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BV prox-regular sweeping process with bounded truncated variation

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ABSTRACT
This paper is devoted to the existence and uniqueness of solutions for perturbed sweeping process measure differential inclusions in infinite dimensional setting. The possibly unbounded moving set is prox-regular and controlled only through the truncated Hausdorff-Pompeiu distance. The normal cone involved is perturbed by a kind of Carathéodory mapping satisfying a time-dependent hypomonotonicity assumption on bounded sets. Various properties of the solution mapping are also provided.

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1. Introduction

Consider a Hilbert space $\mathcal{H}$ and any real $T > 0$. Given $C : [0, T] \rightrightarrows \mathcal{H}$ a nonempty closed convex valued multimapping and $u_0 \in C(0)$, Moreau [1] established in 1971 that there is one and only one absolutely continuous mapping $u : [0, T] \to \mathcal{H}$ such that

$$(\mathcal{P}_{AC}) \begin{cases} -\dot{u}(t) \in N(C(t); u(t)) & \text{a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0, \end{cases}$$

provided that $C(\cdot)$ is absolutely continuous with respect to the Hausdorff-Pompeiu distance, or equivalently for some absolutely continuous function $v : [0, T] \to \mathbb{R}_+ := [0, +\infty[$,

$$\widehat{\text{haus}}(C(s), C(t)) := \max\left\{ \sup_{x \in C(s)} d_{C(t)}(x), \ \sup_{x \in C(t)} d_{C(s)}(x) \right\} \leq |v(s) - v(t)|, \quad (1)$$

for every $s, t \in [0, T]$. Here and below, $N(\cdot;\cdot)$ stands for the normal cone in the sense of convex analysis. Due to its kinematic interpretation (see for instance the

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This paper is dedicated to Boris Mordukhovich on the occasion on his 70th birthday.

introduction of [2]) J.J. Moreau called the differential inclusion $(\mathcal{P}_{AC})$ 'sweeping process' ('processus de rafle' in French). Later, $(\mathcal{P}_{AC})$ has been transformed by J.J. Moreau into the following measure differential inclusion (see, Section 4)

$$(\mathcal{P}_{BV}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) & \text{a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

As above, under an appropriate control on the moving set, namely the finite variation of $C(\cdot)$, or equivalently the existence of a positive Radon measure μ on $[0, T]$ such that

$$\widehat{\text{haus}}(C(s), C(t)) \leq \mu(]s, t]) \quad \text{for all } s, t \in [0, T] \text{with } s \leq t, \tag{2}$$

the problem $(\mathcal{P}_{BV})$ is well-posed (in the sense of existence and uniqueness of a solution).

The crucial role of generalized Cauchy problems $(\mathcal{P}_{AC})$ and $(\mathcal{P}_{BV})$ in numerous applications of mathematics (see, e.g. [3–7]) has led to the development of many variants of the so-called 'Moreau sweeping process' which have their own interest: stochastic [8], perturbed [9], nonconvex [10], state-dependent [11], in Banach spaces [12].

The general study of perturbed sweeping processes probably starts with M.D.P. Monteiro Marques in [9]. It consists in adding a multi-valued term $F(t, u(t))$ (which can be seen in a mechanical point of view as external forces to the system modelized by the considered sweeping process) on the normal cone involved, that is, (in the bounded variation case)

$$(\mathcal{P}_{PBV}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + F(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(t) \in C(t) & \text{for all } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Over the years, such differential inclusions have been at the heart of a large number of works. For existence results (depending on the nature of the perturbation $F(\cdot, \cdot)$ involved) we refer to [13–15] and the references therein. Besides considering a perturbation $F(\cdot, \cdot)$, it is also of interest, for both theoretical and concrete aspects (see, e.g. [16,17]) to relax the convexity assumption on the moving set $C(\cdot)$. The first nonconvex study is due to Valadier [10] with a moving subset $C(t) \subset \mathbb{R}^n$ such that the multimapping $(t, x) \mapsto N^C(C(t); x)$ has a closed graph, where $N^C(\cdot; \cdot)$ denotes the Clarke normal cone. The latter property holds in particular when $C(t) := \mathbb{R}^n \setminus \text{int}K(t)$ for a convex $K(t)$ and such a case has been widely developed in the early nineties (see, e.g. [18,19]). Actually, the existence of solutions for nonconvex sweeping processes still remains a very well-active area of research. It involves large classes of sets coming from variational analysis as prox-regular, subsmooth and α-far (see, e.g. [13,15,20,38] and the references therein). It is worth pointing out that the class of prox-regular sets [21] is known to be the more general ensuring the well-posedness of problem $(\mathcal{P}_{PBV})$ considered with a mapping $F \equiv f$. It is also the suitable class for control problems with sweeping processes (see, e.g. [22–24]).

As mentioned in [25] and in [26], there are many practical situations where an unbounded moving set is not absolutely continuous or of bounded variation with respect to the Hausdorff-Pompeiu distance. An efficient way to weaken the aforementioned control (1) and (2) seems to consist in replacing the Haudorff-Pompeiu $\widehat{\text{haus}}(\cdot,\cdot)$ by the truncated one [26–29], that is, (in the bounded variation framework)

$$\widehat{\text{haus}}_\rho(C(s), C(t)) := \max\left\{ \sup_{x \in C(s)\cap\rho\mathbb{B}} d_{C(t)}(x), \sup_{x \in C(t)\cap\rho\mathbb{B}} d_{C(s)}(x) \right\} \le \mu(]s,t]). \tag{3}$$

In [28], it has been proved that $(\mathcal{P}_{BV})$ with $C(t)$ convex has one and only one solution provided that there are a real $\rho_0 \ge \|u_0\|$ and an extended real $\rho > 0$ such that

$$\left\| \text{proj}_{C(t_k)} \circ \cdots \circ \text{proj}_{C(t_1)}(u_0) \right\| \le \rho_0 \tag{4}$$

for each $t_1 < \cdots < t_k$ and provided that $C(\cdot)$ has a bounded retraction along ρ-truncation (see, e.g. [28]), or equivalently if for some positive Radon measure μ on $[0, T]$

$$\widehat{\text{exc}}_\rho(C(s), C(t)) := \sup_{x \in C(s)\cap\rho\mathbb{B}} d_{C(t)}(x) \le \mu(]s,t]).$$

We also refer to Adly and Le [27] for another existence result in the context of perturbed convex second order sweeping process under (3).

The present paper deals with the existence and uniqueness of a solution for the perturbed sweeping process $(\mathcal{P}_{PBV})$ with $F = f : [0, T] \times \mathcal{H} \to \mathcal{H}$ as a mapping. Here, the moving set $C(\cdot)$ is prox-regular and satisfies (3). The perturbation $f(\cdot,\cdot)$ is measurable in time and uniformly continuous in the state, satisfies for each bounded subset B of $\mathcal{H}$ the hypomonotonicity property for some $l_B \in L^1([0, T], \mathbb{R}_+, \lambda)$,

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \ge -l_B(t) \|x_1 - x_2\|^2 \quad \text{for all } t \in I, x_1, x_2 \in B.$$

For the unperturbed case (that is, $f \equiv 0$), we develop another existence and uniqueness result under the assumption on successive projections (4).

The paper is organized as follows. Section 2 is devoted to recall background in variational analysis and vector measure theory. Section 3 is concerned with various preparatory results which are necessary in order to establish existence and uniqueness of solution for $(\mathcal{P}_{PBV})$ in Section 4 and in Section 5.

2. Preliminaries

Throughout, $I := [T_0, T]$ is an interval of $\mathbb{R}$ with $T_0 < T$ and λ denotes the *Lebesgue measure* on I. The extended real-line is denoted by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, $\mathbb{R}_+ := [0, +\infty[$ is the set of nonnegative reals and $\mathbb{N}$ is the set of the integers starting from 1.

In all the paper, $\mathcal{H}$ is a real Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, the associated norm by $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$ and the closed (resp., open) unit ball centered at zero by $\mathbb{B}$ (resp., $\mathbb{U}$). The closed (resp., open) ball of $\mathcal{H}$ centered at $x \in \mathcal{H}$ of radius $r > 0$ is denoted by $B[x, r]$ (resp., $B(x, r)$). For any subset S of $\mathcal{H}$, $d_S(\cdot)$ (or $d(\cdot, S)$) is the distance function to S, that is,

$$d_S(x) =: d(x, S) := \inf_{y \in S} \|x - y\| \quad \text{for all } x \in \mathcal{H}$$

and the convex (resp., closed convex) hull of S is denoted by $\operatorname{co} S$ (resp., $\overline{\operatorname{co}}\, S$). The multimapping $\operatorname{Proj}_S : \mathcal{H} \rightrightarrows \mathcal{H}$ of nearest points on S is defined by

$$\operatorname{Proj}_S(x) := \left\{ y \in S : \|x - y\| = d_S(x) \right\} \quad \text{for all } x \in \mathcal{H}.$$

If $\operatorname{Proj}_S(x)$ is a singleton for some $x \in \mathcal{H}$, we denote by $\operatorname{proj}_S(x)$ or $\mathrm{P}_S(x)$ the only element of $\operatorname{Proj}_S(x)$, i.e.

$$\operatorname{Proj}_S(x) = \left\{ \operatorname{proj}_S(x) \right\} = \{ \mathrm{P}_S(x) \}.$$

In such a case, one says that $\operatorname{proj}_S(x)$ or $\mathrm{P}_S(x)$ is well-defined.

2.1. Normal cones and subdifferentials

We start by recalling the necessary background on proximal and Clarke normal cones and subdifferentials. In this subsection, $f : U \to \mathbb{R} \cup \{+\infty\}$ is a function defined on a nonempty open subset U of $\mathcal{H}$, finite at $\overline{x} \in U$ and S is a closed subset of $\mathcal{H}$.

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal normal* to S at $x \in S$ whenever there exists a real $r > 0$ such that $x \in \operatorname{Proj}_S(x + r\zeta)$. The set $N^P(S; x)$ (which is a convex cone containing 0 but not necessarily closed) of all proximal normal vectors to S at $x \in S$ is called the *proximal normal cone* of S at x. By convention, if $x \in \mathcal{H} \setminus S$, we put $N^P(S; x) = \emptyset$. It is worth pointing out that for each $u \in \mathcal{H}$ with $\operatorname{Proj}_S(u) \neq \emptyset$,

$$u - \pi \in N^P(S; \operatorname{proj}_S(\pi)) \quad \text{for all } \pi \in \operatorname{Proj}_S(u). \tag{5}$$

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal subgradient* of f at $\overline{x}$ with $f(\overline{x})$ finite, provided there are a real $\sigma \geq 0$ and a real $\eta > 0$ such that

$$\left\langle \zeta, y - \overline{x} \right\rangle \leq f(y) - f(\overline{x}) + \sigma \left\| y - \overline{x} \right\|^2 \quad \text{for all } y \in B(\overline{x}, \eta),$$

which is known to be equivalent to $(\zeta, -1) \in N^P(\operatorname{epi} f; (\overline{x}, f(\overline{x})))$, where $\operatorname{epi} f := \{(x, r) \in \mathcal{H} \times \mathbb{R} : x \in U, f(x) \leq r\}$ is the epigraph of f. The set $\partial_P f(\overline{x})$ of all such proximal subgradients is called the *proximal subdifferential* of f at $\overline{x}$. If f is not finite at $\overline{x} \in U$, one sets $\partial_P f(\overline{x}) := \emptyset$.

The *Clarke normal cone* of S at $x \in S$ is defined by

$$N^C(S;x) := \overline{\mathrm{co}}\left({}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{S \ni u \to x} N^P(S;u) \right),$$

where ${}^{\mathrm{seq}}\operatorname{Lim\,sup}_{S \ni u \to x} N^P(S;u)$ is the sequential limit superior of $N^P(S;\cdot) : \mathcal{H} \rightrightarrows \mathcal{H}$ relative to the set S at x. Recall that the *sequential limit superior* (or *sequential outer limit*) of a multimapping $M : X \rightrightarrows Y$ between two topological spaces X and Y relative to a subset $X_0 \subset X$ at $x \in \mathrm{cl}\, X_0$ (the closure of X_0 in X) is defined as the set

$${}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{X_0 \ni x' \to x} M(x') := \left\{ y \in Y : \exists X_0 \ni x_n \to x, y_n \to y, y_n \in M(x_n)\, \forall n \in \mathbb{N} \right\}.$$

It is clear that the Clarke normal cone is a closed convex cone containing 0. With $N^C(S;x) := \emptyset$ for every $x \in \mathcal{H} \setminus S$, we see that

$$N^P(S;x) \subset N^C(S;x) \quad \text{for all } x \in \mathcal{H}.$$

For f Lipschitz continuous near $\overline{x}$, one defines the *Clarke subdifferential* of f at $\overline{x}$ as the set

$$\partial_C f(\overline{x}) := \overline{\mathrm{co}}\left({}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{x' \to \overline{x}} \partial_P f(x') \right) \supset \partial_P f(\overline{x}).$$

If f is not finite at $\overline{x} \in U$, one sets $\partial_C f(\overline{x}) := \emptyset$. The support function of $\partial_C f(\overline{x})$ (with f Lipschitz continuous near $\overline{x}$) is given by the so-called *Clarke directional derivative* of f at $\overline{x}$, that is, the function $f^o(\overline{x};\cdot) : \mathcal{H} \to \overline{\mathbb{R}}$ defined by

$$f^o(\overline{x};h) := \limsup_{t \downarrow 0, x' \to \overline{x}} t^{-1}(f(x'+th) - f(x')) \quad \text{for all } h \in \mathcal{H}.$$

Recall that the *support function* $\sigma(\cdot, S)$ of S is defined by

$$\sigma(\zeta, S) := \sup_{x \in S} \langle \zeta, x \rangle \quad \text{for all } \zeta \in \mathcal{H}$$

and such a function satisfies (thanks to the Hahn-Banach separation theorem) the following equivalence

$$S_1 \subset S_2 \Leftrightarrow \sigma(\cdot, S_1) \le \sigma(\cdot, S_2), \tag{6}$$

for any two closed convex subsets S_1, S_2 of $\mathcal{H}$. If U is convex and f is Lipschitz continuous near $\overline{x}$ and convex on U, then the *(standard) directional derivative*

$f'(\overline{x}; h) := \lim_{t\downarrow 0} t^{-1}(f(\overline{x}+th) - f(\overline{x}))$ exists for any direction $h \in \mathcal{H}$ and

$$f^o(\overline{x}; h) = f'(\overline{x}; h) \quad \text{for all } h \in \mathcal{H}. \tag{7}$$

If f is convex on some open ball $B(\overline{x}, \delta) \subset U$ for a real $\delta > 0$, one observes that

$$f'(x; h) = \inf_{t\in]0,\delta[} t^{-1}(f(x+th) - f(x)) \quad \text{for all } h \in \mathbb{B}.$$

For any real $\gamma \geq 0$ such that f is γ-Lipschitz near $\overline{x}$, it is known (and not difficult to prove) that $f(\overline{x}; \cdot)$ is finite, sublinear (that is, convex and positively homogeneous) and γ-Lipschitz continuous on $\mathcal{H}$, so in particular

$$\partial_C f(\overline{x}) \subset \gamma\mathbb{B}.$$

It is worth pointing out that the following relations hold true for all $x \in S$:

$$\partial_P d_S(x) = N^P(S; x) \cap \mathbb{B} \quad \text{and} \quad \partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}. \tag{8}$$

For more details on those concepts, we refer to the books [30–32].

