\psi_{m,n}(t) \to 0 \quad \text{as } m,n \to \infty.$$

We derive that for each $t \in I$, $(q_k(t))_{k\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ and then there is $q(t) \in \mathcal{H}$ such that

$$q_k(t) \to q(t).$$

Further, from (5.22), we see that $q(\cdot)$ is a $2\beta(1+\kappa)$-Lipschitz mapping.

Step 6. The mapping $\Phi(\cdot) := (u(\cdot), q(\cdot))$ is a solution of $(\mathcal{P})$.
First, observe (thanks to (5.29)) that we can extract a subsequence (that we do not relabel) $(z_n(\cdot))_{n\in\mathbb{N}}$ which weakly converges in $L^1(I,\mathcal{H},\lambda)$ to a mapping $z(\cdot) \in L^1(I,\mathcal{H},\nu)$. Since $\frac{d\lambda}{d\nu} \in L^\infty(I,\mathbb{R}_+,\nu)$, we get

$$z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot) \to z(\cdot)\frac{d\lambda}{d\nu}(\cdot) \quad \text{weakly in } L^1(I,\mathcal{H},\nu). \tag{5.39}$$

Now, we claim that $u(t) \in C(t,u(t),q(t))$ for every $t \in I$. Indeed, for any $t \in I$ noting first that $\|q_n(\theta_n(t)) - q_n(t)\| \le c(\theta_n(t) - t)$, we see through (ii), (5.32), (5.25) and the $2\beta(1+\kappa)$-Lipschitz property of $q(\cdot)$ that for every $n \in \mathbb{N}$,

$$\begin{aligned}&d\Big(u_n(t), C\big(t,u(t),q(t)\big)\Big) \\ \le& d\Big(u_n(\theta_n(t)), C\big(\theta_n(t), u(\theta_n(t), q_n(\theta_n(t))\big)\Big) + \|u_n(t) - u_n(\theta_n(t))\| \\ &+ \mu(]t,\theta_n(t)]) + L\|u(t) - u_n(\theta_n(t))\| + L'\|q(t) - q_n(\theta_n(t))\| \\ \le& 3\nu(]t,\theta_n(t)]) + L\|u(t) - u_n(\theta_n(t))\| + 2\beta(1+\kappa)L'(\theta_n(t) - t).\end{aligned}$$

Hence, we have the convergence property $d\big(u_n(t), C(t, u(t), q(t))\big) \to 0$ as $n \to \infty$, which ensures (thanks to the closedness of $C(\cdot)$) that

$$u(t) \in C\big(t, u(t), q(t)\big) \quad \text{for all}\, t \in I.$$

From the inequality valid for every $t \in I$

$$d_Q(q(t)) \leq d_Q\big(q_n(\delta_n(t))\big) + \|q(t) - q_n(\delta_n(t))\| = \|q(t) - q_n(\delta_n(t))\|$$

it follows that

$$q(t) \in Q \quad \text{for all}\, t \in I.$$

Now, coming back to (5.31) and (5.39), the sequence $\left(\frac{du_n}{d\nu}(\cdot)+z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot)\right)_{n\in\mathbb{N}}$ weakly converges in $L^1(I, \mathcal{H}, \nu)$ to $\frac{du}{d\nu}(\cdot) + z(\cdot)\frac{d\lambda}{d\nu}(\cdot)$. Then, by Mazur's lemma, we can find for each $n \in \mathbb{N}$

$$\xi_n(\cdot) \in \operatorname{co}\left\{\frac{du_k}{d\nu}(\cdot) + z_k(\cdot)\frac{d\lambda}{d\nu}(\cdot) :\ k \geq n\right\}$$

such that the sequence $(\xi_n(\cdot))_{n\in\mathbb{N}}$ strongly converges in $L^1(I, \mathcal{H}, \nu)$ to $\frac{du}{d\nu}(\cdot)+ z(\cdot)\frac{d\lambda}{d\nu}(\cdot)$. Extracting a subsequence if necessary, we may assume that $(\xi_n(\cdot))_{n\in\mathbb{N}}$ converges ν-almost everywhere to $\frac{du}{d\nu}(\cdot) + z(\cdot)\frac{d\lambda}{d\nu}(\cdot)$. Consequently, for ν-almost every $t \in I$,

