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RESEARCH

History-dependent operators and prox-regular sweeping processes

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Abstract

We consider an abstract inclusion in a real Hilbert space, governed by an almost history-dependent operator and a time-dependent multimapping with prox-regular values. We establish the unique solvability of the inclusion under appropriate assumptions on the data. The proof is based on the arguments of monotonicity, fixed point, and prox-regularity. We then use our result in order to deduce some direct consequences, including an existence and uniqueness result for a class of sweeping processes associated with prox-regular sets. Finally, we provide an example in a finite dimensional case inspired by a rheological model in solid mechanics.

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1 Introduction

A large variety of boundary valued problems arising in mechanics, physics, and engineering sciences lead, in a weak formulation, to nonlinear inclusions. Their solvability involves arguments coming from nonlinear, set-valued, convex, and nonsmooth analysis, among others. Currently, there is a growing interest in the solution of inclusions governed by a special class of operators, the so-called almost history-dependent operators. Such kind of problems arise in the study of different constitutive laws used in the viscoelasticity and viscoplasticity. They also describe the frictional or frictionless contact between a deformable body and an obstacle. References in the field include [1, 5, 19, 20]. There, existence and uniqueness results have been provided by using a fixed point theorem for almost history-dependent operators.

Nevertheless, the inclusions studied in the previously cited papers have been associated with a family of convex sets. Removing the convexity in the study of the corresponding inclusions leads to important mathematical difficulties and gives rise to new and challenging mathematical problems. This can be achieved through the class of prox-regular sets (also known as positively reached, weakly convex, $\mathcal{O}(2)$-convex, φ-convex, proximally smooth (see, e.g., [11] and the references therein)). Recall that a closed set is said to be prox-regular [23] provided that its metric projection is single-valued and continuous on a suitable enlargement of the set. Prox-regular sets share important properties with con-

vex sets, including smoothness of distance function, hypomonotonicity of normals, and global Lipschitz property of metric projection (see Sect. 2). Prox-regularity property has been recognized as a key concept in variational analysis and its applications [16, 24, 28]. The good behavior of metric projection and distance function for prox-regular sets is extensively involved in the study of Moreau's sweeping processes (that is, differential inclusions driven by the normal cone of a moving set [15, 17, 18]) as shown in [3, 6–8, 10, 12–14, 21, 27, 29] and the references therein. However, at the best of our knowledge, there are few literature works concerning history-dependent inclusions governed by prox-regular sets.

The aim of this current paper is to fill this gap. Indeed, here we consider a version of the history-dependent inclusion considered in [19], associated with a family of prox-regular sets. We provide an existence and uniqueness result and, to this end, we use various properties of the prox-regular sets, including the properties of the projection operators and proximal normal cone. Extending the results in [19] to the nonconvex case represents the main trait of novelty in our work.

The structure of the inclusion we consider in this paper is motivated by the study of constitutive laws in solid mechanics, as we show in what follows. Let $d \in \{2, 3\}$ and denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Recall that the inner product and the Euclidean norm on $\mathbb{S}^d$ are defined by

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

where the indices i, j run between 1 and d and the summation convention over repeated indices is used. Consider a constitutive law derived by using the following rheological arguments.

1) The model is obtained by connecting in series an elastic model with a viscoelastic (or viscoplastic) model. Then, at each moment t in the interval of interest I, the strain field satisfies the equality

$$\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}_1(t) + \boldsymbol{\varepsilon}_2(t), \tag{1}$$

where $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ represent the strain field in the elastic and the viscoelastic (or viscoplastic) model, respectively. We refer to $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ as the "regular" and "irregular" strain.

2) The regular strain satisfies the equality

$$\boldsymbol{\varepsilon}_1(t) = B\boldsymbol{\sigma}(t), \tag{2}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t)$ denotes the stress field and $B : \mathbb{S}^d \to \mathbb{S}^d$ represents the compliance operator which could be nonlinear. This operator is supposed to be inversible, and its inverse will be denoted by A, i.e., $B^{-1} = A$.

3) On the other hand, we assume that the irregular strain field is such that

$$\boldsymbol{\varepsilon}_2(t) \in N\big(C(t); \boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)\big), \tag{3}$$

where $C(t)$ is a subset of $\mathbb{S}^d$ to be defined and $\mathcal{R}$ is a memory operator. A popular example of such an operator is given by

$$\mathcal{R}(\sigma) = \int_0^t \mathcal{D}(t-s)\boldsymbol{\sigma}(s)\,ds,$$

where $\mathcal{D}(\cdot)$ denotes a given relaxation tensor. Moreover, for any $\boldsymbol{\omega} \in \mathbb{S}^d$, notation $N(C(t), \boldsymbol{\omega})$ represents a set of $\mathbb{S}^d$ which depends on $C(t)$. Note that (3) shows that at each time moment t the irregular strain depends on the current value of the stress (i.e., $\boldsymbol{\sigma}(t)$) and the history of the stress process (described by the term $\mathcal{R}\boldsymbol{\sigma}(t)$).

We now combine relations (1)–(3) to deduce that

$$\boldsymbol{\varepsilon}(t) \in B\boldsymbol{\sigma}(t) + N\big(C(t); \boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)\big). \tag{4}$$

A concrete example of constitutive law of the form (4) can be obtained by taking $\mathcal{R} \equiv 0$ and $N(C(t), \boldsymbol{\omega}) = \partial\psi_{C(t)}(\boldsymbol{\omega})$ for any $\boldsymbol{\omega} \in \mathbb{S}^d$, where $C(t) \subset \mathbb{S}^d$ is a given convex set (say the von Mises convex) and $\partial\psi_{C(t)}$ represents the Moreau–Rockafellar subdifferential of the indicator function (in the sense of convex analysis) $\psi_{C(t)}$ of the set $C(t)$. This leads to the well-known Hencky law

$$\boldsymbol{\varepsilon}(t) \in B\boldsymbol{\sigma}(t) + \partial\psi_{C(t)}\big(\boldsymbol{\sigma}(t)\big), \tag{5}$$

see, e.g., [22, 26] and the references therein. Note that in (5) we assume that the convex C is time-dependent, and this could arise when C depends on the temperature field, for instance.

Now, we are looking for a stress function $\boldsymbol{\sigma}$ which, applied to the deformable body, keeps it in equilibrium, i.e., the strain field vanishes. Then, at each moment $t \in I$, we have

$$-B\boldsymbol{\sigma}(t) \in N\big(C(t); \boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t)\big)$$

and, using the notation $\boldsymbol{\omega} = \boldsymbol{\varepsilon}_1$ combined with equalities (2) and $B^{-1} = A$, we find that

$$-\boldsymbol{\omega}(t) \in N\big(C(t); A\boldsymbol{\omega}(t) + \mathcal{R}A\boldsymbol{\omega}(t)\big).$$

Therefore, with the notation $\mathcal{R}A\boldsymbol{\omega}(t) = \mathcal{S}\boldsymbol{\omega}(t)$ we arrive at the following inclusion problem.

Problem 1 Find a regular strain function $\boldsymbol{\omega}: I \to \mathbb{S}^d$ such that

$$-\boldsymbol{\omega}(t) \in N\big(C(t); A\boldsymbol{\omega}(t) + \mathcal{S}\boldsymbol{\omega}(t)\big) \quad \text{for all } t \in I. \tag{6}$$

Motivated by the above mechanical problem, in this paper we shall study inclusions of the form (6) in the abstract framework of real Hilbert spaces, under the assumption that $C(t)$ represents a family of prox-regular sets and $\mathcal{S}$ is an almost history-dependent operator.

The rest of the manuscript is organized as follows. In Sect. 2 we recall some notation and preliminaries which are used in the rest of the paper. In Sect. 3 we introduce the abstract history-dependent inclusion and state the main existence and uniqueness result,

Theorem 3.1, together with its consequences. The proof of the theorem is given in Sect. 4 based on a fixed point argument. Finally, in Sect. 5 we provide an example of Problem 1 for which our abstract results work.