2.2. *Vector measures*

In order to define the concept of solutions for sweeping processes with bounded variation, some preliminaries about positive and vector measures are needed. Throughout this subsection, ν and $\hat{\nu}$ are positive Radon measures on $I = [T_0, T]$. For each $t \in I$, $r \in]0, +\infty[$, one sets

$$I(t, r) := I \cap [t-r, t+r], I^+(t, r) := I \cap [t, t+r] \quad \text{and}$$
$$I^-(t, r) := I \cap [t-r, t].$$

For a subset A of I, we denote by $\mathbf{1}_A$ the *characteristic function* (in the sense of measure theory) of A relative to I, i.e. for all $t \in I$,

$$\mathbf{1}_A(t) := \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For any real $p \geq 1$, $L^p(I, \mathcal{H}, \nu)$ stands for the real space of (classes of) ν-measurable mappings from I to $\mathcal{H}$ for which the pth power of their norm value is ν-integrable on I.

The *derivative of the measure* $\hat{\nu}$ *with respect to* ν *is defined as the following limit (with the convention* $\frac{0}{0} = 0$*)*

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r\downarrow 0} \frac{\hat{\nu}(I(t, r))}{\nu(I(t, r))} \tag{9}$$

which exists for ν-almost every $t \in I$. It is worth pointing out that $(d\hat{\nu}/d\nu)(\cdot)$ is a nonnegative Borel function. If $\hat{\nu}$ is the Lebesgue measure on I, that is $\hat{\nu} = \lambda$, the equality (9) gives

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \frac{\lambda(\{t\})}{\nu(\{t\})} = 0 \quad \text{for all } t \in I \text{ with } \nu(\{t\}) > 0, \tag{10}$$

hence

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\nu(\{t\}) = 0 \quad \nu\text{−a.e. } t \in I. \tag{11}$$

Coming back to a general Radon measure $\hat{\nu}$ on I, it is known that the measure $\hat{\nu}$ is absolutely continuous with respect to ν if and only if $\hat{\nu} = (\mathrm{d}\hat{\nu}/\mathrm{d}\nu)(\cdot)\nu$ (i.e. $(\mathrm{d}\hat{\nu}/\mathrm{d}\nu)(\cdot)$ is a density relative to ν). If the latter equality holds, a mapping $u(\cdot) : I \to \mathcal{H}$ is $\hat{\nu}$-integrable on I if and only if $u(\cdot)(\mathrm{d}\hat{\nu}/\mathrm{d}\nu)(\cdot)$ is ν-integrable on I. In such a case, one has

$$\int_I u(t)\,\mathrm{d}\hat{\nu}(t) = \int_I u(t)\frac{\mathrm{d}\hat{\nu}}{\mathrm{d}\nu}(t)\,\mathrm{d}\nu(t). \tag{12}$$

If the two Radon measures ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other one, one says that ν and $\hat{\nu}$ are *absolutely continuously equivalent.*

Now, let us consider a Radon vector measure m on I with values in the real Hilbert space $\mathcal{H}$. The *variation measure* $|m|$ of m is defined for any Borel set $A \subset I$ by

$$|m|\,(A) := \sup_{(B_n)_{n\in\mathbb{N}}\in\mathcal{B}} \sum_{n=1}^{+\infty} \|m(B_n)\|\,,$$

where $\mathcal{B}$ is the set of all sequences $(B_n)_{n\in\mathbb{N}}$ of Borel mutually disjoint subsets of I such that $A = \bigcup_{n\in\mathbb{N}} B_n$. The vector measure m is said to be *absolutely continuous with respect to* ν whenever the positive measure $|m|$ is absolutely continuous with respect to ν. Since $\mathcal{H}$ has the Radon-Nikodým property, under such an absolute continuity assumption, the vector measure m has a density $\zeta : I \to \mathcal{H}$ relative to ν, i.e. $m = \zeta(\cdot)\nu$ (or equivalently, $\zeta(\cdot) \in L^1(I, \mathcal{H}, \nu)$ and for all Borel sets $A \subset I$,

$$m(A) = \int_A \zeta(t)\,\mathrm{d}\nu(t)).$$

In the rest of this section, we focus on mappings with bounded variation. Let $u : I \to \mathcal{H}$ be a mapping. Any $\sigma = (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that $T_0 = t_0 < \cdots < t_k = T$ is called a *subdivision* σ of $[T_0, T] = I$ and to such a subdivision σ, one associates the real $S_\sigma := \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$. If $\mathcal{S}$ denotes the set of all subdivisions of I, one defines the *variation* of u as the extended real

$$\mathrm{var}(u; I) := \sup_{\sigma\in\mathcal{S}} S_\sigma.$$

The mapping u is said to be of *bounded variation on* I if $\mathrm{var}(u; I) < +\infty$. It is well-known that $u(\cdot)$ has one sided limits at each point of I whenever it is of bounded variation on I. In such a case, one sets

$$u(\tau^-) := \lim_{t\uparrow\tau} u(t) \quad \text{for all } \tau \in]T_0, T],$$

where in the whole paper, $t \uparrow \tau$ means $t \to \tau$ with $t < \tau$.

Assume that u is of bounded variation on I and denote by du the differential measure (also called Stieltjes measure) on I with values in $\mathcal{H}$ associated to it (see, e.g. [33]). If in addition, u is right-continuous on I, it is known that

$$u(t) = u(s) + \int_{]s,t]} \mathrm{d}u \quad \text{for all } s, t \in I \text{ with } s \le t.$$

Conversely, if there is a ν-integrable mapping $\hat{u} : I \to \mathcal{H}$ on I satisfying

$$u(t) = u(T_0) + \int_{]T_0,t]} \hat{u}(\tau)\,\mathrm{d}\nu(\tau) \quad \text{for all } t \in I,$$

then $u(\cdot)$ is of bounded variation and right-continuous on I. In such a case, one has

$$|\mathrm{d}u|\,(]s,t]) = \int_{]s,t]} \|\hat{u}(\tau)\|\,\mathrm{d}\nu(\tau) \quad \text{for all } s, t \in I \text{ with } s \le t$$

and du is absolutely continuous with respect to ν and has $\hat{u}(\cdot)$ as a density relative to ν, i.e.

$$\mathrm{d}u = \hat{u}(\cdot)\,\mathrm{d}\nu.$$

According to Moreau and Valadier [34], for ν-almost every $t \in I$, the following limits exists in $\mathcal{H}$,

$$\hat{u}(t) = \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) := \lim_{r\downarrow 0} \frac{\mathrm{d}u(I(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{\mathrm{d}u(I^+(t,r))}{\nu(I^+(t,r))} = \lim_{r\downarrow 0} \frac{\mathrm{d}u(I^-(t,r))}{\nu(I^-(t,r))}.$$

From this, it can be checked that

$$\hat{u}(t) = \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) = \lim_{r\downarrow 0} \frac{\mathrm{d}u(]t-r,t] \cap I)}{\nu(]t-r,t] \cap I)} \quad \nu\text{—a.e. } t \in I. \tag{13}$$

2.3. Bounded variation along ρ-truncation

Let $\rho \in]0, +\infty]$ be a given positive extended real and let S and S' be nonempty subsets of $\mathcal{H}$.

One defines the *ρ-excess* $\mathrm{exc}_\rho(S, S')$ and the *pseudo* ρ-excess $\widehat{\mathrm{exc}}_\rho(S, S')$ of S over S' (also called the *pseudo excess of the* ρ-truncation of S over S') as the extended reals

$$\mathrm{exc}_\rho(S,S') := \sup_{x\in\rho\mathbb{B}} \big(d(x,S') - d(x,S)\big)^+ \quad \text{and} \quad \widehat{\mathrm{exc}}_\rho(S,S') := \sup_{x\in S\cap\rho\mathbb{B}} d(x,S'),$$

so $\widehat{\mathrm{exc}}_\rho(S,S') \le \mathrm{exc}_\rho(S,S')$. It is also known that for any $\rho' \ge 2\rho + d(0,S)$ one also has $\mathrm{exc}_\rho(S,S') \le \widehat{\mathrm{exc}}_{\rho'}(S,S')$, so

$$\widehat{\mathrm{exc}}_\rho(S,S') \le \mathrm{exc}_\rho(S,S') \le \widehat{\mathrm{exc}}_{\rho'}(S,S'). \tag{14}$$

If $\rho = +\infty$, we set by convention $\rho\mathbb{B} = \mathcal{H}$, so in this case both ρ-excess and pseudo ρ-excess of S over S' coincide with the usual *excess of S over S'*, that is,

$$\begin{aligned}\widehat{\mathrm{exc}}_\infty(S,S') &= \sup_{x\in S} d(x,S') =: \mathrm{exc}(S,S') = \sup_{x\in X}\big(d(x,S') - d(x,S)\big)^+\\ &= \mathrm{exc}_\infty(S,S').\end{aligned}$$

It is clear that the ρ-excess $\mathrm{exc}(\cdot,\cdot)$ enjoys the triangle inequality property. It is also readily seen that for every $x' \in \mathcal{H}$,

$$d(x',S') \le d(x',x) + \widehat{\mathrm{exc}}_\rho(S,S') \quad \text{for all } x \in S\cap\rho\mathbb{B},$$

i.e.

$$d(x',S') \le d(x',S\cap\rho\mathbb{B}) + \widehat{\mathrm{exc}}_\rho(S,S') \quad \text{for all } x' \in \mathcal{H}. \tag{15}$$

With the above concept at hand, one can define the *Hausdorff ρ-distance* $\mathrm{haus}_\rho(S,S')$ and the *Hausdorff pseudo ρ-distance* $\widehat{\mathrm{haus}}_\rho(S,S')$ between S and S' as

$$\begin{aligned}\mathrm{haus}_\rho(S,S') &:= \max\big\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S)\big\} = \sup_{x\in\rho\mathbb{B}} |d(x,S') - d(x,S)|,\\ \widehat{\mathrm{haus}}_\rho(S,S') &:= \max\big\{\widehat{\mathrm{exc}}_\rho(S,S'), \widehat{\mathrm{exc}}_\rho(S',S)\big\}.\end{aligned}$$

Clearly, the triangle inequality holds for $\mathrm{haus}_\rho(\cdot,\cdot)$ (while it fails for $\widehat{\mathrm{haus}}_\rho(\cdot,\cdot)$). If $\rho = +\infty$, both $\mathrm{haus}_\rho(S,S')$ and $\widehat{\mathrm{haus}}_\rho(S,S')$ coincide with $\mathrm{haus}(S,S')$, the usual *Hausdorff-Pompeiu distance between S and S'*, i.e.

$$\mathrm{haus}_\infty(S,S') = \widehat{\mathrm{haus}}_\infty(S,S') = \max\big\{\mathrm{exc}(S,S'), \mathrm{exc}(S',S)\big\} =: \mathrm{haus}(S,S').$$

From (14) one sees that for any $\rho' \ge \rho + 2\max\{d(0,S), d(0,S')\}$ one has

$$\widehat{\mathrm{haus}}_\rho(S,S') \le \mathrm{haus}_\rho(S,S') \le \widehat{\mathrm{haus}}_{\rho'}(S,S').$$

In this paper, given a moving set $C : I \rightrightarrows \mathcal{H}$, a mapping $f : I \times \mathcal{H} \to \mathcal{H}$ and $u_0 \in C(T_0)$, we are interested in the study of the following (measure) differential inclusion

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)), \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(T_0) = u_0 \in C(T_0). \end{cases}$$

In order to develop sufficient conditions to ensure the existence of solutions for such a problem, we will assume that there are an extended real $\rho \in]\|u_0\|, +\infty]$ and a positive Radon measure μ on I such that

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) \leq \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t. \tag{16}$$

As we will see below, such an assumption is strongly connected to the concept of multimappings with bounded ρ-variation, which is an extension of the notion of mappings with bounded variation. Let us consider a multimapping $C : I = [T_0, T] \rightrightarrows \mathcal{H}$. To each subdivision $\sigma_0 = (t_0, \ldots, t_k)$ of I (with $k \in \mathbb{N}$), one associates the extended real

$$h_{\sigma_0,\rho} := \sum_{i=0}^{k-1} \mathrm{haus}_\rho(C(t_i), C(t_{i+1})).$$

The *ρ-variation*, or the *variation along ρ-truncation*, of $C(\cdot)$ on I (with respect to $\mathrm{haus}_\rho(\cdot, \cdot)$) is defined as the extended real

$$\mathrm{var}_\rho(C; I) := \sup_{\sigma \in \mathcal{S}} h_{\sigma,\rho},$$

where $\mathcal{S}$ is the set of all subdivisions of I. When $\mathrm{var}_\rho(C; I) < +\infty$, one says that $C(\cdot)$ is of *bounded ρ-variation*, or of *bounded variation along ρ-truncation*, on I (with respect to $\mathrm{haus}_\rho(\cdot, \cdot)$). It is then readily seen that the existence of a positive Radon measure μ on I satisfying (16) with $\mathrm{haus}_\rho(C(s), C(t))$ in place of $\widehat{\mathrm{haus}}_\rho(C(s), C(t))$ entails that

$$\mathrm{var}_\rho(C; I) \leq \mu(]T_0, T]) < +\infty,$$

so $C(\cdot)$ has a bounded variation along ρ-truncation on I (with respect to $\mathrm{haus}_\rho(\cdot, \cdot)$). Furthermore, the mapping $\mathrm{var}_\rho(C; [T_0, \cdot])$ is right-continuous, since for any $\bar{t} \in [T_0, T[$, we have by the triangle inequality for $\mathrm{haus}_\rho(\cdot, \cdot)$ that

$$0 \leq \mathrm{var}_\rho(C; [T_0, t]) - \mathrm{var}_\rho(C; [T_0, \bar{t}]) \leq \mu(]\bar{t}, t]) \quad \text{for all } t \in]\bar{t}, T].$$

Conversely, assume that $C(\cdot)$ has a bounded variation on I along ρ-truncation (with respect to $\mathrm{haus}_\rho(\cdot, \cdot)$) and that the function $\mathrm{var}_\rho(C; [T_0, \cdot])$ is right-continuous on I. Since the latter function is nondecreasing on I, it is of bounded variation on I, so if we denote by $\mu_{C,\rho}$ the differential Radon measure associated with it, we have

$$\mathrm{var}_\rho(C;[T_0,t]) - \mathrm{var}_\rho(C;[T_0,s]) = \mu_{C,\rho}(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \le t.$$

It follows that $C(\cdot)$ satisfies (16) with $\mu = \mu_{C,\rho}$ both for $\mathrm{haus}_\rho(\cdot,\cdot)$ and $\widehat{\mathrm{haus}}_\rho(\cdot,\cdot)$, since

$$\widehat{\mathrm{haus}}_\rho(C(s),C(t)) \le \mathrm{haus}_\rho(C(s),C(t)) \le \mu_{C,\rho}(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \le t.$$

2.4. Prox-regular sets in Hilbert spaces

In addition to the inequality

$$\widehat{\mathrm{haus}}_\rho(C(s),C(t)) \le \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \le t$$

for a given positive Radon measure μ on I and $\rho \in]\|u_0\|,+\infty]$, the multimapping $C(\cdot)$ will be assumed to be uniformly prox-regular valued.

Definition 2.1: Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0,+\infty]$. One says that S is r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $v \in N^P(S;x) \cap \mathbb{B}$ and for all $t \in]0,r[$, one has $x \in \mathrm{Proj}_S(x+tv)$.