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{co}}\left\{\frac{du_k}{d\nu}(\cdot) + z_k(\cdot)\frac{d\lambda}{d\nu}(\cdot) :\ k \geq n\right\}.$$

Thus, for ν-almost every $t \in I$ we have

$$\left\langle h, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \left\langle h, \frac{du_k}{d\nu}(t) + z_k(t)\frac{d\lambda}{d\nu}(t)\right\rangle,$$

for all $h \in \mathcal{H}$. It results from (5.26) and the latter inequality that for ν-almost every $t \in I$

$$\begin{aligned}&\left\langle h, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle\\ &\leq \limsup_{n\to\infty} \sigma\Big(-h, \partial_P d_{C\left(\theta_n(t), u_n(\theta_n(t), q_n(\theta_n(t)))\right)}(u_n(\theta_n(t)))\Big),\end{aligned}$$

for all $h \in \mathcal{H}$. Now, invoking Proposition 4.5, we have for ν-almost every $t \in I$ and for all $h \in \mathcal{H}$,

$$\begin{aligned}\left\langle h, \frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle &\leq \sigma\Big(-h, \partial_P d_{C(t,u(t),q(t))}(u(t))\Big)\\ &= \sigma\Big(h, -\partial_P d_{C(t,u(t),q(t))}(u(t))\Big).\end{aligned}$$

By virtue of Theorem 2.2, the latter inequality holds with the Clarke subdifferential which is always closed and convex. Then, the equivalence in (2.9) and the inclusion in (2.8) guarantee that

$$\frac{du}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -\partial d_{C(t,u(t),q(t))}(u(t))) \subset -N_{C(t,u(t),q(t))}(u(t)) \tag{5.40}$$

for ν-almost every $t \in I$. In the same way, from (5.28) we establish that

$$\dot{q}(t) + f(t, u(t)) \in -N_Q(q(t)) \quad \lambda\text{-a.e. } t \in I. \tag{5.41}$$

Defining $\Phi : I \to \mathcal{H}^2$ by

$$\Phi(t) := (u(t), q(t)) \quad \text{for all } t \in I$$

and taking into account (5.40) and (5.41), we obtain that Φ_1 is a right-continuous mapping with bounded variation on I satisfying

$$\frac{d\Phi_1}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N_{C(t,\Phi_1(t))}(\Phi(t)) \quad \nu\text{-a.e. } t \in I$$

and Φ_2 is absolutely continuous (in fact, Lipschitz continuous on I) with

$$\dot{\Phi}_2(t) + f(t, \Phi_1(t))) \in -N_Q(\Phi_2(t)) \quad \nu\text{-a.e. } t \in I.$$

We claim that $z(t) \in G(t, u(t), q(t))$ for λ-almost every $t \in I$. We may assume that $\varrho_P \neq 0$. Since $(z_k(\cdot))_{k\in\mathbb{N}}$ weakly converges to $z(\cdot)$ in $L^1(I, \mathcal{H}, \lambda)$, the Mazur's lemma (up to a subsequence) allows us to write

$$z(t) \in \bigcap_{n\in\mathbb{N}} \overline{\mathrm{co}}\,\{z_k(t) : k \geq n\} \quad \lambda\text{-a.e. } t \in I.$$

This inclusion along with the fact that

$$z_n(t) \in G(t, u_n(\delta_n(t)), q_n(\delta_n(t))) \quad \text{for all } t \in I, n \in \mathbb{N}$$

furnish a Borel subset $\Omega \subset I$ with $\lambda(I \setminus \Omega) = 0$ such that for all $t \in \Omega$ and all $h \in \mathcal{H}$,

$$\langle h, z(t)\rangle \leq \limsup_{n\to+\infty} \sigma\Big(h, G\big(t, u_n(\delta_n(t)), q_n(\delta_n(t))\big)\Big).$$