2 Notation and preliminaries

In the whole paper, all vector spaces will be real vector spaces. We use $\mathbb{R}_+$ for the set of nonnegative reals, that is, $\mathbb{R}_+ := [0, +\infty)$. The letter T stands for an extended nonnegative real, i.e., $T \in \mathbb{R}_+ \cup \{+\infty\}$ and $I := [0, T] \cap \mathbb{R}_+$. In what follows X is a (real) Hilbert space endowed with its inner product $(\cdot, \cdot)_X$ and its associated norm $\| \cdot \|_X$. The open (resp. closed) ball with respect to the norm $\| \cdot \|_X$ centered at $x \in X$ with radius $r > 0$ is denoted by $B(x, r)$ (resp. $B[x, r]$). The letter $\mathbb{U}_X$ (resp. $\mathbb{B}_X$) stands for the open (resp. closed) unit ball of X centered at the origin 0_X, that is, $\mathbb{U}_X := B(0_X, 1)$ (resp. $\mathbb{B}_X := B[0_X, 1]$). The strong and weak convergences in X will be denoted by $\rightarrow$ and $\rightharpoonup$, respectively, and are considered as $n \to \infty$, even if we do not mention it explicitly. Recall that $\| \cdot \|_X$ enjoys the so-called *sequential Kadec–Klee property*, that is, every sequence $(x_n)_n \subset X$ satisfying $x_n \rightharpoonup x$ along with $\|x_n\|_X \to \|x\|_X$ for some $x \in X$ converges strongly to x.

Projections and nonlinear operators   The *metric projection multimapping* $\mathrm{Proj}_S : X \rightrightarrows X$ associated with a nonempty subset $S \subset X$ is defined as

$$\mathrm{Proj}_S(x) := \left\{ y \in S : d_S(x) = \|x - y\|_X \right\} \quad \text{for all } x \in X,$$

where $d_S(\cdot)$ (or $d(\cdot, S)$) is the *distance function* from S, that is,

$$d_S(x) :=: d(x, S) := \inf_{y \in S} \|x - y\|_X \quad \text{for all } x \in X.$$

When the set $\mathrm{Proj}_S(\overline{x})$ is reduced to a singleton for some vector $\overline{x} \in X$, we say that the metric projection of $\overline{x}$ on S is *well defined*. In such a case, the unique element of $\mathrm{Proj}_S(\overline{x})$ is denoted by $\mathrm{proj}_S(\overline{x})$ or $P_S(\overline{x})$. It is an exercise to check that, for any $x, x' \in X$,

$$x' \in \mathrm{Proj}_S(x) \quad \Longleftrightarrow \quad x' \in S \quad \text{and} \quad (x - x', y - x')_X \leq \frac{1}{2} \|y - x'\|_X^2 \quad \text{for all } y \in S. \quad (7)$$

It is known (and not difficult to establish) that the multimapping $\mathrm{Proj}_S(\cdot)$ is *monotone*, that is,

$$(p_1 - p_2, x_1 - x_2)_X \geq 0,$$

for every $x_1, x_2 \in X$, $p_1 \in \mathrm{Proj}_S(x_1)$, and $p_2 \in \mathrm{Proj}_S(x_2)$.

In the development below, the concept of strong monotonicity of operators will be needed. Recall that an operator $A : X \to X$ is said to be *strongly monotone* with constant $m_A > 0$ provided that

$$(Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \text{for all } u, v \in X. \quad (8)$$

Operators enjoying the Lipschitz property will be also used. We say that the operator $A : Y \to X$ is *Lipschitz continuous* with constant $L_A > 0$ on $Y \subset X$ provided that

$$\|Au - Av\|_X \leq L_A \|u - v\|_X \quad \text{for all } u, v \in Y. \quad (9)$$

The following result on strongly monotone Lipschitz continuous operators will be crucial in our study.

Proposition 2.1 *Let $A : X \to X$ be a strongly monotone Lipschitz continuous operator with respective constants $m_A > 0$ and $L_A > 0$. Then $A : X \to X$ is invertible, and its inverse $A^{-1} : X \to X$ is also strongly monotone and Lipschitz continuous with respective $\frac{m_A}{L_A^2}$ and $\frac{1}{m_A}$, respectively.*

For the proof of Proposition 2.1, we refer to [25, Theorem 1.24].

Proximal normal cone   We now assume that $S \subset X$ and U is a nonempty open subset of X. Then the *proximal normal cone* to S at $x \in X$ is defined as the set

$$N(S;x) := \begin{cases} \{v \in X : \exists r > 0, x \in \mathrm{Proj}_S(x + rv)\} & \text{if } x \in S, \\ \emptyset & \text{otherwise.} \end{cases}$$

For each $x \in S$, it is known that $N(S;x)$ is a convex cone (not necessarily closed), containing 0_X. Further, it can be checked that, for given $(v, x) \in X \times S$, the inclusion $v \in N(S;x)$ holds if and only if there is some real $\sigma \geq 0$ such that

$$\left(v, x' - x\right)_X \leq \sigma \left\| x' - x \right\|_X^2 \quad \text{for all } x' \in S.$$

From the above definition, it is not difficult to see that that for any $v \in X$ with $\mathrm{Proj}_S(v) \neq \emptyset$ the following inclusion holds:

$$v - w \in N(S;w) \quad \text{for all } w \in \mathrm{Proj}_S(v). \tag{10}$$

We conclude this part devoted to the proximal normal cone by recalling that if S is convex, then the following equality holds:

$$N(S;x) = \left\{ v \in X : \left(v, x' - x\right)_X \leq 0, \forall x' \in S \right\} \quad \text{for all } x \in S.$$

Prox-regular sets   We now recall the notion of prox-regular sets in Hilbert spaces. For historical comments, proofs, and further results, we refer to the survey by G. Colombo and L. Thibault [11] (see also the forthcoming monograph [28]) and the references therein.

Definition 2.2 Let S be a nonempty closed subset of X, and let $r \in (0, +\infty]$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever for all $x \in S$, $v \in N(S;x) \cap \mathbb{B}_X$ and $t \in (0, r]$ one has

$$x \in \mathrm{Proj}_S(x + tv).$$

Concerning this definition we have the following comments. First, note that if S is r-prox-regular, then it is r'-prox-regular for any $0 < r' < r$. Further, it is known that the class of ∞-prox-regular subsets of X is nothing but the class of nonempty closed convex sets of X.

Given a nonempty subset $S \subset X$, we denote by $U_r(S)$ and $\mathrm{Enl}_r(S)$ the *r-open and closed enlargement* of S, that is,

$$U_r(S) := \left\{ x \in X : d_S(x) < r \right\}$$

and

$$\mathrm{Enl}_r(S) := \left\{ x \in X : d_S(x) \le r \right\}.$$

Moreover, if $r := +\infty$, we set $1/r := 0$ and $U_r(S) := X$.

The following theorem provides some useful characterizations and properties of uniform prox-regular sets.

Theorem 2.3 *Let S be a nonempty closed subset of X. The following assertions are equivalent for any extended real $r \in (0, +\infty]$.*

(a) *The set S is r-prox-regular.*

(b) *For all $x_1, x_2 \in S$ and $\zeta \in N(S; x_1) \cap \mathbb{B}_X$, one has*

$$(\zeta, x_2 - x_1)_X \le \frac{1}{2r} \|x_1 - x_2\|_X^2.$$

(c) *For all $x_1, x_2 \in S$, $\zeta_1 \in N(S; x_1) \cap \mathbb{B}_X$, and $\zeta_2 \in N(S; x_2) \cap \mathbb{B}_X$, one has*

$$(\zeta_1 - \zeta_2, x_1 - x_2)_X \ge -\frac{1}{r} \|x_1 - x_2\|_X^2.$$

(d) *For any $0 < s < r$, $\mathrm{proj}_S(x)$ is well defined for every $x \in U_s(S)$ and the mapping $\mathrm{proj}_S(\cdot)$ is $(1 - s/r)^{-1}$-Lipschitz continuous therein, i.e.,*

$$\left\| \mathrm{proj}_S(x) - \mathrm{proj}_S(x') \right\|_X \le \frac{1}{1 - s/r} \left\| x - x' \right\|_X \quad \text{for all } x, x' \in U_s(S).$$

(e) *The function $x \mapsto d_S^2(x)$ is $C^{1,1}$ on $U_r(S)$ and*

$$\nabla d_S^2(x) = 2\left(x - \mathrm{proj}_S(x) \right) \quad \text{for all } x \in U_r(S).$$

(f) *For any $0 < s < r$, the function $x \mapsto d_S^2(x) + \frac{s}{r-s} \|x\|_X^2$ is convex on any open convex subset of $U_s(S)$.*

Let $\mathcal{N}$ be any of the normal cones in the sense of the Fréchet, Mordukhovich, or Clarke (see, e.g., [9, 16, 28] for the definitions and basic properties). It is known that assertions (b) and (c) with the truncated normal cone $\mathcal{N}(S; \cdot) \cap \mathbb{B}_X$ in place of the truncated proximal one $N(S; \cdot) \cap \mathbb{B}_X$ are also equivalent to the r-prox-regularity of S. Further, any r-prox-regular set S enjoys the following normal regularity:

$$\mathcal{N}(S; x) = N(S; x) \quad \text{for all } x \in S.$$

Moreover, taking $r = \infty$ in Theorem 2.3 leads to the following result.