The following theorem provides some useful characterizations and properties of uniform prox-regular sets (see, e.g. [21,35]). Before stating it, recall that for any extended real $r > 0$, the *open r-enlargement* of a subset S of $\mathcal{H}$ is defined as

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}.$$

Theorem 2.1: *Let S be a nonempty closed subset of $\mathcal{H}$, $r \in]0,+\infty]$. Consider the following assertions.*

(a) *The set S is r-prox-regular.*
(b) *For all $x_1, x_2 \in S$, for all $v \in N^P(S;x_1)$, one has*

$$\langle v, x_2 - x_1\rangle \le \frac{1}{2r}\|v\|\,\|x_1 - x_2\|^2.$$

(c) *The mapping $\mathrm{proj}_S : U_r(S) \to S$ is well-defined and locally Lipschitz on $U_r(S)$.*
(d) *For all $u \in U_r(S) \setminus S$, one has with $x = \mathrm{proj}_S(u)$*

$$x = \mathrm{proj}_S\left(x + t\frac{u-x}{\|u-x\|}\right) \quad \text{for all } t \in [0,r[.$$

(e) *For all $x, y \in S$ and all $t \in [0,1]$ such that $tx + (1-t)y \in U_r(S)$, one has*

$$d_S(tx + (1-t)y) \le \frac{1}{2r} t(1-t)\,\|x-y\|^2.$$

(f) *For any* $x \in S$, *one has*

$$N^P(S;x) = N^C(S;x) \quad \text{and} \quad \partial_P d_S(x) = \partial_C d_S(x).$$

Then, the assertions (a), (b), (c), (d) *and* (e) *are pairwise equivalent and each one implies* (f).

An r-prox-regular set and its open r-enlargement as well as the property (d) in Theorem 2.1 are illustrated in Figure 1. This set (represented in grey color) looks like a 'tree'.

Theorem 2.1 again allows us to put

$$N(S;x) := N^P(S;x) = N^C(S;x) \quad \text{for all } x \in S,$$

whenever S is a uniform prox-regular set of the real Hilbert space $\mathcal{H}$.

The following proposition provides useful inequalities for proximal subgradients of the distance function associated to a prox-regular set. We refer to [13] for the proof.

Proposition 2.1: *Let* S *be a subset of* $\mathcal{H}$ *which is* r*-prox-regular for some* $r \in]0,+\infty]$. *Let* $x \in S$ *and* $\zeta \in \partial_P d_S(x)$. *Then, for all* $z \in \mathcal{H}$ *such that* $d_S(z) < r$, *one has*

$$\langle \zeta, z - x\rangle \leq \frac{1}{2r}\|z-x\|^2 + \frac{1}{2r}d_S^2(z) + \left(\frac{1}{r}\|z-x\| + 1\right) d_S(z),$$

and

$$\langle \zeta, z - x\rangle \leq \frac{2}{r}\|z-x\|^2 + d_S(z).$$

Before stating the last result of this section, let us recall that a function $f : C \to \mathbb{R} \cup \{+\infty\}$ defined on a nonempty convex subset C of $\mathcal{H}$ is said to be

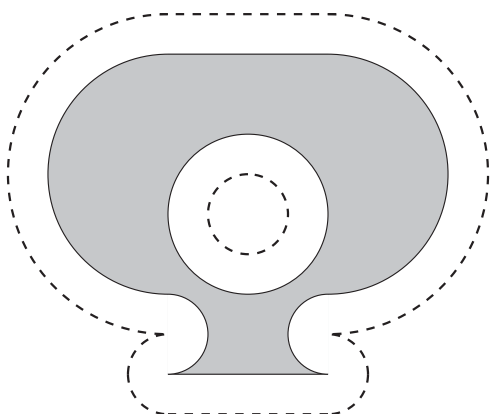

Figure 1. An r-prox-regular set and its open r-enlargement.

σ-semiconvex (on C) for some $\sigma \in \mathbb{R}_+ := [0, +\infty[$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \frac{\sigma}{2} t(1-t) \left\| x - y \right\|^2,$$

for all $x, y \in C$ and for all $t \in]0, 1[$, or equivalently if $f + (\sigma/2)\| \cdot \|^2$ is convex on C.

Theorem 2.2: *Let S be an r-prox-regular subset of $\mathcal{H}$ for some $r \in]0, +\infty]$. Then, for any $s \in]0, r[$, for any nonempty convex set $C \subset U_s(S)$, the function d_S is $(r - s)^{-1}$-semiconvex on C.*

3. Preparatory results

This section is devoted on the one hand to recall some specific results needed in the rest of the paper and on the other hand to establish some other ones. We start with a variant of Gronwall Lemma which is due to M.D.P. Monteiro Marques [9].

Lemma 3.1: *Let ν be a positive Radon measure on $[T_0, T]$, $g, \varphi : [T_0, T] \to \mathbb{R}_+$ two functions such that:*

(i) $g \in L^1([T_0, T], \mathbb{R}_+, \nu)$ *and for some fixed $\theta \in \mathbb{R}_+$, one has*

$$0 \le g(t)\nu(\{t\}) \le \theta < 1 \quad \text{for all } t \in]T_0, T];$$

(ii) $\varphi \in L^\infty([T_0, T], \mathbb{R}_+, \nu)$ *and for some fixed $\alpha \in \mathbb{R}_+$, one has*

$$\varphi(t) \le \alpha + \int_{]T_0,t]} g(s)\varphi(s) \, \mathrm{d}\nu(s) \quad \text{for all } t \in [T_0, T].$$

Then, one has

$$\varphi(t) \le \alpha \exp\left(\frac{1}{1-\theta} \int_{]T_0,t]} g(s) \, \mathrm{d}\nu(s) \right) \quad \text{for all } t \in [T_0, T].$$

The following proposition is due to Moreau [36].

Proposition 3.1: *Let ν be a positive Radon measure on $I = [T_0, T]$, $u(\cdot) : I \to \mathcal{H}$ be a right continuous mapping of bounded variation such that the differential measure $\mathrm{d}u$ has a density $\mathrm{d}u/\mathrm{d}\nu$ relative to ν. Then, the function $\Phi(\cdot) = \|u(\cdot)\|^2 : I \to \mathbb{R}$ is a right continuous function of bounded variation whose differential measure $\mathrm{d}\Phi$ satisfies, in the sense of the ordering of real measures,*

$$\mathrm{d}\Phi \le 2 \left\langle u(\cdot), \frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) \right\rangle \mathrm{d}\nu.$$

Our aim is now to establish the following scalar upper semicontinuity property for prox-regular sets. It is worth pointing out that it recovers the convex case developed in [28, Proposition 4.1].

Proposition 3.2: *Let $C : I = [T_0, T] \rightrightarrows \mathcal{H}$ be an r-prox-regular valued multimapping for some $r \in]0, +\infty]$. Assume that there exist a positive measure μ on I and $\rho \in]0, +\infty]$ such that for all $s, t \in I$ with $s \le t$,*

$$\widehat{\text{exc}}_\rho(C(s), C(t)) \le \mu(]s, t]). \tag{17}$$

Let $\bar{t} \in I$, $\bar{x} \in C(\bar{t}) \cap \rho\mathbb{U}$, $(t_n)_{n\in\mathbb{N}}$ be a sequence of $[\bar{t}, T]$ with $\mu(]\bar{t}, t_n]) \to 0$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}$ with $x_n \to \bar{x}$ and $x_n \in C(t_n)$ for all $n \in \mathbb{N}$. Then, one has

$$\limsup_{n\to+\infty} d^o_{C(t_n)}(x_n; h) \le d^o_{C(\bar{t})}(\bar{x}; h) \quad \text{for all } h \in \mathcal{H},$$

or equivalently

$$\limsup_{n\to+\infty} \sigma(h, \partial_C d_{C(t_n)}(x_n)) \le \sigma(h, \partial_C d_{C(\bar{t})}(\bar{x})) \quad \text{for all } h \in \mathcal{H}.$$

Before giving the proof, we need the following lemmas.

Lemma 3.2: *Let U be an open subset of $\mathcal{H}$, $x \in U$ and $g : U \to \mathbb{R}$ be a function. If there exists a real $\delta > 0$ with $B(x, \delta) \subset U$ and such that g is σ-semiconvex on $B(x, \delta)$ for a real $\sigma \ge 0$, then one has for all $h \in \mathbb{B}$,*

$$g^o(x; h) = \inf_{t\in]0,\delta[} t^{-1}\left(g(x + th) - g(x) + \frac{\sigma}{2}\|x + th\|^2 - \frac{\sigma}{2}\|x\|^2\right) - \sigma\langle x, h\rangle = g'(x; h).$$

Proof: Assume that there exists a real $\delta > 0$ such that $B(x, \delta) \subset U$ and g is σ-semiconvex on $B(x, \delta)$ for a real $\sigma \ge 0$. Fix any $h \in \mathbb{B}$. Set $f := g + (\sigma/2)\|\cdot\|^2$ which is convex on $B(x, \delta)$ according to the σ-semiconvexity on $B(x, \delta)$ of g. From (7), one observes that

$$\begin{aligned} f'(x; h) &= f^o(x; h) \\ &= \limsup_{t\downarrow 0, x'\to x} t^{-1}\left[(g(x' + th) - g(x')) + \frac{\sigma}{2}\left\|x' + th\right\|^2 - \frac{\sigma}{2}\left\|x'\right\|^2\right] \\ &= g^o(x; h) + D\left(\frac{\sigma}{2}\|\cdot\|^2\right)(x)(h). \end{aligned}$$

Since f is convex on $B(x, \delta)$ and $x + th \in B(x, \delta)$ for each $t \in]0, \delta[$, we have

$$f'(x; h) = \inf_{t\in]0,\delta[} t^{-1}(f(x + th) - f(x)).$$

It follows that

$$g^o(x; h) = -\sigma\langle x, h\rangle + \inf_{t\in]0,\delta[} t^{-1}(f(x + th) - f(x)),$$

so the first equality claimed is established. For the second, it remains to see that

$$\begin{aligned} g^o(x;h) &= f'(x;h) - D(\frac{\sigma}{2}\|\cdot\|^2)(x)(h) \\ &= f'(x;h) - \left(\frac{\sigma}{2}\|\cdot\|^2\right)'(x;h) \\ &= \left(f - \frac{\sigma}{2}\|\cdot\|^2\right)'(x;h) \\ &= g'(x;h). \end{aligned}$$

The proof is complete. ■

Lemma 3.3: *Let S be an r-prox-regular subset of $\mathcal{H}$ for some $r \in]0,+\infty]$. Then, for each $s \in]0,r[$, one has for all $(x,h) \in S \times \mathbb{B}$,*

$$\begin{aligned} (d_S)^o(x;h) &= \lim_{t\downarrow 0} t^{-1} d_S(x+th) \\ &= \inf_{t\in]0,s[} t^{-1}[d_S(x+th) + \frac{1}{2(r-s)}(\|x+th\|^2 - \|x\|^2)] \\ &\quad - \frac{1}{r-s}\langle x,h\rangle . \end{aligned}$$

Proof: If $r = +\infty$, we know that S is convex as well as its associated distance function d_S. This justifies the equality claimed. Suppose now $r < +\infty$. Fix any $s \in]0,r[$. Let $(x,h) \in S \times \mathbb{B}$. It is easy to check that $B(x,s) \subset U_s(S)$, so we can apply Theorem 2.2 to get that d_S is $1/(r-s)$-semiconvex on $B(x,s)$. It remains to combine Lemma 3.2 with the equality $d_S(x) = 0$. ■

Now, we are able to prove Proposition 3.2.

Proof of Proposition 3.2.: Fix any $h \in \mathbb{B}$. Let $s \in]0,r[$. Since $C(\bar{t})$ is r-prox-regular, the mapping $\text{proj}_{C(\bar{t})} : U_r(C(\bar{t})) \to \mathcal{H}$ is well-defined and norm-to-norm continuous. In particular, we have

$$\lim_{x\to\bar{x}} \text{proj}_{C(\bar{t})}(x) = \text{proj}_{C(\bar{t})}(\bar{x}) = \bar{x} \in \rho\mathbb{U},$$

so we can find a real $\alpha \in]0,s[$ such that for all $x \in B(\bar{x},\alpha)$,

$$\text{proj}_{C(\bar{t})}(x) \in \rho\mathbb{U} \subset \rho\mathbb{B}.$$

From the latter inclusion and the inequality

$$\left\| x - \text{proj}_{C(\bar{t})}(x) \right\| = d_{C(\bar{t})}(x) \le d_{C(\bar{t})\cap\rho\mathbb{B}}(x) \quad \text{for all } x \in B(\bar{x},\alpha),$$

it is straightforward to check that

$$d_{C(\bar{t})}(x) = d_{C(\bar{t})\cap\rho\mathbb{B}}(x) \quad \text{for all } x \in B(\bar{x},\alpha).$$

Note that for each $n \in \mathbb{N}$, $B(x_n,\alpha) \subset B(x_n,s) \subset U_s(C(t_n))$. According to Lemma 3.3, for each $n \in \mathbb{N}$, we have for all $\tau \in]0,\alpha[$,

$$d^o_{C(t_n)}(x_n;h) \le \tau^{-1} d_{C(t_n)}(x_n+\tau h) - \frac{1}{r-s}\langle x_n, h\rangle + \frac{1}{2(r-s)\tau}(\|x_n+\tau h\|^2 - \|x_n\|^2).$$

Furthermore, for all $n \in \mathbb{N}$, for all $\tau \in]0,\alpha[$,

$$\begin{aligned}\tau^{-1} d_{C(t_n)}(x_n+\tau h) &\le \tau^{-1}\big(\|x_n-\overline{x}\| + d_{C(t_n)}(\overline{x}+\tau h)\big)\\ &\le \tau^{-1}\big(\|x_n-\overline{x}\| + d_{C(\bar{t})\cap\rho\mathbb{B}}(\overline{x}+\tau h) + \mu(]\bar{t},t_n])\big)\\ &\le \tau^{-1}\big(\|x_n-\overline{x}\| + d_{C(\bar{t})}(\overline{x}+\tau h) + \mu(]\bar{t},t_n])\big),\end{aligned}$$

where the second inequality is due to (15) and (17). This entails that for all $\tau \in]0,\alpha[$,

$$\limsup_{n\to+\infty} \tau^{-1} d_{C(t_n)}(x_n+\tau h) \le \tau^{-1} d_{C(\bar{t})}(\overline{x}+\tau h).$$

On the other hand for all $\tau \in]0,\alpha[$,

$$\begin{aligned}&\limsup_{n\to+\infty}\left[-\frac{1}{r-s}\langle x_n,h\rangle + \frac{1}{2(r-s)\tau}(\|x_n+\tau h\|^2 - \|x_n\|^2)\right]\\ &= -\frac{1}{r-s}\langle \overline{x},h\rangle + \frac{1}{2(r-s)\tau}(\|\overline{x}+\tau h\|^2 - \|\overline{x}\|^2)\\ &= \frac{\tau}{2(r-s)}\|h\|^2.\end{aligned}$$

It follows that for all $\tau \in]0,\alpha[$,

$$\limsup_{n\to+\infty} d^o_{C(t_n)}(x_n;h) \le \tau^{-1} d_{C(\bar{t})}(\overline{x}+\tau h) + \frac{\tau}{2(r-s)}\|h\|^2.$$

Since $\overline{x} \in C(\bar{t})$, we have

$$\limsup_{\tau\downarrow 0} \tau^{-1} d_{C(\bar{t})}(\overline{x}+\tau h) \le d^o_{C(\bar{t})}(\overline{x};h).$$

As a consequence, we get

$$\limsup_{n\to+\infty} d^o_{C(t_n)}(x_n;h) \le d^o_{C(\bar{t})}(x;h).$$

The latter inequality being true for any $h \in \mathbb{B}$, the positive homogeneity of the Clarke directional derivative guarantees that it holds for all $h \in \mathcal{H}$. ■

4. Sweeping process with bounded truncated variation

As mentioned above, the aim of this paper is to provide sufficient conditions ensuring the existence and uniqueness of solutions for the bounded variation sweeping process

$$-\,\mathrm{d}u \in N(C(t);u(t)) + f(t,u(t)) \quad \text{and} \quad u(T_0) = u_0. \tag{18}$$

The concept of solution for such measure differential inclusions is developed in details in [28]. For the convenience of the reader, let us recall the definition. As already said in the first sentence of Section 2, λ denotes the *Lebesgue measure* on $I = [T_0, T]$.