Put $\Sigma := \{t \in I : \nu(\{t\}) > 0\}$ and note that Σ is countable (see (2.14)). For each $t \in I \setminus \Sigma$, we have by (5.32)

$$\|u_n(t) - u_n(\delta_n(t))\| \leq 2\nu(]\delta_n(t), t]) \quad \text{for all } n \in \mathbb{N},$$

thus $\|u_n(t) - u_n(\delta_n(t))\| \to 0$ as $n \to +\infty$ (since $\nu(\{t\}) = 0$), which implies that $u_n(\delta_n(t)) \to u(t)$ as $n \to +\infty$ since $u_n(t) \to u(t)$. Then, for each $t \in \Omega \setminus \Sigma$ using the fact that $G(t, \cdot, \cdot)$ is scalarly upper-semicontinuous we get

$$\langle h, z(t)\rangle \leq \sigma(h, G(t, u(t), q(t))) \quad \text{for all } h \in \mathcal{H},$$

which entails $z(t) \in G(t, u(t), q(t))$ by the closedness and convexity of the set $G(t, u(t), q(t))$ and by (2.9). Since the countable set Σ is λ-negligible, it follows that

$$z(t) \in G(t, u(t), q(t)) \quad \lambda\text{-a.e. } t \in I.$$

The proof is then complete. □

Remark 5.2. It is worth pointing out the following feature concerning the measurability of the mapping of minimal norm $m_G(\cdot, x, y)$ involved in the assumption (iv) of Theorem 5.1. By Theorem III-41(2) in [16] this mapping $m_G(\cdot, x, y)$ is λ-Bochner measurable whenever the Hilbert space $\mathcal{H}$ is *separable* and the multimapping $t \mapsto G(t, x, y)$ is Lebesgue measurable in the usual sense that its graph belongs to $\mathcal{L}(I)\otimes\mathcal{B}(\mathcal{H})$, where $\mathcal{L}(I)$ and $\mathcal{B}(\mathcal{H})$ denote respectively the Lebesgue σ-field of I and the Borel σ-field of $\mathcal{H}$. ■

We derive from Theorem 5.1 the case when the measure μ is absolutely equivalent to the Lebesgue measure, say $\mu(]s,t]) = v(t) - v(s)$ for some nondecreasing absolutely continuous function $v : I \to \mathbb{R}_+$.

Corollary 5.3. *Let $C : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings and $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping. Let Q be a closed convex subset of $\mathcal{H}$, $(u_0, q_0) \in \mathcal{H} \times Q$ with $u_0 \in C(T_0, u_0, q_0)$.*

Assume that:

(i) there exists $r \in]0, +\infty]$ such that for every $t \in I$ and every $x, y \in \mathcal{H}$, the set $C(t, x, y)$ is r-prox-regular;

(ii) there exist two reals $L \in]0, 1[$, $L' \geq 0$ and a nondecreasing absolutely continuous function $v : I \to \mathbb{R}$ on I such that

$$\operatorname{haus}\big(C(t_1, x_1, y_1), C(t_2, x_2, y_2)\big) \leq v(t_2) - v(t_1) + L\,\|x_1 - x_2\| + L'\,\|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $x_1, x_2, y_1, y_2 \in \mathcal{H}$;

(iii) the mapping $f(\cdot, x)$ is λ-Bochner measurable on I for each $x \in \mathcal{H}$, there exists a real $\beta \geq 0$ such that

$$\|f(t,x)\| \leq \beta(1 + \|x\|) \quad \textit{for all } t \in I, x \in \mathcal{H},$$

and for each bounded subset B of $\mathcal{H}$ there exists a real $l_B \geq 0$ satisfying

$$\left\|f(t,x) - f(t,x')\right\| \leq l_B \left\|x - x'\right\| \quad \textit{for all}\, t \in I, x, x' \in B;$$

(iv) the multimapping G is nonempty closed convex valued, $G(t, \cdot, \cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $(x, y) \in \mathcal{H}^2$, the mapping $m_G(\cdot, x, y) : I \to \mathcal{H}$ is λ-Bochner measurable on I and there exists a nonnegative function $\alpha(\cdot) \in L^1(I, \lambda, \mathbb{R}_+)$ such that

$$\|m_G(t,x,y)\| = d(0, G(t,x,y)) \leq \alpha(t)(1 + \|x\|),$$

for all $t \in I$, $x, y \in \mathcal{H}$ with $(x, y) \in C(t, x, y) \cap \kappa\mathbb{B} \times Q \cap \kappa'\mathbb{B}$, where

$$\kappa' := \|q_0\| + 2\beta(1 + \kappa)(T - T_0)$$

and

$$\kappa := \big(\|u_0\| + \frac{v(T) - v(T_0)}{1 - L} + s\big)e^s \quad \textit{with} \quad s := \frac{2}{1 - L}\int_{T_0}^{T} \big(\alpha(s) + L'\beta\big)d\lambda(s);$$