Corollary 2.4 *Let S be a nonempty closed subset of X. The following assertions are equivalent.*

  (a) *The set S is convex.*

  (b) *For all $x_1, x_2 \in S$, $\zeta_1 \in N(S; x_1)$, and $\zeta_2 \in N(S; x_2)$, one has*

$$(\zeta_1 - \zeta_2, x_1 - x_2)_X \geq 0.$$

  (c) *For every $x \in X$, $\mathrm{proj}_S(x)$ is well defined and the mapping $\mathrm{proj}_S(\cdot)$ is 1-Lipschitz continuous, i.e.,*

$$\left\| \mathrm{proj}_S(x) - \mathrm{proj}_S(x') \right\|_X \leq \left\| x - x' \right\|_X.$$

  (d) *The function $x \mapsto d_S^2(x)$ is $C^{1,1}$ on X and*

$$\nabla d_S^2(x) = 2\left(x - \mathrm{proj}_S(x) \right) \quad \textit{for all } x \in X.$$

  (e) *The function $x \mapsto d_S^2(x)$ is convex on X.*

We now proceed with two results strongly involved in the proof of our main theorem below. The first one is related to inclusion (10) for prox-regular sets.

Lemma 2.5 ([3]) *Let $r \in (0, +\infty]$, S be an r-prox-regular set of X, and let $x, x' \in X$. If $x - x' \in N(S; x')$ and $\|x - x'\|_X \leq r$ (resp. $\|x - x'\|_X < r$), then $x' \in \mathrm{Proj}_S(x)$ (resp. $x' = \mathrm{proj}_S(x)$).*

The second result deals with some convergence properties of prox-regular sets.

Lemma 2.6 *Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of r-prox-regular subsets of X for some $r \in (0, +\infty]$, and let also S be an r-prox-regular subset of X. Then, for every $x \in U_r(S)$ such that $d(x, S_n) \to d(x, S)$, one has that $\mathrm{proj}_{S_n}(x)$ is well defined for $n \in \mathbb{N}$ large enough and*

$$\mathrm{proj}_{S_n}(x) \to \mathrm{proj}_S(x) \quad \textit{in } X.$$

Proof Fix any $x \in U_r(S)$ and assume that $d(x, S_n) \to d(x, S)$. Take real $0 < s' < r$ such that $d(x, S) < s'$ and take $s \in \mathbb{R}$ such that $s' < s < r$. Since $d(x, S) < s'$, there is an integer $N \in \mathbb{N}$ such that for every integer $n \geq N$ one has $d(x, S_n) < s'$. Set $V := B(x, s - s')$ and fix any integer $n \geq N$. For every $y \in V$, we have

$$d(y, S_n) \leq d(x, S_n) + \|y - x\|_X < s' + s - s' = s,$$

so $V \subset U_s(S_n)$ and, similarly, $V \subset U_s(S)$. Define the function $f_n : V \to \mathbb{R}$ by

$$f_n(y) := d(y, S_n)^2 + \frac{s}{r - s} \|y\|_X^2 \quad \text{for all } y \in V$$

and the function $f : V \to \mathbb{R}$ by

$$f(y) := d(y, S)^2 + \frac{s}{r - s} \|y\|_X^2 \quad \text{for all } y \in V.$$

According to Theorem 2.3, the functions f_n and f are convex and Fréchet differentiable on V. Moreover,

$$\nabla f_n(y) = 2\big(y - \text{proj}_{S_n}(y)\big) + \frac{2s}{r-s}y, \qquad \nabla f(y) = 2\big(y - \text{proj}_S(y)\big) + \frac{2s}{r-s}y \qquad (11)$$

for each $y \in V$. Set $v_n := x - \text{proj}_{S_n}(x)$ for every $n \geq N$. Note that the sequence $(v_n)_{n \geq N}$ is bounded since, by assumption, $(d_{S_n}(x))_{n \geq N}$ converges in $\mathbb{R}$. Let $(v_{s(n)})_{n \geq N}$ be any weakly convergent subsequence of the sequence $(v_n)_{n \geq N}$. Let v be its limit with respect to the weak topology on X. Fix any $z \in V$. Keeping in mind that $f_{s(n)}(\cdot)$ is a convex function, for each integer $n \geq N$, we may write

$$\big(\nabla f_{s(n)}(x), z - x\big)_X \leq f_{s(n)}(z) - f_{s(n)}(x).$$

Then, passing to the limit as $n \to \infty$, we find that

$$\left(2v + \frac{2s}{r-s}x, z - x\right)_X \leq f(z) - f(x).$$

Since $z \in V$ is arbitrary, we get $2v + \frac{2s}{r-s}x \in \partial f(x) = \{\nabla f(x)\}$, i.e.,

$$\nabla f(x) = 2v + \frac{2s}{r-s}x.$$

Coming back to (11), we see that $v = x - \text{proj}_S(x)$. Therefore, the whole sequence $(v_n)_{n \geq N}$ converges weakly in X to $x - \text{proj}_S(x)$. On the other hand, we obviously have

$$\|v_n\|_X = d(x, S_n) \to d(x, S) = \big\|x - \text{proj}_S(x)\big\|_X.$$

These two ingredients allow us to apply the Kadec–Klee property of the norm of X to obtain the strong convergence $v_n \to x - \text{proj}_S(x)$ in X. It results from the above that the whole sequence $(v_n)_{n \geq N}$ converges to $x - \text{proj}_S(x)$ in X, which means that $\text{proj}_{S_n}(x) \to \text{proj}_S(x)$ in X. The proof is then complete. $\qquad \square$

Examples and counter-examples   Theorem 2.3 shows that prox-regular and convex sets share many properties, including the differentiability of distance function, the existence of nearest points, and (hypo)monotonicity of normals, among others. This naturally led several authors to study preservation of prox-regularity under various set operations. In what follows we shall use an example based on the following general result.

Lemma 2.7 ([28]) *If S_1 and S_2 are r-prox-regular sets of X for some $r > 0$, then $S_1 \cup S_2$ is $\min\{g/2, r\}$-prox-regular whenever $g := \inf_{(c_1, c_2) \in S_1 \times S_2} \|c_1 - c_2\| > 0$.*

Remark 2.8 For the convenience of the reader, we also provide the following counter-examples.
  a) [11] Given real $r > 0$, there is an r-prox-regular set S of $\mathbb{R}^2$ such that $Q := S \cap \mathbb{R} \times \{0\}$ fails to be uniformly (even locally !) prox-regular.

b) The inverse image of a uniformly prox-regular set by a continuous linear mapping may fail to be prox-regular. Indeed, the above sets Q and S satisfy $A^{-1}(Q) = S$ with $A : \mathbb{R} \to \mathbb{R}^2$ defined by

$$A(x) := (x, 0) \quad \text{for all } x \in \mathbb{R}.$$

c) [4, Example 7] The direct image of a uniformly prox-regular set by a continuous linear mapping may fail to be prox-regular.

d) [2] The sublevel $\{f \le 0\}$ (resp. the level $\{f = 0\}$) is not prox-regular even for smooth functions. This can be seen in a straightforward way with the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) := xy \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Finally, we recall that, despite the above counter-examples, sufficient conditions ensuring the prox-regularity of $C := \{f_1 \le 0, \ldots, f_p \le 0, h_1 = 0, \ldots, h_q = 0\}$ are developed in [2] in the framework of Hilbert spaces.

History-dependent and almost history-dependent operators   For a normed space $(Y, \| \cdot \|_Y)$, we denote by $C(I; Y)$ the space of continuous functions defined on I with values in Y, i.e.,

$$C(I; Y) = \{v \colon I \to Y : v \text{ is continuous}\}.$$

The case $T > 0$ (i.e., $I = [0, T]$) leads to the space $C([0, T]; Y)$ which is a normed space equipped with the norm $\| \cdot \|_{C([0,T];Y)}$ defined by

$$\|v\|_{C([0,T];Y)} := \max_{t \in [0,T]} \|v(t)\|_Y \quad \text{for all } v \in C\big([0, T]; Y\big).$$

It is well known that $C([0, T]; Y)$ is a Banach space whenever Y is a Banach space. The case $I = \mathbb{R}_+$ leads to the space $C(\mathbb{R}_+; Y)$. If Y is a Banach space, then $C(\mathbb{R}_+; Y)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms.

The vector space of continuously differentiable functions on I with values in Y is denoted by $C^1(I; Y)$. Obviously, for any function $v : I \to Y$, the inclusion $v \in C^1(I; Y)$ holds if and only if $v \in C(I; Y)$ and $\dot{v} \in C(I; Y)$. Here and in what follows, $\dot{v}(\cdot)$ stands for the derivative of the function $v(\cdot)$. For a function $v \in C^1(I; Y)$, the following equality will be used in the next section of this manuscript:

$$v(t) = \int_0^t \dot{v}(s)\, ds + v(0) \quad \text{for all } t \in I.$$

Everywhere below, given two normed spaces Y and Z and an operator $\mathcal{S} \colon C(I; Y) \to C(I; Z)$, for any function $u \in C(I; X)$, we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point $t \in I$, that is, $\mathcal{S}u(t) := (\mathcal{S}u)(t)$.