Definition 4.1: Let $C : I \rightrightarrows \mathcal{H}$ be a uniformly prox-regular valued multimapping, $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping, $u_0 \in C(T_0)$. Let $\mathfrak{e}_f = 0$ if $f \equiv 0$ and $\mathfrak{e}_f = 1$ if $f \not\equiv 0$. Assume that there exist an extended real $\rho \geq \|u_0\|$ and a finite positive Radon measure μ on I such that

$$\widehat{\mathrm{exc}}_\rho(C(s), C(t)) \leq \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \leq t.$$

One says that a mapping $u(\cdot) : I \to \mathcal{H}$ is a solution of the measure differential inclusion

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t);u(t)) + f(t,u(t)) \\ u(T_0) = u_0, \end{cases}$$

whenever:

(a) the mapping $u(\cdot)$ is of bounded variation on I, right-continuous on I and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;
(b) there exists a positive Radon measure ν on I, absolutely continuously equivalent to $\mu + \mathfrak{e}_f\lambda$ and with respect to which the differential measure $\mathrm{d}u$ of u is absolutely continuous with $(\mathrm{d}u/\mathrm{d}\nu)(\cdot)$ as an $L^1(I,\mathcal{H},\nu)$-density and

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \in -N(C(t);u(t)) \quad \nu\text{−a.e.} t \in I. \tag{19}$$

The concept of solution does not depend on the measure ν in the sense that a mapping $u(\cdot) : I \to \mathcal{H}$ satisfying (a) above is a solution of $(\mathcal{P})$ if and only if (19) holds for any positive Radon measure $\hat{\nu}$ which is absolutely continuously equivalent to μ and with respect to which the differential measure $\mathrm{d}u$ of u is absolutely continuous with $(\mathrm{d}u/\mathrm{d}\hat{\nu})(\cdot)$ as an $L^1(I,\mathcal{H},\hat{\nu})$-density.

Our first existence result is concerned with the unperturbed case (i.e. $f \equiv 0$) of (18).

Theorem 4.1: *Let $C : I \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some extended real $r \in]0, +\infty]$, $u_0 \in C(T_0)$. Assume that there exist a positive Radon measure μ on I with $\sup_{\tau \in]T_0,T]} \mu(\{\tau\}) < r/2$, a real $\rho_0 \geq \|u_0\|$, an extended real $\rho > \rho_0$ and a real $\eta > 0$ satisfying:*

(i) *for every $k \in \mathbb{N}$ and for every $t_1, \ldots, t_k \in I$ with $t_0 := T_0 < t_1 < \cdots < t_k$ and $\mu(]t_i, t_{i+1}[) < \eta$ for each $i \in \{0, \ldots, k-1\}$, one has*

$$\left\| \operatorname{proj}_{C(t_k)} \circ \cdots \circ \operatorname{proj}_{C(t_1)}(u_0) \right\| \leq \rho_0, \tag{20}$$

whenever $\operatorname{proj}_{C(t_k)} \circ \cdots \circ \operatorname{proj}_{C(t_1)}(u_0)$ is well-defined;

(ii) *for all $s, t \in I$ with $s \leq t$ and $\mu(]s, t[) < \eta$, one has*

$$\widehat{\operatorname{haus}}_\rho(C(s), C(t)) \leq \mu(]s, t]).$$

Then, there exists one and only one mapping $u : I \to \mathcal{H}$ solution of the measure differential inclusion

$$\begin{cases} -\mathrm{d}u \in N(C(t); u(t)) \\ u(T_0) = u_0 \end{cases} \tag{21}$$

and such that

$$\sup_{t \in]T_0,T]} \left\| u(t) - u(t^-) \right\| < \frac{r}{2}. \tag{22}$$

Furthermore, the solution $u(\cdot) : I \to \mathcal{H}$ satisfies the inequalities

$$\|u(t) - u(s)\| \leq \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t. \tag{23}$$

$$\|u(t)\| \leq \min\{\rho_0, \|u_0\| + \mu(]T_0, T])\} \quad \text{for all } t \in I. \tag{24}$$

Proof: *Uniqueness*. According to [28, Proposition 3.6], there exists at most one mapping $u(\cdot) : I \to \mathcal{H}$ satisfying (21) and (22).

Existence. We distinguish two cases.

Case 1: $\mu(]T_0, T]) = 0$. In such a case, thanks to the fact that $C(\cdot)$ is closed-valued, we have

$$C(T_0) \cap \rho\mathbb{B} \subset C(t) \quad \text{for all } t \in I.$$

It is then clear (keeping in mind the inclusion $u_0 \in C(T_0)$) that the mapping $u : [T_0, T] \to \mathcal{H}$ defined by

$$u(t) := u_0 \quad \text{for all } t \in I$$

satisfies (21)–(24).

Case 2: $\mu(]T_0, T]) > 0$. We develop that case through 5 steps.

Step 1. Time discretization of $I = [T_0, T]$.

Consider the function $v(\cdot) : I \to \mathbb{R}$ defined by

$$v(t) := \mu(]T_0, t]) \quad \text{for all } t \in I$$

and set

$$V := v(T) = \mu(]T_0, T]).$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of $]0, \eta[$ with $\varepsilon_n \downarrow 0$ such that

$$\varepsilon_n + \sup_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2} \quad \text{for all } n \in \mathbb{N}. \tag{25}$$

As in [2], choose for each $n \in \mathbb{N}$, $0 = V_0^n < V_1^n < \cdots < V_{q_n}^n = V$ (with $q_n \in \mathbb{N}$) such that

(a) for all $j \in \{0, \ldots, q_n - 1\}$, $V_{j+1}^n - V_j^n \le \varepsilon_n$;
(b) for all $k \in \mathbb{N}$, $\{V_0^k, \ldots, V_{q_k}^k\} \subset \{V_0^{k+1}, \ldots, V_{q_{k+1}}^{k+1}\}$.

For each $n \in \mathbb{N}$, set $V_{1+q_n}^n := V + \varepsilon_n$ and consider the partition $(J_j^n)_{j\in\{0,\ldots,q_n\}}$ of I where for each $j \in \{0, \ldots, q_n\}$

$$J_j^n := v^{-1}\left(\left[V_j^n, V_{j+1}^n\right[\right) = \{t \in I : V_j^n \le \mu(]T_0, t]) < V_{j+1}^n\}.$$

Observe that $(J_j^m)_{0\le j\le q_m}$ is a refinement of $(J_j^n)_{0\le j\le q_n}$ for all $m, n \in \mathbb{N}$ with $m \ge n$. Using the fact that $v(\cdot)$ is nondecreasing and right-continuous on I, it is not difficult to see that, for each $n \in \mathbb{N}$, $j \in \{0, \ldots, q_n - 1\}$, the set J_j^n is either empty or an interval of the form $[a, b[$ with $a < b$. For each $n \in \mathbb{N}$ we also note that $J_{q_n}^n$ is also of the form $[a, b[\cap I$. This gives for each $n \in \mathbb{N}$ an integer $p(n) \in \mathbb{N}$ and a finite sequence

$$T_0 = t_0^n < \cdots < t_{p(n)}^n = T$$

such that for each integer i with $0 \le i \le p(n) - 2$ there is some $j \in \{0, \ldots, q_n - 1\}$ satisfying $[t_i^n, t_{i+1}^n[= J_j^n$ and such that for $i = p(n) - 1$ the interval $[t_{p(n)-1}, t_{p(n)}[$ is either $J_{q_n}^n \setminus \{T\}$ (if $J_{q_n}^n \neq \{T\}$) or J_k^n for some $k \in \{0, \ldots, q_n - 1\}$. For each integer n, including new points t_j^n if necessary, we may and do suppose that

$$\max_{i\in\{0,\ldots,p(n)-1\}} (t_{i+1}^n - t_i^n) \le \frac{1}{n} \text{ along with } \{t_0^{n+1}, \ldots, t_{p(n+1)}^{n+1}\} \supset \{t_0^n, \ldots, t_{p(n)}^n\}. \tag{26}$$

Note that $(p(n))_{n\in\mathbb{N}}$ is a nondecreasing sequence and that for each $n \in \mathbb{N}$, for all $i \in \{0, \ldots, p(n) - 1\}$ and all $t \in [t_i^n, t_{i+1}^n[$, we have

$$\mu(]t_i^n, t]) = v(t) - v(t_i^n) \le \varepsilon_n < \eta,$$

hence

$$\mu(\left]t_i^n, t_{i+1}^n\right[) \le \varepsilon_n < \eta \quad \text{for all } i \in \{0, \ldots, p(n) - 1\}. \tag{27}$$

Step 2. Construction of finite sequences $(u_i^n)_{0\le i\le p(n)}$ $(n \in \mathbb{N})$.

Fix any $n \in \mathbb{N}$. Put $u_0^n := u_0$. By induction, let us construct a sequence $(u_i^n)_{1\le i\le p(n)}$ such that

$$u_i^n := \operatorname{proj}_{C(t_i^n)}(u_{i-1}^n) \quad \text{for all } i \in \left\{1,\dots,p(n)\right\}. \tag{28}$$

According to the assumption (ii), to the inequality $\|u_0\| \le \rho$ and to the relations (27) and (25), we have

$$\begin{aligned} d_{C(t_1^n)}(u_0^n) &\le \sup_{x\in C(T_0)\cap\rho\mathbb{B}} d_{C(t_1^n)}(x) \\ &\le \widehat{\operatorname{haus}}_\rho(C(T_0), C(t_1^n)) \\ &\le \mu(]T_0, t_1^n]) \le \varepsilon_n + \sup_{s\in]T_0,T]} \mu(\{s\}) < r, \end{aligned}$$

and this allows us to set (thanks to the r-prox-regularity of $C(t_1^n)$) $u_1^n := \operatorname{proj}_{C(t_1^n)}(u_0^n)$. Now, suppose without loss of generality that $p(n) > 1$. Fix any $k \in \{2,\dots,p(n)\}$ and assume that we have constructed $u_1^n,\dots,u_{k-1}^n$ such that

$$u_i^n = \operatorname{proj}_{C(t_i^n)}(u_{i-1}^n) \quad \text{for all } i \in \{1,\dots,k-1\}.$$

Observe that $\operatorname{proj}_{C(t_{k-1}^n)} \circ \dots \circ \operatorname{proj}_{C(t_1^n)}(u_0^n)$ is well-defined, hence by virtue of assumption (i), we get the following inclusion

$$u_{k-1}^n = \operatorname{proj}_{C(t_{k-1}^n)} \circ \dots \circ \operatorname{proj}_{C(t_1^n)}(u_0^n) \in \rho_0\mathbb{B}.$$

As above, using the assumption (ii), the inequality $\|u_{k-1}^n\| \le \rho$ and the relations (27) and (25), we obtain

$$\begin{aligned} d_{C(t_k^n)}(u_{k-1}^n) &\le \sup_{x\in C(t_{k-1}^n)\cap\rho\mathbb{B}} d_{C(t_k^n)}(x) \\ &\le \widehat{\operatorname{haus}}_\rho(C(t_{k-1}^n), C(t_k^n)) \\ &\le \mu(]t_{k-1}^n, t_k^n]) \le \varepsilon_n + \sup_{s\in]T_0,T]} \mu(\{s\}) < r, \end{aligned}$$

which allows us via the r-prox-regularity of $C(t_k^n)$, to put $u_k^n := \operatorname{proj}_{C(t_k^n)}(u_{k-1}^n)$. The induction is then complete. The definition of $(u_i^n)_{1\le i\le p(n)}$ in (28) along with assumption (i) give

$$\left\|u_i^n\right\| \le \rho_0 < \rho \quad \text{for all } i \in \left\{0,\dots,p(n)\right\}. \tag{29}$$

Again, the definition of $(u_i^n)_{0\le i\le p(n)}$ with assumption (ii) and the latter inequality furnish

$$\left\|u_{i+1}^n - u_i^n\right\| = d_{C(t_{i+1}^n)}(u_i^n) \le \widehat{\operatorname{haus}}_\rho(C(t_i^n), C(t_{i+1}^n)) \le \mu(]t_i^n, t_{i+1}^n]), \tag{30}$$

for each $i \in \{0,\dots,p(n)-1\}$.

Step 3. Definition of the sequence $(u_n(\cdot))_{n\in\mathbb{N}}$.

Fix any integer $n \geq 1$. If $\mu(]t_i^n, t_{i+1}^n]) = 0$ for some $i \in \{0, \dots, p(n) - 1\}$, then

$$\widehat{\text{haus}}_\rho(C(t_i^n), C(t_{i+1}^n)) = 0,$$

in particular $C(t_i^n) \cap \rho\mathbb{B} \subset C(t_{i+1}^n)$, so $u_i^n = u_{i+1}^n$. As in Moreau [2], let us define the mapping $u_n(\cdot) : I \to \mathcal{H}$ by $u_n(T) = u_{p(n)}^n$ and for all $i \in \{0, \dots, p(n) - 1\}$,

$$u_n(t) = u_i^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ if } \mu(]t_i^n, t_{i+1}^n]) = 0,$$

and

$$u_n(t) = u_i^n + \frac{\mu(]t_i^n, t])}{\mu(]t_i^n, t_{i+1}^n])}(u_{i+1}^n - u_i^n) \quad \text{for all } t \in [t_i^n, t_{i+1}^n[\text{ if } \mu(]t_i^n, t_{i+1}^n]) > 0.$$

According to the development above, we can write

$$u_n(t) = u_i^n = u_{i+1}^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n] \text{ if } \mu(]t_i^n, t_{i+1}^n]) = 0. \tag{31}$$

Furthermore, it is not difficult to check that

$$u_n(t) = u_i^n + \frac{\mu(]t_i^n, t])}{\mu(]t_i^n, t_{i+1}^n])}(u_{i+1}^n - u_i^n) \quad \text{for all } t \in [t_i^n, t_{i+1}^n] \text{ if } \mu(]t_i^n, t_{i+1}^n]) > 0. \tag{32}$$

On the other hand, by (29), we have

$$\|u_n(t)\| \leq \rho_0 \quad \text{for all } t \in I. \tag{33}$$

From (32) and (31), note that

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \zeta_n(s)\, \mathrm{d}\mu(s), \tag{34}$$

where $\zeta_n(T_0) = 0$ and for each $i \in \{0, \dots, p(n) - 1\}$,

$$\zeta_n(t) = 0 \quad \text{for all } t \in]t_i^n, t_{i+1}^n] \text{ if } \mu(]t_i^n, t_{i+1}^n]) = 0$$

and

$$\zeta_n(t) = \frac{u_{i+1}^n - u_i^n}{\mu(]t_i^n, t_{i+1}^n])} \quad \text{for all } t \in]t_i^n, t_{i+1}^n] \text{ if } \mu(]t_i^n, t_{i+1}^n]) > 0.$$

As a consequence, $u_n(\cdot)$ is right-continuous and with bounded variation on I. Moreover, the equality (34) says that $\zeta_n(\cdot)$ is a density of $u_n(\cdot)$ relative to μ. So, it follows that $(\mathrm{d}u_n/\mathrm{d}\mu)(\cdot)$ exists as a density of $u_n(\cdot)$ relative to μ and

$$\frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t) = \zeta_n(t) \quad \mu\text{−a.e.}t \in I.$$

From the definition of $\zeta_n(\cdot)$ and (30), it is readily seen that

$$\left\| \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t) \right\| = \|\zeta_n(t)\| \le 1 \quad \mu\text{−a.e.}t \in I. \tag{35}$$

Using (35) and the fact that $\zeta_n(\cdot)$ is a density of u_n relative to μ, we have for all $s, t \in I$ with $s \le t$,

$$\|u_n(t) - u_n(s)\| = \left\| \int_{]s,t]} \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(w)\,\mathrm{d}\mu(w) \right\| \le \mu(]s,t]). \tag{36}$$

Now, let us define $\theta_n : I \to I$ by $\theta_n(T_0) = T_0$ and for all $t \in I$ by

$$\theta_n(t) = t^n_{i+1} \quad \text{if} t \in]t^n_i, t^n_{i+1}] \text{ with } i \in \left\{0, \ldots, p(n) - 1\right\}.$$

Combining (5), the definition of $\zeta_n(\cdot)$, (28), (35) and (8), we have

$$\zeta_n(t) = \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t) \in -N^P\big(C(\theta_n(t)); u_n(\theta_n(t))\big) \cap \mathbb{B} = -\partial_P d_{C(\theta_n(t))}\big(u_n(\theta_n(t))\big), \tag{37}$$

for μ-almost every $t \in I$.