(v) for every bounded subset B of $\mathcal{H}$ and every $t \in I$, the set $C(t, B \times \kappa'\mathbb{B}) \cap B$ is relatively compact.

Then, there exists a solution $\Phi(\cdot) : I \to \mathcal{H}^2$ of the mixed partially differential inclusion

$$\begin{cases} -\dot{\Phi}(t) \in N_{C(t,\Phi(t))\times Q}(\Phi(t)) + G(t, \Phi(t)) \times \{f(t, \Phi_1(t))\}, \\ \Phi(T_0) = (u_0, q_0). \end{cases}$$

Further, Φ_2 is $2\beta(1 + \kappa)$-Lipschitz continuous on I.

Proof. Consider the (unique) positive Radon measure μ on I satisfying

$$\mu(]s,t]) = v(t) - v(s) \quad \text{for all } s,t \in I \text{ with } s < t.$$

Since the measure μ is non-atomic, Proposition 4.6 says that the assumption (v) is equivalent to the relative compactness of $C(I, B \times \kappa'\mathbb{B}) \cap B$ for every bounded subset B of $\mathcal{H}$. It remains to apply Theorem 5.1 to complete the proof. □

We deduce from the latter theorem the case where $f \equiv 0$ and $Q = \{0\}$, that is, the existence of solutions for $(\mathcal{FSP})$.

Corollary 5.4. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be two multimappings, $u_0 \in \mathcal{H}$ with $u_0 \in C(T_0, u_0)$.*

Assume that:

(i) there exists $r \in]0,+\infty]$ such that for every $t \in I$ and every $x \in \mathcal{H}$, the set $C(t,x)$ is r-prox-regular;

(ii) there exist a real $L \in]0,1[$, and a positive Radon measure μ on I with $\sup_{s \in]T_0,T]} \mu(\{s\}) < (1-L)r$ such that

$$\operatorname{haus}\big(C(t_1,x_1), C(t_2,x_2)\big) \le \mu(]t_1,t_2]) + L\,\|x_1 - x_2\|$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $x_1, x_2 \in \mathcal{H}$;

(iii) the multimapping G is nonempty closed convex valued, $G(t,\cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $x \in \mathcal{H}$, the mapping $m_G(\cdot,x) : I \to \mathcal{H}$ is Lebesgue measurable on I and there exists a nonnegative function $\alpha(\cdot) \in L^1(I,\lambda,\mathbb{R}_+)$ such that

$$\|m_G(t,x)\| = d(0, G(t,x)) \le \alpha(t)(1 + \|x\|),$$

for all $t \in I$, $x \in \mathcal{H}$ with $x \in C(t,x) \cap \kappa\mathbb{B}$, where

$$\kappa := \left(\|u_0\| + \frac{\mu(]T_0,T]) + 2\int_{T_0}^{T} \alpha(s)\, d\lambda(s)}{1-L} \right);$$

(iv) for every bounded subset B of $\mathcal{H}$, the set $C(I,B) \cap B$ is relatively compact.

Then, there exists a mapping $u(\cdot) : I \to \mathcal{H}$ such that

$$\begin{cases} -du \in N_{C(t,u(t))}(u(t)) + G(t,u(t)), \\ u(T_0) = u_0. \end{cases}$$

Proof. It suffices to apply the latter theorem with $Q := \{0\}$ and $f \equiv 0$ (as said above) along with the multimappings $\widehat{C} : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ and $\widehat{G} : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ defined by

$$\widehat{C}(t,x,y) := C(t,x) \quad \text{and} \quad \widehat{G}(t,x,y) := G(t,x) \quad \text{for all } (t,x,y) \in I \times \mathcal{H}^2.$$

□

6. Reduction of second order sweeping process ($\mathcal{SSP}$) to the first order one ($\mathcal{FSP}$)

Our aim here is to derive existence of solution for the problem ($\mathcal{SSP}$) with bounded variation via a reduction to a first order state-dependent sweeping process. As mentionned in the very introduction of the present paper, such a reduction have been only observed ([43, 53]) in the absolutely continuous and finite dimensional setting.