We end this section with two important classes of operators defined on the space of continuous functions.

Definition 2.9 Let $(Y, \|\cdot\|_Y)$ (resp. $(Z, \|\cdot\|_Z)$) be a normed space, and let K (resp. K') be a nonempty closed subset of Y (resp. Z). An operator $\mathcal{S}\colon C(I;K) \to C(I;K')$ is called:

a) history-dependent (h.d. for short), if for any nonempty compact set $\mathcal{J} \subset I$, there exists $L_{\mathcal{J}}^{\mathcal{S}} > 0$ such that, for all $u_1, u_2 \in C(I;K)$ and all $t \in \mathcal{J}$,

$$\left\| \mathcal{S}u_1(t) - \mathcal{S}u_2(t) \right\|_Z \leq L_{\mathcal{J}}^{\mathcal{S}} \int_0^t \left\| u_1(s) - u_2(s) \right\|_Y ds. \tag{12}$$

b) almost history-dependent (a.h.d. for short), if for any nonempty compact set $\mathcal{J} \subset I$, there exist $l_{\mathcal{J}}^{\mathcal{S}} \in [0,1)$ and $L_{\mathcal{J}}^{\mathcal{S}} > 0$ such that, for all $u_1, u_2 \in C(I;K)$ and all $t \in \mathcal{J}$,

$$\left\| \mathcal{S}u_1(t) - \mathcal{S}u_2(t) \right\|_Z \leq l_{\mathcal{J}}^{\mathcal{S}} \left\| u_1(t) - u_2(t) \right\|_Y + L_{\mathcal{J}}^{\mathcal{S}} \int_0^t \left\| u_1(s) - u_2(s) \right\|_Y ds. \tag{13}$$

The next fixed point result makes clear the interest of such operators.

Theorem 2.10 *Let K be a nonempty closed subset of a Banach space Y, and let $\Lambda : C(I;K) \to C(I;K)$ be an almost history-dependent operator. Then Λ has a unique fixed point, i.e., there exists a unique element $\eta^* \in C(I;K)$ such that $\Lambda\eta^* = \eta^*$.*

A proof of Theorem 2.10 can be found in [26, pp. 41–45]. There, the main properties of history-dependent and almost history-dependent operators are presented together with various examples and applications.

3 Problem statement and main results

In this section we state an existence and uniqueness result for a time-dependent inclusion involving nonlinear operators. To this end we consider a nonempty closed bounded subset $K \subset X$, a multimapping $C : I \rightrightarrows X$, and two operators $A : X \to X$ and $\mathcal{S} : C(I;X) \to C(I;K)$. As usual, $\mathrm{Im}(C)$ denotes the *range* of $C(\cdot)$, that is,

$$\mathrm{Im}(C) := \bigcup_{t\in I} C(t).$$

With the above data and notation at hand, we introduce the following inclusion problem.

Problem 2 Find a continuous function $u : I \to X$ such that

$$-u(t) \in N\big(C(t); Au(t) + \mathcal{S}u(t)\big) \quad \text{for all } t \in I. \tag{14}$$

In the study of (14) we consider the following assumptions.

$(\mathcal{C})$ The multimapping $C : I \rightrightarrows X$ has r-prox-regular values for some real $r \in (0, +\infty]$ and, for every $t \in I$ and every sequence $(t_n)_{n\geq 1}$ of I converging to t, one has

$$d\big(u, C(t_n)\big) \to d\big(u, C(t)\big) \quad \text{for all } u \in U_r\big(C(t)\big). \tag{15}$$

$(\mathcal{A})$ The mapping A is m_A-strongly monotone and L_A-Lipschitz continuous for some reals $m_A, L_A > 0$.

(S) For any nonempty compact set $\mathcal{J} \subset I$, there exist $l_{\mathcal{J}}^{\mathcal{S}} > 0$ and $L_{\mathcal{J}}^{\mathcal{S}} > 0$ such that, for all $u_1, u_2 \in C(I; X)$ and $t \in \mathcal{J}$, inequality (13) holds.

Note that, using Lemma 2.6, it follows that the Wijsman-type convergence (15) is equivalent to the convergence in X

$$\text{proj}_{S_n}(x) \to \text{proj}_S(x) \quad \text{for all } x \in U_r(S).$$

Our main result in the study of Problem 2 that we state here and prove in the next section is the following.

Theorem 3.1 *Assume* $(\mathcal{C})$, $(\mathcal{A})$, *and* $(\mathcal{S})$. *Assume also that*

$$m_A < \min\{L_A, L_A^{2/3}\}, \tag{16}$$

$$s := \sup_{x \in \text{Im}(C) - K} \|A^{-1}(x)\|_X < \frac{m_A^3}{L_A^2}\left(1 - \sqrt{1 - \frac{m_A^4}{L_A^4}}\right) r \tag{17}$$

along with

$$\frac{m_A}{2} > \frac{2sL_A^2}{r}. \tag{18}$$

In addition, assume that for any nonempty compact set $\mathcal{J} \subset I$ *the following smallness condition holds:*

$$\left(\frac{1}{2m_A} + \frac{2s}{r}\right)^{1/2} l_{\mathcal{J}}^{\mathcal{S}} < \left(\frac{m_A}{2} - \frac{2sL_A^2}{r}\right)^{1/2}. \tag{19}$$

Then Problem 2 has at least a solution $u(\cdot)$. *Moreover, the solution takes values in* $s\mathbb{B}_X := \{sb : b \in \mathbb{B}_X\}$ *and is the unique solution of Problem 2 with this property.*

Remark 3.2 As mentioned in the introduction, Problem 2 has been already studied in [19] under the assumption that $C(t)$ is a nonempty closed convex moving set (that is, ∞-prox-regular). It should be noted that if $r = \infty$, then estimate (18) obviously holds, (17) means that the operator A^{-1} is bounded on the set $\text{Im}(C) - K$, and (19) becomes

$$l_{\mathcal{J}}^{\mathcal{S}} < m_A.$$

This inequality plays a crucial role for the well-posedness of Problem 2 driven by a convex set $C(t)$ studied in [19]. There, the set K is possibly unbounded (say $K = X$), and there is no need to assume the boundedness of the operator A^{-1}.

In the rest of this section we present some consequences of Theorem 3.1.

Corollary 3.3 *Assume* $(\mathcal{C})$, $(\mathcal{A})$, (16), (17), *and* (18). *Assume also that* $\mathcal{S} : C(I; X) \to C(I; K)$ *is a history-dependent operator. Then Problem 2 has at least a solution* $u(\cdot)$. *Moreover, the solution takes values in* $s\mathbb{B}_X := \{sb : b \in \mathbb{B}_X\}$ *and is the unique solution of Problem 2 with this property.*

Proof Since $\mathcal{S}$ is a history-dependent operator, Definition 2.9(a) guarantees that condition $(\mathcal{S})$ holds with $l_{\mathcal{J}}^{\mathcal{S}} = 0$ for any compact $\mathcal{J} \subset I$. We deduce from here that in this case the smallness condition (19) is satisfied. Therefore, Corollary 3.3 is a direct consequence of Theorem 3.1. $\qquad\square$

Theorem 3.1 allows us to obtain an existence and uniqueness result for a first order sweeping process. To present it, besides the data C, A, and $\mathcal{S}$ and their associated assumptions $(\mathcal{C})$, $(\mathcal{A})$, and $(\mathcal{S})$, respectively, we consider an operator $B : X \to K$ and an element u_0 such that:

   $(\mathcal{B})$ $B : X \to X$ is a Lipschitz continuous operator with values in $Y \subset X$.
   $(\mathcal{K})$ $Y + K \subset K$.
   $(\mathcal{U})$ $u_0 \in X$.

We are now in a position to introduce the following sweeping process.

Problem 3 Find a continuously differentiable function $u : I \to X$ such that

$$
\begin{cases}
-\dot{u}(t) \in N(C(t); A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)) & \text{for all } t \in I, \\
u(0) = u_0.
\end{cases}
$$

Our first result in this section is the following.