Step 4. Cauchy property of the sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(I, \mathcal{H})$ (the real Banach space of bounded mappings endowed with the norm of uniform convergence).

From the definition of $\theta_n(\cdot)$ and $u_n(\cdot)$ (with $n \in \mathbb{N}$) and from (28) and (33), we get

$$u_n(\theta_n(t)) \in C(\theta_n(t)) \cap \rho_0\mathbb{B} \quad \text{for all } t \in I, \text{ all } n \in \mathbb{N}. \tag{38}$$

Let us set for all $t \in I$, for all $n \in \mathbb{N}$,

$$\gamma_n(t) := \mu(]t, \theta_n(t)]).$$

For each $t \in I$, noting that $\theta_n(t) \downarrow t$ since $0 \le \theta_n(t) - t \le t^n_{i+1} - t^n_i \le 1/n$ by (26), we see that

$$\gamma_n(t) \to 0 \quad \text{as} n \to \infty. \tag{39}$$

Fix for a moment any $t \in I$, any $n, m \in \mathbb{N}$. We observe that (see (15))

$$\begin{aligned} d\big(u_m(t), C(\theta_n(t))\big) &\le d\big(u_m(t), C(\theta_m(t)) \cap \rho\mathbb{B}\big) + \widehat{\mathrm{exc}}_\rho\big(C(\theta_m(t)), C(\theta_n(t))\big) \\ &\le d\big(u_m(\theta_m(t)), C(\theta_m(t)) \cap \rho\mathbb{B}\big) + \|u_m(t) - u_m(\theta_m(t))\| \\ &\quad + \mu\big(]t, \max\{\theta_n(t), \theta_m(t)\}]\big). \end{aligned}$$

Using (38), (36), (27) and (25), the latter inequality entails

$$\begin{aligned} d\big(u_m(t), C(\theta_n(t))\big) &\le \mu(]t, \theta_m(t)]) + \mu(]t, \max\{\theta_n(t), \theta_m(t)\}]) \\ &\le 2\gamma_m(t) + \gamma_n(t) \\ &\le 2\max\left\{\varepsilon_{n,}, \varepsilon_m\right\} + 2 \sup_{\tau \in]T_0, T]} \mu(\{\tau\}) < r. \end{aligned} \tag{40}$$

On the other hand, the inequality (35) entails straightforwardly

$$\begin{aligned} \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(t) - u_m(t) \right\rangle &= \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(t) - u_n(\theta_n(t)) \right\rangle \\ &\quad + \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\ &\le \|u_n(t) - u_n(\theta_n(t))\| + \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle, \end{aligned}$$

for μ-almost every $t \in I$. According to (37), (38) and (40), we can apply Proposition 2.1 to obtain

$$\begin{aligned} &\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n\big(\theta_n(t)\big) - u_m(t) \right\rangle \\ &\quad \le \frac{1}{2r}\left\|u_m(t) - u_n\big(\theta_n(t)\big)\right\|^2 + \frac{1}{2r} d^2_{C(\theta_n(t))}(u_m(t)) \\ &\quad + \left(\frac{1}{r}\left\|u_n\big(\theta_n(t)\big) - u_m(t)\right\| + 1\right) d_{C(\theta_n(t))}(u_m(t)) \\ &\le \frac{1}{2r}\Big(\|u_n(t) - u_m(t)\| + \left\|u_n\big(\theta_n(t)\big) - u_n(t)\right\|\Big)^2 \\ &\quad + \frac{1}{2r} d^2_{C(\theta_n(t))}(u_m(t)) \\ &\quad + \left[\frac{1}{r}\Big(\left\|u_n\big(\theta_n(t)\big) - u_n(t)\right\| + \|u_n(t) \right. \\ &\quad \left. - u_m(t)\|\Big) + 1\right] d_{C(\theta_n(t))}(u_m(t)). \end{aligned}$$

Hence, coming back to (36) and (40), we get

$$\begin{aligned} &\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n\big(\theta_n(t)\big) - u_m(t) \right\rangle \\ &\quad \le \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + \gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\ &\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big). \end{aligned}$$

We deduce from (35) and (36) that

$$\begin{aligned}\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(t) - u_m(t) \right\rangle &= \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(t) - u_n(\theta_n(t)) \right\rangle \\ &\quad + \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\ &\le \gamma_n(t) + \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + \gamma_n(t)\big)^2 \\ &\quad + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\ &\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right] \\ &\quad \big(\gamma_n(t) + 2\gamma_m(t)\big).\end{aligned}$$

Interchanging n and m yields

$$\begin{aligned}\left\langle \frac{\mathrm{d}u_m}{\mathrm{d}\mu}(t), u_m(t) - u_n(t) \right\rangle &\le \gamma_m(t) + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + \gamma_m(t)\big)^2 \\ &\quad + \frac{1}{2r}\big(\gamma_m(t) + 2\gamma_n(t)\big)^2 \\ &\quad + \left[\frac{1}{r}\big(\gamma_m(t) + \|u_m(t) - u_n(t)\|\big) + 1\right] \\ &\quad \big(\gamma_m(t) + 2\gamma_n(t)\big).\end{aligned}$$

Since the sequences $(u_k(\cdot))_{n\in\mathbb{N}}$ and $(\gamma_k(\cdot))_{k\in\mathbb{N}}$ are uniformly bounded, one can choose a real $A \ge 0$ such that for all $t \in I$, for all $n, m \in \mathbb{N}$,

$$\begin{aligned}&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\mu}(t), u_n(t) - u_m(t) \right\rangle \\ &\quad \le \frac{1}{r}\|u_n(t) - u_m(t)\|^2 + \frac{A}{2}[(\gamma_n(t) + \gamma_m(t))^2 + \gamma_n(t) + \gamma_m(t)]. \qquad (41)\end{aligned}$$

Fix any $m, n \in \mathbb{N}$. Let us define the function $\phi_{n,m} : I \to [0, +\infty[$ by

$$\phi_{n,m}(t) := \|u_n(t) - u_m(t)\|^2 \quad \text{for all } t \in I,$$

and let us apply Proposition 3.1 to get

$$d\phi_{n,m} \le 2\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\mu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\mu}(t), u_n(\cdot) - u_m(\cdot) \right\rangle \mathrm{d}\mu.$$

Putting the latter inequality, (41) and $\phi_{n,m}(T_0) = 0$ together, we have

$$\phi_{n,m}(t) \le \int_{]T_0,t]} \frac{2}{r}\phi_{n,m}(s)\,\mathrm{d}\mu(s) + \alpha_{n,m},$$

where

$$\alpha_{n,m} := A \int_{]T_0,T]} [(\gamma_n(s) + \gamma_m(s))^2 + \gamma_n(s) + \gamma_m(s)] \, d\mu(s).$$

Applying Lemma 3.1 with a real κ such that

$$\frac{2}{r} \sup_{s \in]T_0,T]} \mu(\{s\}) \leq \kappa < 1$$

we get

$$\phi_{n,m}(t) \leq \alpha_{n,m} \exp\left(\frac{2}{r(1-\kappa)} \mu(]T_0, T])\right),$$

so kepping in mind that $\phi_{n,m}(T_0) = 0$ it results that

$$\sup_{t \in [T_0,T]} \phi_{n,m}(t) = \sup_{t \in]T_0,T]} \phi_{n,m}(t) \leq \alpha_{n,m} \exp\left(\frac{2}{r(1-\kappa)} \mu(]T_0, T])\right).$$

By the Lebesgue dominated convergence theorem, the fact that $(\gamma_n(\cdot))_{n \in \mathbb{N}}$ is uniformly bounded by $\mu(]T_0, T])$ with $\lim_{n \to +\infty} \gamma_n(t) = 0$ for each $t \in I$ (see (39)), it ensues that $\lim_{n,m \to +\infty} \alpha_{n,m} = 0$. As a consequence, $(u_n(\cdot))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(I, \mathcal{H})$, so there is some mapping $u : I \to \mathcal{H}$ such that

$$u_n(\cdot) \to u(\cdot) \text{ uniformly on } I.$$

Thanks to (35), we may suppose that $((du_n/d\mu)(\cdot))_{n \in \mathbb{N}}$ converges weakly in $L^2(I, \mathcal{H}, \mu)$ to some mapping $g(\cdot) \in L^2(I, \mathcal{H}, \mu)$. Thus, we have for any $t \in I$

$$\int_{]T_0,t]} \frac{du_n}{d\mu}(s) \, d\mu(s) \to \int_{]T_0,t]} g(s) \, d\mu(s) \text{ weakly in } \mathcal{H}.$$

Since $u_n(t) \to u(t)$ for each $t \in I$, the weak convergence above gives

$$u(t) = u_0 + \int_{]T_0,t]} g(s) \, d\mu(s) \quad \text{for all } t \in I.$$

Hence, $u(\cdot)$ is right-continuous with bounded variation on I and du has $g(\cdot) \in L^2(I, \mathcal{H}, \mu)$ as a density relative to μ. A direct consequence is that

$$\frac{du_n}{d\mu}(\cdot) \to \frac{du}{d\mu}(\cdot) \text{ weakly in } L^2(I, \mathcal{H}, \mu),$$

which entails

$$\frac{du_n}{d\mu}(\cdot) \to \frac{du}{d\mu}(\cdot) \text{ weakly in } L^1(I, \mathcal{H}, \mu).$$

Step 5. The mapping $u(\cdot)$ satisfies (21)–(24).

First, note that for all $t \in I$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\|u_n(\theta_n(t)) - u(t)\| &\le \|u_n(\theta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\| \\ &\le \mu(]t, \theta_n(t)]) + \|u_n(t) - u(t)\|,\end{aligned}$$

where the second inequality is due to (36). Since $u_n(t) \to u(t)$ and $\mu(]t, \theta_n(t)])$ $\to 0$ for each $t \in I$ (see (39)), the latter inequality entails that

$$u_n(\theta_n(t)) \to u(t) \quad \text{for all } t \in I.$$

From (15) and (38), we have

$$\begin{aligned}d_{C(t)}(u_n(\theta_n(t))) &\le d_{C(\theta_n(t)) \cap \rho\mathbb{B}}(u_n(\theta_n(t))) + \widehat{\text{exc}}_\rho(C(\theta_n(t)), C(t)) \\ &\le \widehat{\text{haus}}_\rho(C(t), C(\theta_n(t))) \\ &\le \mu(]t, \theta_n(t)]),\end{aligned}$$

for all $n \in \mathbb{N}$ and all $t \in I$. Since $\mu(]t, \theta_n(t)]) \to 0$ by (39), passing to the limit and keeping in mind that $C(\cdot)$ is closed-valued, we obtain

$$u(t) \in C(t) \quad \text{for all } t \in I.$$

Now, we apply a classical technique due to C. Castaing ([37]). Thanks to Mazur's lemma, there exists a sequence $(z_n(\cdot))_{n\in\mathbb{N}}$ which converges strongly in $L^1(I, \mathcal{H}, \mu)$ to $(\mathrm{d}u/\mathrm{d}\mu)(\cdot)$ with

$$z_n(\cdot) \in \mathrm{co}\left\{\frac{\mathrm{d}u_k}{\mathrm{d}\mu}(\cdot) : k \ge n\right\} \quad \text{for all } n \in \mathbb{N}. \tag{42}$$

Extracting a subsequence if necessary, we may suppose that

$$z_n(t) \to \frac{\mathrm{d}u}{\mathrm{d}\mu}(t) \quad \mu\text{−a.e. } t \in I.$$

Combining the inclusion (42) with the latter convergence, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}\mu}(t) \in \bigcap_{n\in\mathbb{N}} \overline{\mathrm{co}}\left\{\frac{\mathrm{d}u_k}{\mathrm{d}\mu}(t) : k \ge n\right\} \quad \mu\text{−a.e.} t \in I.$$

Such an inclusion yields for μ-almost every $t \in I$ that

$$\left\langle \xi, \frac{\mathrm{d}u}{\mathrm{d}\mu}(t)\right\rangle \le \inf_{n\in\mathbb{N}} \sup_{k\ge n} \left\langle \xi, \frac{\mathrm{d}u_k}{\mathrm{d}\mu}(t)\right\rangle \quad \text{for all } \xi \in \mathcal{H}.$$

Coming back to (37), it follows that, for μ-almost every $t \in I$,

$$\left\langle \xi, \frac{\mathrm{d}u}{\mathrm{d}\mu}(t)\right\rangle \le \limsup_{n\to+\infty} \sigma\left(\xi, -\partial_P d_{C(\theta_n(t))}\left(u_n(\theta_n(t))\right)\right) \quad \text{for all } \xi \in \mathcal{H}.$$

Through Proposition 3.2, the latter inequality entails for μ-almost every $t \in I$,

$$\left\langle \xi, \frac{\mathrm{d}u}{\mathrm{d}\mu}(t) \right\rangle \leq \sigma(\xi, -\partial_C d_{C(t)}(u(t))) \quad \text{for all } \xi \in \mathcal{H}.$$

Thanks to (6) and the fact that the Clarke subdifferential is closed and convex

$$\left\{ \frac{\mathrm{d}u}{\mathrm{d}\mu}(t) \right\} \subset \overline{\mathrm{co}}\left(-\partial_C d_{C(t)}(u(t))\right) = -\partial_C d_{C(t)}(u(t)) \quad \mu\text{-a.e.} t \in I$$

It remains to invoke (8) to obtain

$$\frac{\mathrm{d}u}{\mathrm{d}\mu}(t) \in -N(C(t); u(t)) \quad \mu\text{-a.e.} t \in I.$$

Consequently, the mapping $u : I \to \mathcal{H}$ satisfies (21), i.e.

$$\begin{cases} -\mathrm{d}u \in N(C(t); u(t)) \\ u(T_0) = u_0. \end{cases}$$

Now, we are going to show (22)–(24). Taking the limit in (36) gives

$$\|u(t) - u(s)\| \leq \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \leq t, \tag{43}$$

which is the inequality (23). Again, passing to the limit in (33), we have

$$\|u(t)\| \leq \rho_0 \quad \text{for all } t \in I.$$

The relation (24) is then a direct consequence of the latter inequality and (43). Using again (43), we have for every $s < t$

$$\|u(t) - u(s)\| \leq \sup_{\tau \in]T_0,T]} \mu(\{\tau\}) + \mu(]s,t[).$$

Taking the limit as $s \uparrow t$ in the latter inequality gives the desired relation (22). The proof is then complete. ∎

The case $\rho = +\infty$ in the latter theorem is of a great interest.