Let us start this section with the following result which can be seen as a counterpart of Proposition 3.3.

Proposition 6.1. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings. Let also $(u_0, v_0) \in \mathcal{H}^2$ with $v_0 \in C(T_0, u_0)$. Assume that there exist a real $L' \geq 0$ and a positive Radon measure μ on I such that*

$$\operatorname{haus}\big(C(s,x), C(t,y)\big) \leq \mu\,(]s,t]) + L'\|x-y\|,$$

for all $s, t \in I$ with $s < t$ and $x, y \in \mathcal{H}$. If $\Phi(\cdot)$ is a solution of the first order mixed state-dependent Moreau sweeping process (associated to μ) with outward normal at the velocity inside the set

$$(\mathcal{P}) \begin{cases} -d\Phi \in N_{C(t,\Phi_2(t))\times\mathcal{H}}(\Phi(t)) + F(t,\Phi(t)) \times \{-\Phi_1(t)\} \\ \Phi(T_0) = (v_0, u_0), \end{cases}$$

then the mapping $\Phi_2(\cdot)$ is a solution of the second order sweeping process (associated to μ)

$$(\mathcal{Q}) \begin{cases} -d\dot{\Phi}_2(t) \in N_{C(t,\Phi_2(t))}(\dot{\Phi}_2(t)) + F(t,\dot{\Phi}_2(t),\Phi_2(t)) \\ \Phi_2(T_0) = u_0, \dot{\Phi}_2(T_0) = v_0, \end{cases}$$

and $\dot{\Phi}_2 = \Phi_1$ λ-almost everywhere on I.

Proof. Assume that $\Phi(\cdot) : I \to \mathcal{H}^2$ is a solution of ($\mathcal{P}$). Let us define the multimappings $S_C, G_F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ by setting for every $(t,x,y) \in I \times \mathcal{H}^2$,

$$S_C(t,x,y) := C(t,y) \times \mathcal{H} \quad \text{and} \quad G_F(t,x,y) := F(t,x,y) \times \{-x\}.$$

Observe that $(v_0, u_0) \in S_C(T_0, \Phi(T_0)) = C(T_0, \Phi_2(T_0)) \times \mathcal{H}$. Coming back to Definition 3.4 and according to the definition of S_C and G_F, there is a λ-Bochner integrable mapping $z(\cdot) : I \to \mathcal{H}^2$ with

$$z(t) \in G_F(t,\Phi(t)) = F(t,\Phi_1(t),\Phi_2(t)) \times \{-\Phi_1(t))\} \quad \lambda\text{-a.e. } t \in I \tag{6.1}$$

and such that Φ_2 is absolutely continuous on I with

$$\dot{\Phi}_2(t) - \Phi_1(t) = 0 \quad \text{a.e. } t \in I, \tag{6.2}$$

and there exists also a positive Radon measure ν on I, absolutely continuously equivalent to $\mu + \varrho_{G_F}\lambda$ and with respect to which the differential measure $d\Phi_1$ of Φ_1 is absolutely continuous with $\frac{d\Phi_1}{d\nu}(\cdot)$ as an $L^1(I, \mathcal{H}, \nu)$-density along with for ν-almost every $t \in I$

$$\frac{d\Phi_1}{d\nu}(t) + z_1(t)\frac{d\lambda}{d\nu}(t) \in -N_{C(t,\Phi_2(t))}(\Phi_1(t)).$$

All together means that $\Phi_2(\cdot)$ satisfies the second order sweeping process $(\mathcal{Q})$ associated to μ and with outward normal at the velocity inside the set. The proof is then complete. $\square$

Now, we are able to reduce the second order sweeping process with outward normal at the velocity to the Moreau's one.