Corollary 3.4 *Assume that* $(\mathcal{C})$, $(\mathcal{A})$, $(\mathcal{S})$, $(\mathcal{B})$, $(\mathcal{K})$, $(\mathcal{U})$, (16), (17), (18), *and* (19). *Then Problem* 2 *has at least a solution* $u(\cdot)$. *Moreover, its derivative* $\dot{u}(\cdot)$ *takes values in* $s\mathbb{B}_X :=$ $\{sb : b \in \mathbb{B}_X\}$ *and* $u(\cdot)$ *is the unique solution of Problem* 3 *with this property.*

Proof We use assumption $(\mathcal{K})$ to introduce the operator $\widetilde{\mathcal{S}} : C(I; X) \to C(I; K)$ defined by

$$
\widetilde{\mathcal{S}}v(t) := B\left(\int_0^t v(s)\,ds + u_0 \right) + \mathcal{S}v(t) \quad \text{for all } t \in I, \text{all } v \in C(I; X). \tag{20}
$$

Next, we consider the auxiliary problem of finding a function $v : I \to X$ such that

$$
-v(t) \in N\big(C(t); Av(t) + \widetilde{\mathcal{S}}v(t)\big) \quad \text{for all } t \in I. \tag{21}
$$

Let $L_B > 0$ be a Lipschitz constant of the operator B. We use assumptions $(\mathcal{S})$ and $(\mathcal{B})$ to see that, for any nonempty compact set $\mathcal{J} \subset I$, any functions $v_1, v_2 \in C(I; X)$, and any $t \in I$, the following inequality holds:

$$
\big\| \widetilde{\mathcal{S}}v_1(t) - \widetilde{\mathcal{S}}v_2(t) \big\|_X \leq l_{\mathcal{J}}^{\mathcal{S}} \big\| v_1(t) - v_2(t) \big\|_X + \big(L_B + L_{\mathcal{J}}^{\mathcal{S}} \big) \int_0^t \big\| v_1(s) - v_2(s) \big\|_X ds.
$$

Therefore, we are in a position to apply Theorem 3.1 in order to obtain the existence of a unique function $v \in C(I; X)$ with values in $s\mathbb{B}_X$, which satisfies the time-dependent inclusion (21). Denote by $u : I \to X$ the function defined by

$$
u(t) := u_0 + \int_0^t v(s)\,ds \quad \text{for all } t \in I. \tag{22}
$$

Then (20)–(22) imply that u is a solution of Problem 3 with regularity $u \in C^1(I; X)$. This proves the existence part of the theorem. The uniqueness part follows from the unique solvability of auxiliary problem (21), guaranteed by Theorem 3.1. □

A direct consequence of Corollary 3.4 is the following.

Corollary 3.5 *Assume* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{U})$, (16), (17), *and* (18). *Assume also that* $\mathcal{S}$: $C(I; X) \to C(I; K)$ *is a history-dependent operator. Then Problem 2 has at least a solution* $u(\cdot)$. *Moreover, its derivative* $\dot{u}(\cdot)$ *takes values in* $s\mathbb{B}_X := \{sb : b \in \mathbb{B}_X\}$ *and* $u(\cdot)$ *is the unique solution of Problem 2 with this property.*

The proof of Corollary 3.5 follows from arguments similar to those used in the proof of Corollary 3.3 and, therefore, we skip it.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps that we present below. We start with a fixed point result for the projection mapping on a prox-regular set.

Lemma 4.1 *Let* C *be an* r-*prox-regular subset of* X *for some* $r \in (0, +\infty)$, *and let* $B : X \to X$ *be an* m_B-*strongly monotone and* L_B-*Lipschitz continuous operator with* $m_B < \min\{L_B, L_B^2\}$. *Assume that*

$$\alpha := \sup_{x \in C-K} \|B(x)\|_X < \rho(1 - \sqrt{1 - \rho m_B})r =: \beta \quad \text{with } \rho := \frac{m_B}{L_B^2}. \tag{23}$$

Then the following statements hold for any $\eta \in K$.

(a) *For all* $z \in C$ *and all* $\theta \in (0, 1]$, *one has* $z - \theta B(z - \eta) \in \mathrm{Enl}_{\theta\alpha}(C) \subset U_r(C)$.

(b) *For every* $s \in [\rho\alpha, \beta]$, *the mapping* $C \ni z \mapsto \mathrm{proj}_C(z - \rho B(z - \eta))$ *is a contraction on* C *of constant* $(1 - s/r)^{-1}(1 - m_B\rho)^{1/2}$.

(c) *There exists a unique element* $z_\eta \in C$ *such that*

$$z_\eta = \mathrm{proj}_C\big(z_\eta - \rho B(z_\eta - \eta)\big) = \mathrm{proj}_C\big(z_\eta - B(z_\eta - \eta)\big). \tag{24}$$

Proof (a) Fix $\eta \in K$. Note that inclusion (a) follows directly from the estimates

$$d_C\big(z - \theta B(z - \eta)\big) \leq d_C(z) + \theta \|B(z - \eta)\|_X \leq \theta \sup_{x \in C-K} \|B(x)\|_X = \theta\alpha < r,$$

valid for every $z \in C$ and every $\theta \in (0, 1]$. This allows us to consider the mapping $\Lambda : C \to C$ defined by

$$\Lambda z := \mathrm{proj}_C\big(z - \rho B(z - \eta)\big) \quad \text{for all } z \in C, \tag{25}$$

where $\delta := m_B\rho = \frac{m_B^2}{L_B^2} \in (0, 1)$.

(b) We claim that Λ is a contraction on C. First, note that

$$1 + \rho^2 L_B^2 - 2\rho m_B = 1 - \delta > 0. \tag{26}$$

Pick any $s \in (\rho\alpha, \beta]$. We have $z - \rho B(z - \eta) \in \mathrm{Enl}_{\rho\alpha}(C) \subset U_s(C)$ for all $z \in C$. Fix any $z_1, z_2 \in C$ and set $u_i := z_i - \eta$ for each $i \in \{1, 2\}$. Using the definition of Λ in (25) combined with the $\kappa := (1 - s/r)^{-1}$-Lipschitz property of $\mathrm{proj}_C(\cdot)$ on the s-open enlargement $U_s(C)$ of C, the m_B-strong monotonicity of B, and (26), we see that

$$
\begin{aligned}
\|\Lambda z_1 - \Lambda z_2\|_X^2 &\leq \kappa^2 \left\| \left(z_1 - \rho B(z_1 - \eta)\right) - \left(z_2 - \rho B(z_2 - \eta)\right) \right\|_X^2 \\
&= \kappa^2 \left\| (u_1 - u_2) - \rho(Bu_1 - Bu_2) \right\|_X^2 \\
&= \kappa^2 \|u_1 - u_2\|_X^2 - 2\rho\kappa^2(u_1 - u_2, Bu_1 - Bu_2)_X + (\rho\kappa)^2 \|Bu_1 - Bu_2\|_X^2 \\
&\leq \kappa^2 \left(1 + \rho^2 L_B^2 - 2\rho m_B\right) \|u_1 - u_2\|_X^2 \\
&= \kappa^2(1 - \delta) \|u_1 - u_2\|_X^2 = \kappa^2(1 - \delta) \|z_1 - z_2\|_X^2.
\end{aligned}
$$

On the other hand, an elementary computation shows that

$$
\begin{aligned}
\kappa^2(1 - \delta) < 1 \quad &\Longleftrightarrow \quad 1 - \frac{m_B^2}{L_B^2} < \left(1 - \frac{s}{r}\right)^2 \\
&\Longleftrightarrow \quad s^2 - 2rs + \frac{m_B^2 r^2}{L_B^2} > 0 \\
&\Longleftrightarrow \quad (s - \gamma_1)(s - \gamma_2) > 0
\end{aligned}
$$

with $\gamma_1 := r(1 + \sqrt{1 - \delta}) > r(1 - \sqrt{1 - \delta}) =: \gamma_2 > \beta$. Combining the above equivalences with inclusion $s \in (0, \beta]$, we deduce that $\kappa^2(1 - \delta) < 1$, which is the desired inequality. Therefore, the mapping Λ is a contraction on the nonempty closed subset C of the Hilbert space X with constant $\kappa(1 - \delta)^{\frac{1}{2}}$. From the arbitrariness of $s \in (\rho\alpha, \beta]$, it is easy to see that Λ is a contraction on C with constant $(1 - \rho\alpha/r)^{-1}(1 - \delta)^{\frac{1}{2}}$.