Corollary 4.1: *Let $C : I \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some extended real $r \in]0, +\infty]$, $u_0 \in C(T_0)$. Assume that there exist a positive Radon measure μ on I with $\sup_{\tau \in]T_0,T]} \mu(\{\tau\}) < r/2$, a real $\rho_0 \geq \|u_0\|$ and a real $\eta > 0$ satisfying* (i) *of Theorem* 4.1 *and such that for all $s,t \in I$ with $s \leq t$ and $\mu(]s,t[) < \eta$, one has*

$$\widehat{\mathrm{haus}}(C(s), C(t)) \leq \mu(]s,t]).$$

Then, there exists one and only one mapping $u : I \to \mathcal{H}$ satisfying (21)–(24) of Theorem 4.1.

Another direct consequence of Theorem 4.1 is the case where the positive Radon measure $\mu(]s,t]) = v(t) - v(s)$ for some nondecreasing absolutely continuous function $v(\cdot) : I \to \mathbb{R}$, that is μ is absolutely continuous with respect to λ.

Corollary 4.2: *Let $C : I \rightrightarrows \mathcal{H}$ be a multimapping with r-prox-regular values for some extended real $r \in]0,+\infty]$, $u_0 \in C(T_0)$. Assume that there exist a nondecreasing absolutely continuous function $v(\cdot) : I \to \mathbb{R}$ on I, a real $\rho_0 \geq \|u_0\|$, an extended real $\rho > \rho_0$ and a real $\eta > 0$ satisfying:*

(i) *for every $k \in \mathbb{N}$ and for every $t_1, \ldots, t_k \in I$ with $t_0 := T_0 < t_1 < \cdots < t_k$ and $v(t_{i+1}) - v(t_i) < \eta$ for each $i \in \{0, \ldots, k-1\}$, one has*

$$\left\| \mathrm{proj}_{C(t_k)} \circ \cdots \circ \mathrm{proj}_{C(t_1)}(u_0) \right\| \leq \rho_0,$$

whenever $\mathrm{proj}_{C(t_k)} \circ \cdots \circ \mathrm{proj}_{C(t_1)}(u_0)$ is well-defined;

(ii) *for all $s, t \in I$ with $s \leq t$ and $v(t) - v(s) < \eta$, one has*

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) \leq v(t) - v(s).$$

Then, there exists one and only one mapping $u : I \to \mathcal{H}$ satisfying

$$\begin{aligned} &-\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ &u(t) \in C(t) \quad \text{for all } t \in I, \\ &u(T_0) = u_0, \end{aligned}$$

as well as the inequalities

$$\|u(t) - u(s)\| \leq v(t) - v(s) \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

and

$$\|u(t)\| \leq \min\{\rho_0, \|u_0\| + v(T) - v(T_0)\} \quad \text{for all } t \in I.$$

Now, in addition to the convex situations in [28] we provide another situation where the inequality (20) in Theorem 4.1 holds true.

Proposition 4.1: *Let S be an r-prox-regular subset of $\mathcal{H}$ with $r \in]0,+\infty[$, $u_0 \in S$, $\zeta : I \to \mathcal{H}$ be a right continuous mapping of bounded variation on I with $\zeta(T_0) = 0$, $\mathrm{var}(\zeta; I) < r$ and $\sup_{s \in]T_0,T]} \|\zeta(s) - \zeta^-(s)\| < r/2$. The following hold:*

(a) *For all $t \in I$, $S + \zeta(t)$ is r-prox-regular.*

(b) *$\sup_{s \in]T_0,T]} |d\zeta|(\{s\}) < r/2$ and for all $s, t \in I$ with $s < t$,*

$$\widehat{\mathrm{haus}}_\rho(S + \zeta(s), S + \zeta(t)) \leq \mathrm{haus}_\rho(S + \zeta(s), S + \zeta(t)) \leq |d\zeta|(]s,t]).$$

(c) *For all* $t_1, \ldots, t_k \in I$ *with* $k \in \mathbb{N}$ *and* $t_1 < \cdots < t_k$, *one has*

$$\left\| \mathrm{proj}_{S+\zeta(t_k)} \circ \cdots \circ \mathrm{proj}_{S+\zeta(t_1)}(u_0) \right\| \le \mathrm{var}(\zeta, I) + \|u_0\|$$

whenever $\mathrm{proj}_{S+\zeta(t_k)} \circ \cdots \circ \mathrm{proj}_{S+\zeta(t_1)}(u_0)$ *is well-defined.*

Proof: Let $C(\cdot) = S + \zeta(\cdot) : I \rightrightarrows \mathcal{H}$.

(a) Fix any $t \in [T_0, T]$. Let $c_1, c_2 \in C(t)$ and $\tau \in [0,1]$ such that $\tau c_1 + (1 - \tau)c_2 \in U_r(C(t))$. There are $s_1, s_2 \in S$ such that

$$c_1 = s_1 + \zeta(t) \quad \text{and} \quad c_2 = s_2 + \zeta(t).$$

It is readily seen that

$$d(\tau c_1 + (1-\tau)c_2, C(t)) = d(\tau s_1 + (1-\tau)s_2, S),$$

so $\tau s_1 + (1-\tau)s_2 \in U_r(S)$. Combining the r-prox-regularity of S with Theorem 2.1(e), we get

$$d(\tau c_1 + (1-\tau)c_2), C(t)) = d(\tau s_1 + (1-\tau)s_2, S) \le \frac{1}{2r}\tau(1-\tau)\|s_1 - s_2\|^2$$
$$= \frac{1}{2r}\tau(1-\tau)\|c_1 - c_2\|^2 .$$

Applying Theorem 2.1(e) again, we obtain the r-prox-regularity of $C(t)$.

(b) According to the inequality $\sup_{s\in]T_0,T]} \|\zeta(s) - \zeta^-(s)\| < r/2$, we have

$$\sup_{s\in]T_0,T]} |d\zeta|(\{s\}) = \sup_{s\in]T_0,T]} \|d\zeta(\{s\})\| < \frac{r}{2}.$$

On the other hand, we observe that

$$\mathrm{haus}_\rho(C(s), C(t)) \le \|\zeta(s) - \zeta(t)\| = \|(d\zeta)(]s,t])\| \le |d\zeta|(]s,t]),$$

for all $s, t \in I$ with $s \le t$.

(c) Fix any $k \in \mathbb{N}$, $t_1, \ldots, t_k \in I$ with $t_1 < \cdots < t_k$ such that $\mathrm{proj}_{C(t_k)} \circ \cdots \circ \mathrm{proj}_{C(t_1)}$ is well-defined. Set $t_0 := T_0$ and $\rho_0 := \mathrm{var}(\zeta; I) + \|u_0\|$. Let us set for each $i \in \{1, \ldots, k\}$,

$$u_i := \mathrm{proj}_{C(t_i)}(u_{i-1})$$

and let us show by induction that

$$\|u_i - u_{i-1}\| \le \|\zeta(t_i) - \zeta(t_{i-1})\| \quad \text{for all } i \in \{1, \ldots, k\}.$$

From the inclusion $u_0 \in S$ and the equality $\zeta(T_0) = 0$, we observe that

$$d_{C(t_1)}(u_0) \le \|\zeta(t_1)\| = \|\zeta(t_1) - \zeta(T_0)\| \le \mathrm{var}(\zeta; I) < r,$$

so we have

$$\|u_1 - u_0\| \leq \|\zeta(t_1) - \zeta(t_0)\|.$$

If $k=1$, the proof is complete. Hence, we may assume that $k>1$. Fix any $n \in \{1, \ldots, k-1\}$ and assume that

$$\|u_i - u_{i-1}\| \leq \|\zeta(t_i) - \zeta(t_{i-1})\| \quad \text{for all } i \in \{1, \ldots, n\}.$$

According to the inclusion $u_n - \zeta(t_n) + \zeta(t_{n+1}) \in C(t_{n+1})$, we have

$$d_{C(t_{n+1})}(u_n) \leq \|\zeta(t_{n+1}) - \zeta(t_n)\| \leq \operatorname{var}(\zeta; I) < r,$$

and then we obtain

$$\|u_{n+1} - u_n\| \leq \|\zeta(t_{n+1}) - \zeta(t_n)\|.$$

Consequently, the induction is complete. It remains to see that

$$\begin{aligned}
\|u_{k+1}\| &\leq \|u_{k+1} - u_k\| + \|u_k\| \\
&\leq \|\zeta(t_{k+1}) - \zeta(t_k)\| + \|u_k - u_{k-1}\| + \|u_{k-1}\| \\
&\leq \|\zeta(t_{k+1}) - \zeta(t_k)\| + \|\zeta(t_k) - \zeta(t_{k-1})\| + \|u_{k-1}\| \\
&\leq \sum_{i=0}^{k} \|\zeta(t_{i+1}) - \zeta(t_i)\| + \|u_0\|. \\
&\leq \operatorname{var}(\zeta; I) + \|u_0\| = \rho_0
\end{aligned}$$

to finish the proof

∎

The following result is related to the jumps of the solution of (21).

Proposition 4.2: *Under the assumptions of Theorem* 4.1, *the solution* $u(\cdot) : I \to \mathcal{H}$ *of the sweeping process satisfies the following properties*

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \text{and} \quad u(t) = \operatorname{proj}_{C(t)}(u(t^-)) \quad \textit{for all } t \in]T_0, T].$$

Proof: Fix any $t \in]T_0, T]$. The first inequality is a direct consequence of the inequality (23) of Theorem 4.1. Then, if $\mu(\{t\}) = 0$, we have $u(t) = u(t^-)$, so $u(t) = \operatorname{proj}_{C(t)}(u(t^-))$. This justifies that we may assume $\mu(\{t\}) > 0$. From (13), we get

$$\frac{du}{d\mu}(t) = \lim_{\varepsilon \downarrow 0} \frac{du(]t-\varepsilon, t] \cap I)}{\mu(]t-\varepsilon, t] \cap I)} = \frac{u(t) - u(t^-)}{\mu(\{t\})}.$$

Combining the latter equality with the fact that $u(\cdot)$ satisfies (21), we obtain

$$\frac{du}{d\mu}(t) = \frac{u(t) - u(t^-)}{\mu(\{t\})} \in -N(C(t); u(t))$$

or equivalently (keeping in mind that $N(\cdot;\cdot)$ is a cone)

$$u(t^-) - u(t) \in N(C(t); u(t)).$$

On the other hand, thanks to the inequality

$$\left\| u(t) - u(t^-) \right\| \leq \sup_{s \in]T_0, T]} \mu(\{s\}) < \frac{r}{2},$$

we can apply [14, Proposition 3.5] to get

$$u(t) = \text{proj}_{C(t)}(u(t^-)).$$

The proof is then complete. ∎

5. Perturbed sweeping process

Requiring a stronger inequality on ρ_0 in Theorem 4.1, namely $\rho_0 > \|u_0\| + \mu(]T_0, T])$, one can remove the assumption (20) and consider a perturbation $f(\cdot,\cdot)$ of the normal cone involved as in (18), that is, the perturbed sweeping process

$$-du \in N(C(t); u(t)) + f(t, u(t)) \quad \text{and} \quad u(T_0) = u_0.$$

The main result of this section is the following:

Theorem 5.1: *Let $C : I \rightrightarrows \mathcal{H}$ be an r-prox-regular valued multimapping for some extended real $r \in]0, +\infty]$ and $u_0 \in C(T_0)$. Let also $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping with $f \not\equiv 0$, μ be a positive Radon measure on I with $\sup_{\tau \in]T_0, T]} \mu(\{\tau\}) < r/2$. Assume:*

(i) *the mapping $f(\cdot, x)$ is measurable for every $x \in \bigcup_{t \in I} C(t)$, and for each bounded subset B of $\mathcal{H}$ the mapping $f(t, \cdot)$ is uniformly continuous on B for every $t \in I$ and there exists a function $l_B \in L^1(I, \mathbb{R}_+, \lambda)$ such that*

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \geq -l_B(t) \|x_1 - x_2\|^2 \quad \textit{for all } t \in I, x_1, x_2 \in B;$$

(ii) *there exists $\alpha(\cdot) \in L^1(I, \mathbb{R}_+, \lambda)$ with $1 - 2\int_{T_0}^{T} \alpha(s)\, d\lambda(s) > 0$ such that*

$$\|f(t, x)\| \leq \alpha(t)(1 + \|x\|) \quad \text{for all } t \in I, x \in \mathcal{H};$$

(iii) *there exist a real* $\rho_0 > \|u_0\| + \mu(]T_0, T])$*, an extended real* $\rho \geq (\rho_0 + 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s))/(1 - 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s))$ *and a real* $\eta > 0$ *such that*

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) \leq \mu(]s, t]),$$

for all $s, t \in I$ *with* $s \leq t$ *and* $\mu(]s, t[) < \eta$.

Then, there exists one and only one mapping $u : I \to \mathcal{H}$ *satisfying*

$$\begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases} \tag{44}$$

and

$$\sup_{t \in]T_0, T]} \left\| u(t) - u(t^-) \right\| < \frac{r}{2}. \tag{45}$$

Proof: The proof is quite similar to that of Theorem 4.1. Let us focus only on the differences.

Uniqueness. It is a direct consequence of [28, Proposition 3.16].

Existence. Choose any real $\omega \in]0, 1]$ such that

$$(2 + \omega) \sup_{\tau \in]T_0, T]} \mu(\{\tau\}) < r \tag{46}$$

and define the constant

$$\kappa := \frac{\|u_0\| + \mu(]T_0, T]) + 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s)}{1 - 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s)}$$

and the positive Radon measure $\nu := \mu + \omega^{-1}(1 + (2 + \kappa)\alpha(\cdot))\lambda$ on I. The constant κ is well-defined and non-negative, and we note by the above inequalities that

$$\kappa < \frac{\rho_0 + 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s)}{1 - 2\int_{T_0}^{T} \alpha(s)\,\mathrm{d}\lambda(s)} \leq \rho. \tag{47}$$

As in the proof of the previous theorem, we proceed in 5 steps.

Step 1. Time discretization of $I = [T_0, T]$.