Theorem 6.2. *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ be two multimappings, $(u_0, v_0) \in \mathcal{H}^2$ with $v_0 \in C(T_0, u_0)$.*

Assume that:
(i) there exists $r \in]0, +\infty]$ such that for every $t \in I$ and every $y \in \mathcal{H}$, the set $C(t, y)$ is r-prox-regular;
(ii) there exist a real $L' \geq 0$ and a positive Radon measure μ on I such that

$$\operatorname{haus}\big(C(t_1, y_1), C(t_2, y_2)\big) \leq \mu(]t_1, t_2]) + L' \|y_1 - y_2\|,$$

for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $y_1, y_2 \in \mathcal{H}$;
(iii) the multimapping F is nonempty closed convex valued, $F(t, \cdot, \cdot)$ is scalarly upper semicontinuous for each $t \in I$, and for each $(x, y) \in \mathcal{H}^2$, the mapping $m_F(\cdot, x, y) : I \to \mathcal{H}$ is λ-Bochner measurable on I and there exists a nonnegative function $\alpha(\cdot) \in L^1(I, \lambda, \mathbb{R}_+)$ such that

$$\|m_F(t, x, y)\| = d(0, F(t, x, y)) \leq \alpha(t)(1 + \|x\|),$$

for all $t \in I$, $x, y \in \mathcal{H}$ with $(x, y) \in C(t, y) \cap \kappa\mathbb{B} \times Q \cap \kappa'\mathbb{B}$, where

$$\kappa' := \|q_0\| + 2(1 + \kappa)(T - T_0)$$

and

$$\kappa := \big(\|u_0\| + \mu(]T_0, T]) + s\big)e^s \quad \text{with} \quad s := 2 \int_{T_0}^{T} \big(\alpha(s) + L'\beta\big) d\lambda(s);$$

(iv) for every bounded subset B of $\mathcal{H}$, the set $C(I, B \times \kappa'\mathbb{B}) \cap B$ is relatively compact.

Then, there exists a solution $u(\cdot) : I \to \mathcal{H}$ of the second order sweeping process (with outward normal at the velocity inside the set)

$$\begin{cases} -d\dot{u} \in N_{C(t, u(t))}(\dot{u}(t)) + F(t, \dot{u}(t), u(t)) \\ u(T_0) = u_0, \dot{u}(T_0) = v_0. \end{cases}$$

Proof. As above, we define the multimappings $S_C, G_F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}^2$ by setting

$$S_C(t, x, y) := C(t, y) \times \mathcal{H} \quad \text{and} \quad G_F(t, x, y) := F(t, x, y) \times \{-x\},$$

for every $(t, x, y) \in I \times \mathcal{H}^2$. Obviously, $G_F(\cdot, \cdot, \cdot)$ is nonempty closed convex valued and $S_C(t, \cdot, \cdot)$ is scalarly upper semicontinuous for every $t \in I$. Then, all conditions of Theorem 5.1 are satisfied and this guarantees the existence of a mapping $\Phi(\cdot)$ satisfying

$$\begin{cases} -d\Phi \in N_{S_C(t, \Phi(t))}(\Phi(t)) + G_F(t, \Phi(t)) \\ \Phi(T_0) = (v_0, u_0). \end{cases}$$

It remains to apply Proposition 6.1 to complete the proof. □

Acknowledgment. The first author has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No 823731 CONMECH.

References

[1] S. Adly, *A Variational Approach to Nonsmooth Dynamics, Applications in Unilateral Mechanics and Electronics*, SpringerBriefs in Mathematics. Springer, Cham, 2017.

[2] S. Adly, B.K. Le, *Unbounded second-order state-dependent Moreau's sweeping processes in Hilbert spaces*, J. Optim. Theory Appl. 169 (2016), 407-423.

[3] S. Adly, T. Haddad, *An implicit sweeping process approach to quasistatic evolution variational inequalities*, SIAM J. Math. Anal. 50 , 761-778.

[4] S. Adly, F. Nacry, *An existence result for discontinuous second-order nonconvex state-dependent sweeping processes*, Appl. Math. Optim. 79 (2019), 515-546.

[5] F. Aliouane, D. Azzam-Laouir, C. Castaing, M.D.P. Monteiro Marques, *Second order time and state dependent sweeping process in Hilbert space*, J. Optim. Theory Appl. 182 (2019), 153188.