(c) The classical Banach fixed point theorem then guarantees the existence of unique $z_\eta \in C$ such that $\Lambda z_\eta = z_\eta$. Now, putting together this equality and the inclusion (10), we get

$$
-\rho B(z_\eta - \eta) \in N(C; z_\eta)
$$

or, equivalently (keeping in mind that $N(C; z_\eta)$ is a cone in X),

$$
-B(z_\eta - \eta) \in N(C; z_\eta).
$$

It remains to observe that (23) and assumption $\rho \in (0, 1)$ (coming from inequality $m_B < L_B^2$) imply that

$$
-B(z_\eta - \eta) = \left(z_\eta - B(z - \eta)\right) - z_\eta \in N(C; z_\eta) \cap r\mathbb{U}_X.
$$

This inclusion and Lemma 2.5 entail that

$$
z_\eta = \mathrm{proj}_C\left(z_\eta - B(z - \eta)\right),
$$

which concludes the proof.   □

We now use Lemma 4.1 to deduce the following result.

Lemma 4.2 *Assume* $(\mathcal{C})$ *and let* $B : X \to X$ *be an* m_B-*strongly monotone and* L_B-*Lipschitz continuous operator with* $m_B < \min\{L_B, L_B^2\}$. *Moreover, assume that*

$$\alpha := \sup_{x \in \mathrm{Im}(C) - K} \|B(x)\|_X < \rho(1 - \sqrt{1 - \rho m_B})r =: \beta \quad with \ \rho := \frac{m_B}{L_B^2}.$$

Then, for any $\eta \in C(I; K)$, *there exists a unique continuous function* $z_\eta : I \to X$ *such that*

$$z_\eta(t) = \mathrm{proj}_{C(t)}\big(z_\eta(t) - B\big(z_\eta(t) - \eta(t)\big)\big) \quad for \ all \ t \in I.$$

Proof Let $\eta \in C(I; K)$. Thanks to Lemma 4.1, we know that for every $t \in I$ there exists a unique element $z_\eta(t) \in C(t)$ such that

$$z_\eta(t) = \mathrm{proj}_{C(t)}\big(z_\eta(t) - B\big(z_\eta(t) - \eta(t)\big)\big) = \mathrm{proj}_{C(t)}\big(z_\eta(t) - \rho B\big(z_\eta(t) - \eta(t)\big)\big).$$

This justifies the claimed existence and uniqueness property. It remains to establish that $z_\eta(\cdot)$ is a continuous function. Fix $t \in I$ and consider a sequence $(t_n)_{n \in \mathbb{N}}$ of elements of I which converges to t. Due to the closedness of I, we obviously have $t \in I$. For each $n \in \mathbb{N}$, denote $C_n := C(t_n)$, $\eta_n := \eta(t_n)$, $\zeta_n := z_\eta(t_n)$, and $\omega_n := \zeta_n - \rho B(\zeta_n - \eta_n)$. Set also $C_\infty := C(t)$, $\eta_\infty := \eta(t)$, $\zeta_\infty := z_\eta(t)$, and $\omega_\infty := \zeta_\infty - \rho B(\zeta_\infty - \eta_\infty)$. With the above notation at hand, it is clear that for every integer $n \in \mathbb{N}$ we have

$$\zeta_\infty = \mathrm{proj}_{C_\infty}(\omega_\infty) \quad and \quad \zeta_n = \mathrm{proj}_{C_n}(\omega_n),$$

hence,

$$\|\zeta_\infty - \zeta_n\|_X \leq \big\|\mathrm{proj}_{C_\infty}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_\infty)\big\|_X + \big\|\mathrm{proj}_{C_n}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_n)\big\|_X. \tag{27}$$

We now estimate each of the two terms in the right-hand side of (27). We start by setting

$$u_\infty := \zeta_\infty - \eta_\infty \quad and \quad u_n := \zeta_n - \eta_n \quad for \ all \ n \in \mathbb{N}. \tag{28}$$

It is readily seen that

$$\|u_\infty - u_n\|_X \leq \|\zeta_\infty - \zeta_n\|_X + \|\eta_\infty - \eta_n\|_X \quad for \ all \ n \in \mathbb{N}. \tag{29}$$

Set $\delta := \rho m_B$. Fix any real $\varepsilon > 0$ with $\rho\alpha + \varepsilon < \beta$ and let $s \in (\rho\alpha + \varepsilon, \beta)$. Since $\zeta_\infty \in C_\infty \subset U_r(C_\infty)$, we can use assumption $(\mathcal{C})$ to see that $d(\zeta_\infty, C_n) \to d(\zeta_\infty, C_\infty) = 0$. Thus, we can find some integer $N \geq 1$ such that

$$d(\zeta_\infty, C_n) \leq \varepsilon \quad for \ all \ n \geq N.$$

Fix for a moment an integer $n \geq N$. We easily observe that

$$d_{C_n}(\omega) \leq d_{C_n}(\zeta_\infty) + \rho\big\|B(\zeta_\infty - \eta_\infty)\big\|_X \leq \varepsilon + \rho\alpha < s.$$

On the other hand, Lemma 4.1(a) guarantees that

$$d_{C_n}(\omega_n) \leq \rho\alpha < s$$

and the r-prox-regularity of C_n implies that

$$\left\| \mathrm{proj}_{C_n}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_n) \right\|_X \leq \kappa \|\omega_\infty - \omega_n\|_X$$

with $\kappa := (1 - s/r)^{-1}$. Using this inequality, the definition of ω_∞, ω_n, and (28), we see that

$$
\begin{aligned}
\left\| \mathrm{proj}_{C_n}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_n) \right\|_X
&\leq \kappa \left\| \zeta_\infty - \rho B(\zeta_\infty - \eta_\infty) - \zeta_n + \rho B(\zeta_n - \eta_n) \right\|_X \\
&= \kappa \left\| u_\infty - u_n + \rho(Bu_n - Bu_\infty) + \eta_\infty - \eta_n \right\|_X \\
&\leq \kappa \left\| u_\infty - u_n + \rho(Bu_n - Bu_\infty) \right\|_X + \|\eta_\infty - \eta_n\|_X. \quad (30)
\end{aligned}
$$

Moreover, using the m_B-strong monotonicity of B and its L_B-Lipschitz property yields

$$
\begin{aligned}
\left\| (u_\infty - u_n) - \rho(Bu_\infty - Bu_n) \right\|_X^2
&\leq \|u_\infty - u_n\|_X^2 - 2\rho(u_\infty - u_n, Bu_\infty - Bu_n)_X + \rho^2 \|Bu_\infty - Bu_n\|_X^2 \\
&\leq \left(1 + \rho^2 L_B^2 - 2\rho m_B \right) \|u_\infty - u_n\|_X^2 = (1 - \delta)\|u_\infty - u_n\|_X^2,
\end{aligned}
$$

or, equivalently,

$$\left\| u_\infty - u_n - \rho(Bu_\infty - Bu_n) \right\|_X \leq \sqrt{(1-\delta)} \|u_\infty - u_n\|_X. \quad (31)$$

Finally, letting $L := \kappa\sqrt{1 - \delta}$ and taking into account inequalities (31), (30), and (29) it follows that

$$\left\| \mathrm{proj}_{C_n}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_n) \right\|_X \leq L\|\zeta_\infty - \zeta_n\|_X + (1 + L)\|\eta_\infty - \eta_n\|_X.$$

Noting that $L \in (0, 1)$ (see Lemma 4.1(b)) and coming back to inequality (27), we see that

$$(1 - L)\|\zeta_n - \zeta_\infty\|_X \leq \left\| \mathrm{proj}_{C_\infty}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_\infty) \right\|_X + (1 + L)\|\eta_n - \eta_\infty\|_X. \quad (32)$$

Next, using inequality

$$d_D(\omega_\infty) \leq d_D(\zeta_\infty) + \rho \left\| B(\zeta_\infty - \theta) \right\|_X = \rho \left\| B(\zeta_\infty - \theta) \right\|_X \leq \rho\alpha < r,$$

it follows that $\omega_\infty \in U_r(C_\infty)$. Therefore, using assumption $(\mathcal{C})$ and Theorem 2.6, we obtain that

$$\left\| \mathrm{proj}_{C_\infty}(\omega_\infty) - \mathrm{proj}_{C_n}(\omega_\infty) \right\|_X \to 0.$$

It remains to use (32) and the continuity of the function $\eta : I \to K$ to see that $\zeta_n = z(t_n) \to z(t) = \zeta_\infty$ in X, as $n \to \infty$. This shows that the function $z_\eta : I \to X$ is continuous and concludes the proof. □

The next step is the following.