Consider any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of $]0, \eta[$ with $\varepsilon_n \downarrow 0$ and construct (as in the proof of Theorem 4.1) for each $n \in \mathbb{N}$ an integer $p(n) \geq 1$ and a discretization

$(t_i^n)_{0\leq i\leq p(n)}$ of $I=[T_0,T]$ such that

$$T_0 = t_0^n < \cdots < t_{p(n)}^n = T$$

and

$$\nu(]t_i^n, t_{i+1}^n[) \leq \varepsilon_n < \eta \text{ for each } i \in \{0,\ldots,p(n)-1\}. \tag{48}$$

In particular, note that for each $n\in\mathbb{N}$,

$$\mu(]t_i^n, t_{i+1}^n[) + t_{i+1}^n - t_i^n \leq \varepsilon_n \quad \text{for each } i \in \{0,\ldots,p(n)-1\}.$$

Noting that $\int_Q \alpha(s)\,\mathrm{d}\lambda(s) \to 0$ as $\lambda(Q)\to 0$ and setting for every $n\in\mathbb{N}$

$$\alpha_i^n := \int_{t_i^n}^{t_{i+1}^n} \alpha(s)\,\mathrm{d}\lambda(s) \quad \text{for each } i \in \{0,\ldots,p(n)-1\},$$

we can choose by inequality (46) some $N\in\mathbb{N}$ such that for every integer $n\geq N$,

$$\Big(2+\frac{\omega}{2}\Big)\varepsilon_n + \Big(1+\frac{\omega}{2}\Big)\sup_{s\in]T_0,T]}\mu(\{s\}) < \frac{r}{2} \quad \text{and} \quad 2(1+\kappa)\max_{0\leq i\leq p(n)-1}\alpha_i^n < 1. \tag{49}$$

Step 2. Construction of finite sequences $(u_i^n)_{0\leq i\leq p(n)}$ $(n\geq N)$.

Fix any $n\in\mathbb{N}$ with $n\geq N$ and set $u_0^n := u_0$. We proceed to construct by induction a sequence $(u_i^n)_{1\leq i\leq p(n)}$ of $\mathcal{H}$ such that for each $i\in\{1,\ldots,p(n)\}$,

$$u_i^n := \mathrm{proj}_{C(t_i^n)}\Big(u_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f(s,u_{i-1}^n)\,\mathrm{d}\lambda(s)\Big), \tag{50}$$

$$\|u_i^n\|(1-2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)) \leq \|u_0^n\| + \mu(]T_0,T]) + 2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)$$

and

$$\left\| u_i^n - \Big(u_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f(s,u_{i-1}^n)\,\mathrm{d}\lambda(s)\Big)\right\| \leq \mu(]t_{i-1}^n, t_i^n]) + (1+\|u_{i-1}^n\|)\alpha_{i-1}^n. \tag{51}$$

Using the assumption (ii), the inequality $\|u_0^n\|\leq\rho$, the assumption (iii), the inequalities $\omega^{-1}\geq 1$ and $\kappa\geq\|u_0^n\|$, the definition of ν, (48) and (49), we see that

$$\begin{aligned}
d_{C(t_1^n)}\Big(u_0^n - \int_{t_0^n}^{t_1^n} f(s,u_0^n)\,\mathrm{d}\lambda(s)\Big) &\le d_{C(t_1^n)}(u_0^n) + (1+\|u_0^n\|)\int_{t_0^n}^{t_1^n}\alpha(s)\,\mathrm{d}\lambda(s)\\
&\le \widehat{\mathrm{exc}}_\rho(C(t_0^n),C(t_1^n)) + (1+\|u_0^n\|)\alpha_0^n\\
&\le \mu(]t_0^n,t_1^n]) + (1+\|u_0^n\|)\alpha_0^n\\
&\le \mu(]t_0^n,t_1^n[) + \sup_{s\in]T_0,T]}\mu(\{s\}) + (1+\|u_0^n\|)\alpha_0^n\\
&\le \mu(]t_0^n,t_1^n[) + \sup_{s\in]T_0,T]}\mu(\{s\}) + \omega^{-1}(1+\kappa)\alpha_0^n\\
&\le \nu(]t_0^n,t_1^n[) + \sup_{s\in]T_0,T]}\mu(\{s\})\\
&\le \varepsilon_n + \sup_{s\in]T_0,T]}\mu(\{s\}) < r. \qquad (52)
\end{aligned}$$

The latter inequality along with the r-prox-regularity of $C(t_1^n)$ allows us to set

$$u_1^n := \mathrm{proj}_{C(t_1^n)}\big(u_0^n - \int_{t_0^n}^{t_1^n} f(s,u_0^n)\,\mathrm{d}\lambda(s)\big).$$

Then, coming back to (52), we obtain that

$$\left\| u_1^n - \left(u_0^n - \int_{t_0^n}^{t_1^n} f(s,u_0^n)\,\mathrm{d}\lambda(s)\right)\right\| \le \mu(]t_0^n,t_1^n]) + (1+\|u_0^n\|)\alpha_0^n.$$

The latter inequality entails through the assumption (ii)

$$\begin{aligned}
\|u_1^n\| &\le \|u_0^n\| + \int_{t_0^n}^{t_1^n}\|f(s,u_0^n)\|\ \mathrm{d}\lambda(s) + \mu(]t_0^n,t_1^n]) + (1+\|u_0^n\|)\alpha_0^n\\
&\le \|u_0^n\| + \mu(]t_0^n,t_1^n]) + 2(1+\|u_0^n\|)\alpha_0^n\\
&\le \|u_0^n\| + \mu(]T_0,T]) + 2(1+\|u_0^n\|)\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s),
\end{aligned}$$

which gives

$$\begin{aligned}
&\max\left\{\|u_0^n\|,\|u_1^n\|\right\}\\
&\quad\le \|u_0^n\| + \mu(]T_0,T]) + 2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s) + 2\|u_0^n\|\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)\\
&\le \|u_0^n\| + \mu(]T_0,T]) + 2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)\\
&\quad + 2\max\{\|u_0^n\|,\|u_1^n\|\}\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s),
\end{aligned}$$

and hence

$$\max\left\{\left\|u_0^n\right\|,\left\|u_1^n\right\|\right\}(1-2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)) \le \left\|u_0^n\right\| + \mu(]T_0,T]) + 2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s).$$

Assume that $p(n) > 1$ (otherwise, there is nothing to prove). Fix any $k \in \{2,\ldots,p(n)\}$. Suppose that we have constructed $u_1^n,\ldots,u_{k-1}^n \in \mathcal{H}$ such that for each $i \in \{1,\ldots,k-1\}$,

$$u_i^n := \mathrm{proj}_{C(t_i^n)}\big(u_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f(s,u_{i-1}^n)\,\mathrm{d}\lambda(s)\big),$$

$$\|u_i^n\|(1-2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s)) \le \|u_0^n\| + \mu(]T_0,T]) + 2\int_{T_0}^{T}\alpha(s)\,\mathrm{d}\lambda(s) \tag{53}$$

and

$$\left\|u_i^n - \big(u_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f(s,u_{i-1}^n)\,\mathrm{d}\lambda(s)\big)\right\| \le \mu(]t_{i-1}^n,t_i^n]) + (1+\left\|u_{i-1}^n\right\|)\alpha_{i-1}^n. \tag{54}$$

From (53) we note that $\|u_i^n\| \le \kappa$ for each $i \in \{1,\ldots,k-1\}$. As above, according to the assumption (ii), the inequality $\|u_{k-1}^n\| \le \kappa \le \rho$, the assumption (iii), the definition of ν, (48) and (49), we have

$$\begin{aligned} d_{C(t_k^n)}&\left(u_{k-1}^n - \int_{t_{k-1}^n}^{t_k^n} f(s,u_{k-1}^n)\,\mathrm{d}\lambda(s)\right) \\ &\le d_{C(t_k^n)}(u_{k-1}^n) + (1+\left\|u_{k-1}^n\right\|)\int_{t_{k-1}^n}^{t_k^n}\alpha(s)\,\mathrm{d}\lambda(s) \\ &\le \widehat{\mathrm{exc}}_\rho(C(t_{k-1}^n),C(t_k^n)) + (1+\left\|u_{k-1}^n\right\|)\alpha_{k-1}^n \\ &\le \mu(]t_{k-1}^n,t_k^n]) + (1+\left\|u_{k-1}^n\right\|)\alpha_{k-1}^n \\ &\le \mu(]t_{k-1}^n,t_k^n[) + \sup_{s\in]T_0,T]}\mu(\{s\}) + (1+\kappa)\alpha_{k-1}^n \\ &\le \nu(]t_{k-1}^n,t_k^n[) + \sup_{s\in]T_0,T]}\mu(\{s\}) \\ &\le \varepsilon_n + \sup_{s\in]T_0,T]}\mu(\{s\}) < r. \end{aligned} \tag{55}$$

Thanks to the r-prox-regularity of $C(t_k^n)$, we can set

$$u_k^n := \mathrm{proj}_{C(t_k^n)}\left(u_{k-1}^n - \int_{t_{k-1}^n}^{t_k^n} f(s,u_{k-1}^n)\,\mathrm{d}\lambda(s)\right).$$

From the definition of u_k^n and (55), it is clear that

$$\left\| u_k^n - \left(u_{k-1}^n - \int_{t_{k-1}^n}^{t_k^n} f(s, u_{k-1}^n)\, \mathrm{d}\lambda(s) \right) \right\| \le \mu(]t_{k-1}^n, t_k^n]) + (1 + \|u_{k-1}^n\|)\alpha_{k-1}^n.$$

It is straightforward from the latter inequality and (54) that for each $i \in \{1, \dots, k\}$,

$$\|u_i^n\| \le \|u_{i-1}^n\| + \mu(]t_{i-1}^n, t_i^n]) + 2(1 + \|u_{i-1}^n\|)\alpha_{i-1}^n.$$

It follows that

$$\begin{aligned} \|u_k^n\| &\le \|u_{k-1}^n\| + \mu(]t_{k-1}^n, t_k^n]) + 2\left(1 + \max_{0\le i\le k-1} \|u_i^n\|\right) \alpha_{k-1}^n \\ &\le \|u_{k-2}^n\| + \mu(]t_{k-2}^n, t_k^n]) + 2\left(1 + \max_{0\le i\le k-1} \|u_i^n\|\right) (\alpha_{k-2}^n + \alpha_{k-1}^n) \\ &\vdots \\ &\le \|u_0^n\| + \mu(]T_0, T]) + 2\left(1 + \max_{0\le i\le k-1} \|u_i^n\|\right) \sum_{j=0}^{k-1} \alpha_j^n, \end{aligned}$$

thus $\|u_k^n\| \le \|u_0^n\| + \mu(]T_0, T]) + 2\int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s) + 2\max_{0\le i\le k} \|u_i^n\| \int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s)$. This and the induction assumption (53) for every $i \in \{1, \dots, k-1\}$ ensure that

$$\begin{aligned} \max_{0\le i\le k} \|u_i^n\| &\le \|u_0^n\| + \mu(]T_0, T]) + 2\int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s) \\ &\quad + 2\max_{0\le i\le k} \|u_i^n\| \int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s), \end{aligned}$$

or equivalently

$$\max_{0\le i\le k} \|u_i^n\| \left(1 - 2\int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s)\right) \le \|u_0^n\| + \mu(]T_0, T]) + 2\int_{T_0}^T \alpha(s)\, \mathrm{d}\lambda(s).$$

The induction is then complete.

From the inequality $\|u_0\| \le \kappa$ and from (53) we note that

$$\|u_i^n\| \le \kappa \quad \text{for all } i \in \{0, 1, \dots, p(n)\}. \tag{56}$$

From the latter inequality, from (51) and from the definition of ν we also note that

$$\begin{aligned} &\left\| u_i^n - \left(u_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f(s, u_{i-1}^n)\, \mathrm{d}\lambda(s) \right) \right\| \le \mu(]t_{i-1}^n, t_i^n]) \\ &\quad + (1 + \kappa)\alpha_{i-1}^n \le \nu(]t_{i-1}^n, t_i^n]). \end{aligned} \tag{57}$$

Step 3. Definition of the sequence $(u_n(\cdot))_{n\ge N}$.

Fix any integer $n \geq N$. Let us define the mapping $u_n : I \to \mathcal{H}$ by $u_n(T) = u^n_{p(n)}$ and

$$u_n(t) = u^n_i + \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])}\Big(u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(s, u^n_i)\, \mathrm{d}\lambda(s)\Big) - \int_{t^n_i}^{t} f(s, u^n_i)\, \mathrm{d}\lambda(s),$$

for all $t \in [t^n_i, t^n_{i+1}[$ with $i \in \{0, \ldots, p(n) - 1\}$. It is not difficult to check that

$$u_n(t) = u^n_i + \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])}\Big(u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(s, u^n_i)\, \mathrm{d}\lambda(s)\Big) - \int_{t^n_i}^{t} f(s, u^n_i)\, \mathrm{d}\lambda(s),$$

for all $t \in [t^n_i, t^n_{i+1}]$ with $i \in \{0, \ldots, p(n) - 1\}$. Further, rewriting $u_n(t)$ for each $t \in [t^n_i, t^n_{i+1}]$ ($i \in \{0, \ldots, p(n) - 1\}$) in the form

$$u_n(t) = \left(1 - \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])}\right) u^n_i + \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])} u^n_{i+1} + \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])} \int_{t^n_i}^{t^n_{i+1}} f(s, u^n_i)\, \mathrm{d}\lambda(s) - \int_{t^n_i}^{t} f(s, u^n_i)\, \mathrm{d}\lambda(s),$$

we see from (56), from assumption (ii) and from the second inequality in (49) that for all $t \in I$

$$\|u_n(t)\| \leq \kappa + 2(1 + \kappa) \max_{0 \leq j \leq p(n)-1} \alpha^n_j < \kappa + 1. \tag{58}$$

Now, define $\delta_n(\cdot) : I \to I$ by $\delta_n(T) = T$ and

$$\delta_n(t) := t^n_i \quad \text{if}\, t \in [t^n_i, t^n_{i+1}[\text{with}\ i \in \left\{0, \ldots, p(n) - 1\right\},$$

and define also $\zeta_n(\cdot) : I \to \mathcal{H}$ by $\zeta_n(T_0) = 0$ and

$$\zeta_n(t) := \frac{u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(s, u^n_i)\, \mathrm{d}\lambda(s)}{\nu(]t^n_i, t^n_{i+1}])} \text{for all}\ t \in]t^n_i, t^n_{i+1}]. \tag{59}$$

By the very definition of $u_n(\cdot)$, $\delta_n(\cdot)$ and $\zeta_n(\cdot)$, we have

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \zeta_n(s)\, \mathrm{d}\nu(s) - \int_{]T_0,t]} f\big(s, u_n(\delta_n(s))\big)\, \mathrm{d}\lambda(s).$$

Using the definition of ν we see that λ is absolutely continuous with respect to ν an it has $\mathrm{d}\lambda/\mathrm{d}\nu$ as a density in $L^\infty(I, \mathbb{R}_+, \nu)$ relative to ν. Then we can write (see (12))

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \left(\zeta_n(s) - f\big(s, u_n(\delta_n(s))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) \right) \mathrm{d}\nu(s),$$

and hence $u_n(\cdot)$ is right-continuous and with bounded variation on I. Moreover, the latter equality says that $\zeta_n(\cdot) - f(\cdot, u_n(\delta_n(\cdot)))(\mathrm{d}\lambda/\mathrm{d}\nu)(\cdot)$ is a density of $u_n(\cdot)$ relative to ν. So, it follows that $(\mathrm{d}u_n/\mathrm{d}\nu)(\cdot)$ exists as a density of $u_n(\cdot)$ relative to ν and

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t, u_n(\delta_n(t))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \zeta_n(t) \quad \nu\text{−a.e.} t \in I. \tag{60}$$

We deduce from this equality, the definition of $\zeta_n(\cdot)$ and (57) that

$$\left\| \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t, u_n(\delta_n(t))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \right\| = \|\zeta_n(t)\| \leq 1 \quad \nu\text{−a.e.} t \in I. \tag{61}$$

Since the measure $\kappa\alpha(\cdot)\lambda$ is absolutely continuous with respect to ν, it has $\mathrm{d}(\kappa\alpha(\cdot)\lambda)/\mathrm{d}\nu$ a density relative to ν and then we have by the definition of ν

$$0 \leq (2+\kappa)\alpha(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \frac{\mathrm{d}((2+\kappa)\alpha(\cdot)\lambda)}{\mathrm{d}\nu}(t) \leq \omega \quad \nu\text{−a.e.} t \in I.$$

Putting the latter inequality, (61), assumption (ii) and (58) together, we get

$$\left\| \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) \right\| \leq 1 + \left\| f(t, u_n(\delta_n(t))) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \right\| \leq 1 + (2+\kappa)\alpha(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \leq 1+\omega, \tag{62}$$

and this ensures that for all $s, t \in I$ with $s \leq t$,

$$\|u_n(t) - u_n(s)\| = \left\| \int_{]s,t]} \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(w)\,\mathrm{d}\nu(w) \right\| \leq (1+\omega)\nu(]s,t]) \leq 2\nu(]s,t]). \tag{63}$$

Besides $\delta_n(\cdot)$, let us define $\theta_n : I \to I$ by $\theta_n(T_0) = T_0$ and for all $t \in I$ by

$$\theta_n(t) = t_{i+1}^n \quad \text{if } t \in]t_i^n, t_{i+1}^n] \text{with } i \in \big\{0, \ldots, p(n)-1\big\}.$$

Combining (5) and the definition of ζ_n, we see that

$$\zeta_n(t) \in -N^P(C(\theta_n(t); u_n(\theta_n(t))) \quad \nu\text{−a.e. } t \in I.$$

With (8) and (61), this entails that

$$\zeta_n(t) \in -\partial_P d_{C(\theta_n(t))}(u_n(\theta_n(t))) \quad \nu\text{−a.e. } t \in I. \tag{64}$$

Step 4. The sequence $(u_n)_{n\geq N}$ of $\mathcal{B}(I, \mathcal{H})$ (the real Banach space of bounded mappings endowed with the norm of uniform convergence) has the Cauchy property.