[6] D. Azzam-Laouir, S. Izza, L. Thibault, *Mixed semicontinuous perturbation of nonconvex state-dependent sweeping process*, Set-Valued Var. Anal. 22 (2014), 271-283.

[7] M.V. Balashov, G.E. Ivanov, *Properties of the metric projection on weakly vial-convex sets and parametrization of set-valued mappings with weakly convex images*, Math. Notes 80, 461467 (2006).

[8] H. Benabdellah, *Existence of solutions to the nonconvex sweeping process*, J. Differential Equations 164 (2000), 286-295.

[9] F. Bernicot, J. Venel, *Sweeping process by prox-regular sets in Riemannian Hilbert manifolds*, J. Differential Equations 259 (2015), 4086-4121.

[10] M. Bounkhel, A.-Y. Rabab, *First and second order convex sweeping processes in reflexive smooth Banach spaces*, Set-Valued Var. Anal. 18 (2010), 151-182.

[11] M. Bounkhel, L. Thibault, *On various notions of regularity of sets in nonsmooth analysis*, Nonlinear Anal. Ser. A: Theory Methods 48 (2002), 223-246.

[12] T.H. Cao and B.S. Mordukhovich. Optimal control of a nonconvex perturbed sweeping process, to appear in J. Differential Equations.

[13] C. Castaing, *Version aléatoire de problème de rafle par un convexe variable*, C.R. Acad. Sci. Paris, Sér. A 277 (1973), 1057-1059.

[14] C. Castaing, *Equation différentielle multivoque avec contrainte sur l'état dans les espaces de Banach*, Travaux Sém. Anal. Convexe Montpellier (1978), Exposé 13.

[15] C. Castaing, A.G. Ibrahim, M. Yarou *Some contributions to nonconvex sweeping process*, J. Nonlinear Convex Anal. 10 (2009), 1-20.

[16] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lectures in Mathematics No 580, Springer-Verlag, Berlin, 1977.

[17] K. Chraibi, *Etude théorique et numérique de problèmes d'évolution en présence de liaisons unilatérales et de frottement*, Ph.D. thesis, Université de Montpellier, 1987.

[18] D.S. Clark, *Short proof of a discrete Gronwall inequality.* Elsevier Science, Volume 16, Issue 3, March 1987, Pages 279-281.

[19] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Second Edition, Classics in Applied Mathematics, 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[20] F.H. Clarke, R.J. Stern, P.R. Wolenski, *Proximal smoothness and the lower-C2 property*, J. Convex Anal. 2 (1995), 117-144.

[21] G. Colombo, V.V. Goncharov, *The sweeping processes without convexity*, Set-Valued Anal. 7 (1999), 357-374.

[22] G. Colombo, L. Thibault, *Prox-regular sets and applications*, Handbook of nonconvex analysis and applications 99-182 (2010), Int. Press, Somerville, MA.

[23] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[24] N. Dinculeanu, *Vector Measures, Pergamon*, Oxford, 1967.

[25] S. Di Marino, B. Maury, F. Santambrogio, *Measure sweeping process*, J. Convex Anal 23 (2016), 567-601.

[26] J. Diestel, Jr. J.J. Uhl, Vector Measures, Mathematical Surveys, American Mathematical Society, Providence, R.I., 1977.

[27] J.F. Edmond, L. Thibault, *BV solutions of nonconvex sweeping process differential inclusions with perturbation*, J. Differential Equations 226 (2006), 135-179.

[28] T. Haddad, *Nonconvex differential variational inequality and state-dependent sweeping process*, J. Optim. Theory Appl. 159 (2013), 386-398.

[29] T. Haddad, J. Noel, L. Thibault, *Perturbed sweeping process with a subsmooth set depending on the state*, Linear and Nonlinear Analysis 2 (2016), 155-174.

[30] A. Jourani, E. Vilches, *Positively α-far sets and existence results for generalized perturbed sweeping processes*, J. Convex Anal. 23 (2016), 775-821.

[31] M. Kunze, M.D.P. Monteiro Marques, *On parabolic quasi-variational inequalities and state-dependent sweeping processes*, Topol. Methods Nonlinear Anal. 12 (1998), 179-191.