Lemma 4.3 *Assume* $(\mathcal{C})$, $(\mathcal{A})$, (16), (17), *and* (18). *Then, for any* $\eta \in C(I;K)$, *there exists a unique continuous function* $u_\eta : I \to X$ *such that*

$$-u_\eta(t) \in N\big(C(t); Au_\eta(t) + \eta(t)\big) \cap s\mathbb{B}_X \quad \text{for all } t \in I.$$

Proof Using Proposition 2.1, it follows that the operator A^{-1} is $m_{A^{-1}} := \frac{m_A}{L_A^2}$-strongly monotone and $L_{A^{-1}} := \frac{1}{m_A}$-Lipschitz continuous. Note that $m_{A^{-1}} < \min\{L_{A^{-1}}, L_{A^{-1}}^2\}$ and, moreover,

$$s < \rho(1 - \sqrt{1 - \rho m_{A^{-1}}})r \quad \text{with } \rho := \frac{m_{A^{-1}}}{L_{A^{-1}}^2}.$$

Let $\eta \in C(I;K)$ and denote by $z_\eta \in C(I;X)$ the function obtained in Lemma 4.2 with $B := A^{-1}$. Then

$$z_\eta(t) = \text{proj}_{C(t)}\big(z_\eta(t) - u_\eta(t)\big) \quad \text{for all } t \in I$$

with $u_\eta \in C(I;X)$ defined by

$$u_\eta(t) := A^{-1}\big(z_\eta(t) - \eta(t)\big) \in s\mathbb{B}_X \quad \text{for all } t \in I.$$

It follows from the definition of proximal normal cone that

$$\big(z_\eta(t) - u_\eta(t)\big) - z_\eta(t) = -u_\eta(t) \in N\big(C(t); z_\eta(t)\big) \quad \text{for all } t \in I,$$

and this concludes the proof of the existence part of the lemma.

Now, let $u_1, u_2 : I \to X$ be two functions such that

$$-u_1(t) \in N\big(C(t); Au_1(t) + \eta(t)\big) \cap s\mathbb{B}_X \quad \text{and} \quad -u_2(t) \in N\big(C(t); Au_2(t) + \eta(t)\big) \cap s\mathbb{B}_X$$

for every $t \in I$. Fix any $t \in I$. Then, for each $i \in \{1,2\}$, we have

$$Au_i(t) + \eta(t) \in C(t)$$

along with

$$\big(u_i(t), Au_i(t) + \eta(t) - v\big)_X \le \frac{\|u_i(t)\|_X}{2r}\big\|Au_i(t) + \eta(t) - v\big\|_X^2 \quad \text{for all } v \in C(t).$$

This implies that

$$\big(u_1(t), Au_1(t) + \eta(t) - \big(Au_2(t) + \eta(t)\big)\big)_X \le \frac{s}{2r}\big\|Au_1(t) - Au_2(t)\big\|_X^2,$$

$$\big(u_2(t), Au_2(t) + \eta(t) - \big(Au_1(t) + \eta(t)\big)\big)_X \le \frac{s}{2r}\big\|Au_1(t) - Au_2(t)\big\|_X^2,$$

and adding these inequalities yields

$$\big(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\big)_X \le \frac{s}{r}\big\|Au_1(t) - Au_2(t)\big\|_X^2.$$

Using now the m_A-strong monotonicity and the L_A-Lipschitz property of the operator A, we get

$$m_A \|u_1(t) - u_2(t)\|_X^2 \le \frac{sL_A^2}{r} \|u_1(t) - u_2(t)\|_X^2.$$

We now use the the assumption $m_A > \frac{sL_A^2}{r}$ to deduce that $u_1(t) = u_2(t)$.   □

Lemma 4.3 allows us to consider the operator $\Lambda : C(I;K) \to C(I;K)$ defined by

$$\Lambda\eta := \mathcal{S}u_\eta \quad \text{for all } \eta \in C(I;K). \tag{33}$$

We have the following result.

Lemma 4.4 *Assume* $(\mathcal{C})$, $(\mathcal{A})$, $(\mathcal{H})$, (16), (17), (18), *and* (19). *Then the operator* Λ *has a unique fixed point* $\eta^* \in C(I;K)$.

Proof According to Theorem 2.10, it is enough to prove that the operator $\Lambda : C(I;K) \to C(I;K)$ is an almost history-dependent operator. Let $\eta_1, \eta_2 \in C(I;K)$. Using Lemma 4.3, we find two continuous functions $u_1 := u_{\eta_1} : I \to s\mathbb{B}_X$ and $u_2 := u_{\eta_2} : I \to s\mathbb{B}_X$ such that

$$-u_1(t) \in N\big(C(t); Au_1(t) + \eta_1(t)\big) \quad \text{and} \quad -u_2(t) \in N\big(C(t); Au_2(t) + \eta_2(t)\big) \tag{34}$$

for all $t \in I$. Let $\mathcal{J}$ be a nonempty compact subset of I and let $t \in \mathcal{J}$. Using (33) and assumption $(\mathcal{S})$ yields

$$\big\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\big\|_X = \big\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\big\|_X$$

$$\le l_{\mathcal{J}}^{\mathcal{S}} \big\|u_1(t) - u_2(t)\big\|_X + L_{\mathcal{J}}^{\mathcal{S}} \int_0^t \big\|u_1(s) - u_2(s)\big\|_X \, ds. \tag{35}$$

On the other hand, from (34) we see that

$$Au_i(t) + \eta_i(t) \in C(t)$$

for each $i \in \{1, 2\}$ and, therefore,

$$\big(u_i(t), Au_i(t) + \eta_i(t) - v\big)_X \le \frac{s}{2r} \big\|Au_i(t) + \eta_i(t) - v\big\|_X^2 \quad \text{for all } v \in C(t).$$

Taking $i = 1$ and $v := Au_2(t) + \eta_2(t)$ in the above estimate yields

$$\big(u_1(t), Au_1(t) + \eta_1(t) - Au_2(t) - \eta_2(t)\big)_X \le \frac{s}{2r} \big\|Au_1(t) + \eta_1(t) - Au_2(t) - \eta_2(t)\big\|_X^2.$$

Similarly, taking $i = 2$ and $v := Au_1(t) + \eta_1(t)$, we get

$$\big(u_2(t), Au_2(t) + \eta_2(t) - Au_1(t) - \eta_1(t)\big)_X \le \frac{s}{2r} \big\|Au_2(t) + \eta_2(t) - Au_1(t) - \eta_1(t)\big\|_X^2.$$

Adding the previous two inequalities, we find that

$$\left(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\right)_X$$
$$\leq \left(u_1(t) - u_2(t), \eta_2(t) - \eta_1(t)\right)_X + \frac{s}{r}\left\|Au_2(t) + \eta_2(t) - Au_1(t) - \eta_1(t)\right\|_X^2.$$

Fix $\varepsilon > 0$ such that $m_A - \frac{\varepsilon^2}{2} - \frac{2sL_A^2}{r} > 0$. Then, using the Cauchy–Schwarz inequality and the elementary inequality $2ab \leq (\varepsilon a)^2 + (\frac{b}{\varepsilon})^2$ valid for every real a, b, we obtain that

$$\left(u_1(t) - u_2(t), \eta_2(t) - \eta_1(t)\right)_X \leq \frac{\varepsilon^2}{2}\left\|u_1(t) - u_2(t)\right\|_X^2 + \frac{1}{2\varepsilon^2}\left\|\eta_1(t) - \eta_2(t)\right\|_X^2.$$

On the other hand, note that

$$\left\|Au_1(t) + \eta_1(t) - Au_2(t) - \eta_2(t)\right\|_X^2 = \left\|Au_1(t) - Au_2(t)\right\|_X^2 + \left\|\eta_1(t) - \eta_2(t)\right\|_X^2$$
$$+ 2\left(Au_1(t) - Au_2(t), \eta_1(t) - \eta_2(t)\right)_X$$
$$\leq L_A^2\left\|u_1(t) - u_2(t)\right\|_X^2 + \left\|\eta_1(t) - \eta_2(t)\right\|_X^2$$
$$+ 2L_A\left\|u_1(t) - u_2(t)\right\|_X\left\|\eta_1(t) - \eta_2(t)\right\|_X$$
$$\leq L_A^2\left\|u_1(t) - u_2(t)\right\|_X^2 + \left\|\eta_1(t) - \eta_2(t)\right\|_X^2$$
$$+ L_A^2\left\|u_1(t) - u_2(t)\right\|_X^2 + \left\|\eta_1(t) - \eta_2(t)\right\|_X^2$$
$$= 2L_A^2\left\|u_1(t) - u_2(t)\right\|_X^2 + 2\left\|\eta_1(t) - \eta_2(t)\right\|_X^2.$$

Therefore, setting $a_\varepsilon := m_A - \frac{\varepsilon^2}{2} - \frac{2sL_A^2}{r}$, $b_\varepsilon := \frac{1}{2\varepsilon^2} + \frac{2s}{r}$ and taking into account the strong monotonicity of the operator A, we find that

$$\left\|u_1(t) - u_2(t)\right\|_X^2 \leq \frac{b_\varepsilon}{a_\varepsilon}\left\|\eta_1(t) - \eta_2(t)\right\|_X^2.$$

Now, choosing $\varepsilon := \sqrt{m_A}$, we get

$$\left\|u_1(t) - u_2(t)\right\|_X \leq c\left\|\eta_1(t) - \eta_2(t)\right\|_X$$

with $c := (\frac{1}{2m_A} + \frac{2s}{r})^{1/2}(\frac{m_A}{2} - \frac{2sL_A^2}{r})^{-1/2}$. Substituting this inequality in (35) yields

$$\left\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\right\|_X = \left\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\right\|_X$$
$$\leq cl_\mathcal{J}^\mathcal{S}\left\|\eta_1(t) - \eta_2(t)\right\|_X + cL_\mathcal{J}^\mathcal{S}\int_0^t\left\|\eta_1(s) - \eta_2(s)\right\|_X ds.$$

We now invoke the smallness assumption (19) to obtain that the operator Λ is an almost history-dependent operator. It remains to apply Theorem 2.10 to complete the proof. $\square$

We are now in a position to provide the proof of Theorem 3.1.

Proof Let $\eta^* \in C(I;K)$ be the fixed point of the operator Λ, and let $u^\star := u_{\eta^\star} \in C(I;X)$ be the function given by Lemma 4.3 with $\eta := \eta^\star$. So, we have

$$-u^\star(t) \in N\big(C(t); Au(t) + \eta(t)\big) \cap s\mathbb{B}_X \quad \text{for all } t \in I.$$

This inclusion combined with equality $\eta^\star = \Lambda\eta^\star = \mathcal{S}u^\star$ implies that

$$-u^\star(t) \in N\big(C(t); Au^\star(t) + \mathcal{S}u^\star(t)\big) \cap s\mathbb{B}_X \quad \text{for all } t \in I,$$

which shows that $u^\star$ is a solution to Problem 2. This proves the existence part of Theorem 3.1. The uniqueness part is a direct consequence of the uniqueness of the fixed point of the operator Λ. $\qquad\square$

5 An example

In this section we provide an example of Problem 2 for which our abstract results work. To this end we consider two elements $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{S}^d$ and three real constants g_1, g_2, and k such that

$$g_1, g_2, k > 0, \tag{36}$$

$$k > g_2, \tag{37}$$

$$\|a_1 - a_2\| > g_1 + k, \tag{38}$$

$$\frac{\|\boldsymbol{a}_1\| + g_1}{\|\boldsymbol{a}_1 - \boldsymbol{a}_2\| - g_1 - g_2} < \frac{\sqrt{7}}{32}. \tag{39}$$

We now introduce the sets

$$C_1 = \big\{\boldsymbol{x} \in \mathbb{S}^d : \|\boldsymbol{x} - \boldsymbol{a}_1\| \le g_1\big\}, \tag{40}$$

$$C_2 = \big\{\boldsymbol{x} \in \mathbb{S}^d : \|\boldsymbol{x} - \boldsymbol{a}_2\| \le g_2\big\}, \tag{41}$$

$$C(t) = C_1 \cup C_2 \quad \forall t \in I, \tag{42}$$

$$K = \big\{\boldsymbol{x} \in \mathbb{S}^d : \|\boldsymbol{x} - \boldsymbol{a}_2\| \le k\big\}. \tag{43}$$

For the sake of simplicity, we only consider the setting where $C(t)$ is autonomous, i.e., independent of time t. Nevertheless, we mention that the result below in this section can be easily extended to the case when g_1, g_2 are real-valued positive functions depending on $t \in I$ and, in this case, the set $C(t)$ will depend on t. We denote by $P_K : \mathbb{S}^d \to K$ the projection operator on the closed convex set K and consider the following inclusion problem.

Problem 4 Find a continuous function $\boldsymbol{u} : I \to X$ such that

$$-\boldsymbol{u}(t) \in N\left(C(t); \boldsymbol{u}(t) + P_K\left(\int_0^t \boldsymbol{u}(s)\,ds\right)\right) \quad \text{for all } t \in I.$$

We have the following existence and uniqueness result.

Theorem 5.1 *Assume* (36)–(39). *Then Problem* 4 *has at least a solution* $\boldsymbol{u}(\cdot)$. *Moreover, the solution takes values in* $s\mathbb{B}_{\mathbb{S}^d} := \{s\boldsymbol{b} : \boldsymbol{b} \in \mathbb{B}_{\mathbb{S}^d}\}$ *with* $s := \|\boldsymbol{a}_1\| + g_1$ *and is the unique solution of Problem* 4 *with this property.*

Proof We apply Corollary 3.3 on the space $X = \mathbb{S}^d$ with $C(t)$ defined by (40)–(42), K defined by (43), and the operators A, $\mathcal{S}$ given by

$$A\boldsymbol{u} = \boldsymbol{u} \quad \forall \boldsymbol{u} \in \mathbb{S}^d, \tag{44}$$

$$\mathcal{S}\boldsymbol{u}(t) = P_K\left(\int_0^t \boldsymbol{u}(s)\,ds\right) \quad \forall \boldsymbol{u} \in C(I; \mathbb{S}^d). \tag{45}$$

First, we remark that the sets C_1 and C_2 are convex and, therefore they are ∞-prox regular. Moreover, using assumptions (37), (38), we see that

$$g := \inf_{(\boldsymbol{c}_1, \boldsymbol{c}_2) \in C_1 \times C_2} \|\boldsymbol{c}_1 - \boldsymbol{c}_2\| > 0 = \|\boldsymbol{a}_1 - \boldsymbol{a}_2\| - g_1 - g_2 > 0.$$

Therefore, Lemma 2.7 guarantees that for each $t \in I$ the set $C(t)$ is r-prox-regular with

$$r = \frac{1}{2}\left(\|\boldsymbol{a}_1 - \boldsymbol{a}_2\| - g_1 - g_2\right). \tag{46}$$

In addition, since $C(t)$ does not depend on t, it follows that convergence (15) holds. We conclude from here that assumption $(\mathcal{C})$ is satisfied.

On the other hand, it is obvious to see that the operator A satisfies condition $(\mathcal{A})$ with $m_A = 1$ and $L_A = 1 + \varepsilon$ for any $\varepsilon > 0$. Moreover, the operator $\mathcal{S}$ defined by (45) is history-dependent operator (keeping in mind that P_K is 1-Lipschitz on $\mathbb{S}^d$). In addition, inequality (16) is obviously satisfied.

We now show that with a convenient choice of ε, conditions (17) and (18) are satisfied. To this end we use (37) and (38) to see that $\mathrm{Im}(C) - K = (C_1 \cup C_2) - K = C_1$ and, therefore,

$$s := \sup_{\boldsymbol{x} \in \mathrm{Im}(C) - K} \|A^{-1}\boldsymbol{x}\| = \sup_{\boldsymbol{x} \in C_1} \|\boldsymbol{x}\| = \|\boldsymbol{a}_1\| + g_1. \tag{47}$$

Hence, using (46), (47) and equalities $m_A = 1$, $L_A = 1 + \varepsilon$, it follows that conditions (17), (18) are equivalent with the inequalities

$$\frac{\|\boldsymbol{a}_1\| + g_1}{\|\boldsymbol{a}_1 - \boldsymbol{a}_2\| - g_1 - g_2} < \frac{1}{2(1 + \varepsilon)^2}\left(1 - \sqrt{1 - \frac{1}{(1 + \varepsilon)^4}}\right), \tag{48}$$

$$\frac{\|\boldsymbol{a}_1\| + g_1}{\|\boldsymbol{a}_1 - \boldsymbol{a}_2\| - g_1 - g_2} < \frac{1}{8(1 + \varepsilon)^2}, \tag{49}$$

respectively. Next, an elementary calculus reveals that

$$x > \frac{\sqrt{7}}{4} \quad \Longrightarrow \quad \frac{\sqrt{7}}{32} < \frac{x}{8} < \frac{x}{2}\left(1 - \sqrt{1 - x^2}\right),$$

and, using this inequality with $x = \frac{1}{(1+\varepsilon)^2}$, we deduce that

$$0 < \varepsilon < \frac{2}{\sqrt{\sqrt{7}}} - 1 \quad \Longrightarrow \quad \frac{\sqrt{7}}{32} < \frac{1}{8(1 + \varepsilon)^2} < \frac{1}{2(1 + \varepsilon)^2}\left(1 - \sqrt{1 - \frac{1}{(1 + \varepsilon)^4}}\right). \tag{50}$$

Choose now $0 < \varepsilon < \frac{2}{\sqrt[4]{7}} - 1$. Then, using (39) and (50), we find that (48) and (49) hold, which implies that conditions (17) and (18) are satisfied, too. Theorem 5.1 is now a direct consequence of Corollary 3.3.   □

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Not applicable.

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