[32] B. Maury, J. Venel, *A discrete contact model for crowd motion*, ESAIM Math. Model. Numer. Anal. 45 (2011), no 1, 145-168.

[33] M.D.P. Monteiro Marques, *Perturbations convexes semi-continues supérieurement de problèmes d'évolution dans les espaces de Hilbert*, Travaux Sém. Anal. Convexe Montpellier (1984), Exposé 2.

[34] M.D.P. Monteiro Marques, *Differential inclusions in nonsmooth mechanical problems. Shocks and dry friction*, Progress in Nonlinear Differential Equations and their Applications, 9. Birkhuser Verlag, Basel, 1993.

[35] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation I: basic theory*, Grundlehren der Mathematischen Wissenschaften, vol. 330, Springer-Verlag, Berlin, 2006.

[36] J.J. Moreau, *An introduction to unilateral dynamics*, Frémond, M., Maceri, F. (eds.) Novel Approaches in Civil Engineering, Springer, Berlin, 2002.

[37] J.J. Moreau, *Rafle par un convexe variable I*, Travaux Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[38] J.J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), 347-374.

[39] J.J. Moreau, M.Valadier, *A chain rule involving vector functions of bounded variation*, J. Funct. Anal. 74 (1987), 333-345.

[40] F. Nacry, *Truncated nonconvex state-dependent sweeping process: implicit and semi-implicit adapted Moreau's catching-up algorithms*, J. Fixed Point Theory Appl. 20 (2018), Art 121, 32 pp.

[41] F. Nacry, L. Thibault, *Regularization of sweeping process: old and new*, Pure and Applied Functional Analysis 4 (2019), 59-117.

[42] F. Nacry, L. Thibault, *BV prox-regular sweeping process with bounded truncated variation*, Optimization 69 (2020), 1391-1437.

[43] J. Noel, *Second-order general perturbed sweeping process differential inclusion*, J. Fixed Point Theory Appl. 20 (2018), Art 133, 21 pp.

[44] J. Noel, L. Thibault, *Nonconvex sweeping process with a moving set depending on the state*, Vietnam Journal of Mathematics 42 (2014), 595-612.

[45] L. Paoli, *Mathematical aspects of vibro-impact problems*, Advanced topics in nonsmooth dynamics, Transactions of the European Network for Nonsmooth Dynamics, Springer, Cham, 2018.

[46] R. A. Poliquin, R. T. Rockafellar *Prox-regular functions in variational analysis*, Trans. Amer. Math. Soc. 348 (1996), 1805-1838.

[47] R. A. Poliquin, R. T. Rockafellar, L. Thibault, *Local differentialbility of distance functions,* Trans. Amer. Math. Soc. 352 (2000), 5231-5249.

[48] R.T. Rockafellar, R. J-B. Wets, *Variational Analysis*, Springer-Verlag, 1997.

[49] L. Thibault, *Sweeping process with regular and nonregular sets*, J. Differential Equations 193 (2003), 1-26.

[50] L. Thibault, *Moreau sweeping process with bounded truncated retraction*, J. Convex Anal. 23 (2016), 1051-1098.

[51] L. Thibault, *Unilateral Variational Analysis in Banach Spaces*, to appear.

[52] M. Valadier, *Quelques problèmes d'entraînement unilatéral en dimension finie*, Travaux Sém. Anal. Convexe Montpellier (1988), Exposé 8.

[53] M.F. Yarou, *Reduction approach to second order perturbed state-dependent sweeping process,* Creat. Math. Inform., 28 (2019), 215-221.

(Florent Nacry) Université Perpignan Via Domitia, Laboratoire de Mathématiques et Physique, 52 Avenue Paul Alduy, 66860 Perpignan.
E-mail address: `florent.nacry@univ-perp.fr`

(Jimmy Noel) Université Montpellier, Institut Montpelliérain Alexander Grothendieck, 34095 Montpellier Cedex 5, France
E-mail address: `noel.jimmy.j@gmail.com`

(Lionel Thibault) Université Montpellier, Institut Montpelliérain Alexander Grothendieck, 34095 Montpellier Cedex 5, France
E-mail address: `lionel.thibault@umontpellier.fr`