Set for every $n \in \mathbb{N}$ and every $t \in I$,

$$\gamma_n(t) := 2\nu(]t, \theta_n(t)]).$$

Observe first from (50) and (56) (keeping in mind (47)) that

$$u_n(\theta_n(t)) \in C(\theta_n(t)) \cap \rho\mathbb{B} \quad \text{for all } n \in \mathbb{N}, t \in I. \tag{65}$$

Fix any $m, n \in \mathbb{N}$ with $m, n \geq N$, $t \in I$. As in the proof of Theorem 4.1, we have

$$\begin{aligned} d\big(u_m(t), C(\theta_n(t))\big) &\leq d\big(u_m(t), C(\theta_m(t)) \cap \rho\mathbb{B}\big) + \widehat{\mathrm{exc}}_\rho\big(C(\theta_m(t)), C(\theta_n(t))\big) \\ &\leq d\big(u_m(\theta_m(t)), C(\theta_m(t)) \cap \rho\mathbb{B}\big) + \|u_m(t) - u_m(\theta_m(t))\| \\ &\quad + \mu\big(]t, \max\{\theta_n(t), \theta_m(t)\}]\big). \end{aligned}$$

By virtue of (65), (63), (48) and (49), we observe that

$$d\big(u_m(t), C(\theta_n(t))\big) \leq (1+\omega)\nu(]t, \theta_m(t)]) + \nu(]t, \max\{\theta_n(t), \theta_m(t)\}]) \tag{66}$$

$$\begin{aligned} &\leq (2+\omega)\nu(]t, \max\{\theta_n(t), \theta_m(t)\}]) \\ &\leq (2+\omega)\max\{\varepsilon_m, \varepsilon_n\} + (2+\omega)\sup_{\tau \in]T_0, T]} \mu(\{\tau\}) < r. \end{aligned} \tag{67}$$

Further, (66) also ensures that

$$d\big(u_m(t), C(\theta_n(t))\big) \leq 2\max\{\gamma_m(t), \gamma_n(t)\}. \tag{68}$$

Now, for all $t \in I$, all $n, m \in \mathbb{N}$ with $n, m \geq N$ set $\gamma_{m,n}(t) := \max\{\gamma_m(t), \gamma_n(t)\}$,

$$\begin{aligned} F_n(t) &:= f\big(t, u_n(\delta_n(t))\big) \quad \text{and} \\ \varphi_{n,m}(t) &:= \big\|F_n(t) - f\big(t, u_n(t)\big)\big\| \, \|u_n(t) - u_m(t)\|. \end{aligned} \tag{69}$$

Fix any integers $n \geq N$ and $m \geq N$. Using (64), (65), (67), Proposition 2.1, (63) and (68), we have for ν-almost every $t \in I$,

$$\begin{aligned} &\big\langle \zeta_n(t), u_n\big(\theta_n(t)\big) - u_m(t) \big\rangle \\ &\quad \leq \frac{1}{2r}\big\|u_m(t) - u_n\big(\theta_n(t)\big)\big\|^2 + \frac{1}{2r} d^2_{C(\theta_n(t))}(u_m(t)) \\ &\quad + \left(\frac{1}{r}\big\|u_n\big(\theta_n(t)\big) - u_m(t)\big\| + 1\right) d_{C(\theta_n(t))}(u_m(t)) \\ &\leq \frac{1}{2r}\Big(\|u_n(t) - u_m(t)\| + \big\|u_n\big(\theta_n(t)\big) - u_n(t)\big\|\Big)^2 + \frac{1}{2r} d^2_{C(\theta_n(t))}(u_m(t)) \\ &\quad + \left[\frac{1}{r}\Big(\big\|u_n\big(\theta_n(t)\big) - u_n(t)\big\| + \|u_n(t) - u_m(t)\|\Big) + 1\right] d_{C(\theta_n(t))}(u_m(t)) \\ &\leq \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + \gamma_n(t)\big)^2 + \frac{2}{r}(\gamma_{m,n}(t))^2 \\ &\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(2\gamma_{m,n}(t)\big). \end{aligned}$$

The latter inequality gives through (61) and (63)

$$
\begin{aligned}
&\langle \zeta_n(t), u_n(t) - u_m(t)\rangle \\
&= \big\langle \zeta_n(t), u_n(t) - u_n\big(\theta_n(t)\big)\big\rangle + \big\langle \zeta_n(t), u_n\big(\theta_n(t)\big) - u_m(t)\big\rangle \\
&\le \big\| u_n(t) - u_n\big(\theta_n(t)\big)\big\| + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + \gamma_n(t)\big)^2 + \frac{2}{r}\big(\gamma_{m,n}(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(2\gamma_{m,n}(t)\big) \\
&\le 2\nu\big(\,]t, \theta_n(t)]\,\big) + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + \gamma_n(t)\big)^2 + \frac{2}{r}\big(\gamma_{m,n}(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(2\gamma_{m,n}(t)\big),
\end{aligned}
$$

for ν-almost every $t \in I$. On the other hand, the definition of $F_n(\cdot)$ and (60) ensure that for ν-almost every $t \in I$,

$$
\begin{aligned}
\left\langle \frac{du_n}{d\nu}(t), u_n(t) - u_m(t)\right\rangle \le{}& \left\langle F_n(t)\frac{d\lambda}{d\nu}(t), u_m(t) - u_n(t)\right\rangle \\
&+ 2\nu\big(\,]t, \theta_n(t)]\,\big) + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + \gamma_n(t)\big)^2 \\
&+ \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\big) + 1\right]\big(2\gamma_{m,n}(t)\big) \\
&+ \frac{2}{r}\big(\gamma_{m,n}(t)\big)^2.
\end{aligned}
$$

Since m and n are arbitrarily chosen, we also have for ν-almost every $t \in I$

$$
\begin{aligned}
\left\langle \frac{du_m}{d\nu}(t), u_m(t) - u_n(t)\right\rangle \le{}& \left\langle F_m(t)\frac{d\lambda}{d\nu}(t), u_n(t) - u_m(t)\right\rangle + 2\nu\big(\,]t, \theta_m(t)]\,\big) \\
&+ \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + \gamma_m(t)\big)^2 \\
&+ \left[\frac{1}{r}\big(\gamma_m(t) + \|u_m(t) - u_n(t)\|\big) + 1\right]\big(2\gamma_{m,n}(t)\big) \\
&+ \frac{2}{r}\big(\gamma_{m,n}(t)\big)^2.
\end{aligned}
$$

Hence, by adding both latter inequalities, we deduce that for ν-almost every $t \in I$,

$$
\begin{aligned}
&\left\langle \frac{du_n}{d\nu}(t) - \frac{du_m}{d\nu}(t), u_n(t) - u_m(t)\right\rangle \\
&\quad \le \frac{d\lambda}{d\nu}(t)\,\langle F_n(t) - F_m(t), u_m(t) - u_n(t)\rangle + 2\nu\big(\,]t, \theta_n(t)]\,\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\nu\big(\,]t,\theta_m(t)]\,\big) + \frac{1}{2r}\big(\,\|u_m(t) - u_n(t)\| + \gamma_n(t)\big)^2 \\
&+ \frac{1}{2r}\big(\,\|u_n(t) - u_m(t)\| + \gamma_m(t)\big)^2 + \frac{4}{r}\big(\gamma_{m,n}(t)\big)^2 \\
&+ \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(2\gamma_{m,n}(t)\big) \\
&+ \left[\frac{1}{r}\big(\gamma_m(t) + \|u_m(t) - u_n(t)\|\,\big) + 1\right]\big(2\gamma_{m,n}(t)\big). \qquad (70)
\end{aligned}
$$

Writing for all $t \in I$,

$$
\begin{aligned}
&\langle F_n(t) - F_m(t), u_m(t) - u_n(t)\rangle \\
&\quad = \big\langle f(t, u_n(t)) - f(t, u_m(t)), u_m(t) - u_n(t)\big\rangle \\
&\quad + \big\langle F_n(t) - f(t, u_n(t)), u_m(t) - u_n(t)\big\rangle \\
&\quad\quad + \big\langle f(t, u_m(t)) - F_m(t), u_m(t) - u_n(t)\big\rangle,
\end{aligned}
$$

we can apply assumption (i) with $B := (1+\kappa)\mathbb{B}$ (see (58)) to get

$$
\langle F_n(t) - F_m(t), u_m(t) - u_n(t)\rangle \le \varphi_{m,n}(t) + \varphi_{n,m}(t) + l_B(t)\,\|u_m(t) - u_n(t)\|^2 . \qquad (71)
$$

Combining (70) and (71), we obtain for ν-almost every $t \in I$, and for all integers $n, m \ge N$

$$
\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_n(t) - u_m(t)\right\rangle \\
&\quad \le \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\big(l_B(t)\,\|u_n(t) - u_m(t)\|^2 + \varphi_{n,m}(t) + \varphi_{m,n}(t)\big) \\
&\quad + 2\nu\big(\,]t,\theta_n(t)]\,\big) + 2\nu\big(\,]t,\theta_m(t)]\,\big) \\
&\quad\quad + \frac{1}{2r}\big(\,\|u_m(t) - u_n(t)\| + \gamma_n(t)\big)^2 \\
&\quad + \frac{1}{2r}\big(\,\|u_n(t) - u_m(t)\| + \gamma_m(t)\big)^2 + \frac{4}{r}\big(\gamma_{m,n}(t)\big)^2 \\
&\quad\quad + \left[\frac{1}{r}\big(\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right](2\gamma_{m,n}(t)) \\
&\quad\quad + \left[\frac{1}{r}\big(\gamma_m(t) + \|u_m(t) - u_n(t)\|\,\big) + 1\right](2\gamma_{m,n}(t)).
\end{aligned}
$$

Consequently, for ν-almost every $t \in I$, for all $n, m \ge N$ (with $\beta := 1 + \kappa$)

$$
\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_n(t) - u_m(t)\right\rangle \\
&\quad \le \left(l_B(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) + \frac{1}{r}\right)\|u_n(t) - u_m(t)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d\lambda}{d\nu}(t)\varphi_{n,m}(t)+\frac{d\lambda}{d\nu}(t)\varphi_{m,n}(t)+\frac{4}{r}\big(\gamma_{m,n}(t)\big)^2\\
&\quad+2\nu\big(\,]t,\theta_n(t)]\,\big)+2\nu\big(\,]t,\theta_m(t)]\,\big)\\
&+\frac{1}{2r}\big(\gamma_n^2(t)+4\beta\gamma_n(t)+\gamma_m^2(t)+4\beta\gamma_m(t)\big)\\
&\quad+\left[\frac{1}{r}\big(\gamma_n(t)+2\beta\big)+1\right](2\gamma_{m,n}(t))+\left[\frac{1}{r}\big(\gamma_m(t)+2\beta\big)+1\right](2\gamma_{m,n}(t)).
\end{aligned}
\tag{72}
$$

Now denote by N_ν the complement in I of the set of points $t \in I$ where $(d\lambda/d\nu)(t)$ exists in $\mathcal{H}$ as in (9), so $\nu(N_\nu) = 0$. Let us show that for every $t \in I \setminus N_\nu$

$$
\lim_{n\to+\infty}\Big(F_n(t)-f\big(t,u_n(t)\big)\Big)\frac{d\lambda}{d\nu}(t)=0. \tag{73}
$$

Fix any $t \in I \setminus N_\nu$ and consider two cases.

Case 1: $\nu(\{t\}) > 0$.

In this case, we know (see (10)) that $(d\lambda/d\nu)(t) = 0$, so the desired above limit property is evident.

Case 2: $\nu(\{t\}) = 0$.

Then, the inequality $\|u_n(t) - u_n(\delta_n(t))\| \le 2\nu(]\delta_n(t), t])$ by (63) and the convergence $\delta_n(t) \to t$ imply $u_n(\delta_n(t)) - u_n(t) \to 0$. It follows that $F_n(t) - f(t, u_n(t)) \to 0$ according to the uniform continuity of $f(t, \cdot)$ over B, which again confirms (73).

For each $t \in I \setminus N_\nu$, it results from (73) and from the definition of $\varphi_{n,m}$ in (69) that $(d\lambda/d\nu)(t)\varphi_{n,m}(t) \to 0$ as $n, m \to +\infty$, since the sequence $(u_n(t))_{n\in\mathbb{N}}$ is bounded by (58). By the assumption (ii) and the boundedness of $(u_n(\cdot))_n$ in the space $\mathcal{B}(I, \mathcal{H})$ (due to (58) again), the Lebesgue dominated convergence theorem ensures that

$$
\int_{]T_0,T]}\frac{d\lambda}{d\nu}(t)\varphi_{n,m}(t)\,d\nu(t)\to 0 \text{as} n,m\to+\infty.
$$

Note also that $\nu(]t, \theta_n(t)]) \to 0$ as $n \to +\infty$. For all $n, m \in \mathbb{N}$, setting

$$
\begin{aligned}
A_{n,m}:=\frac{1}{2}\int_{]T_0,T]}\Bigg\{&\frac{d\lambda}{d\nu}(t)\varphi_{n,m}(t)+\frac{d\lambda}{d\nu}(t)\varphi_{m,n}(t)\\
&+2\nu\big(\,]t,\theta_n(t)]\,\big)+2\nu\big(\,]t,\theta_m(t)]\,\big)\\
&+\frac{1}{2r}\big(\gamma_n^2(t)+4\beta\gamma_n(t)+\gamma_m^2(t)+4\beta\gamma_m(t)\big)+\frac{4}{r}\big(\gamma_{m,n}(t)\big)^2\\
&+\left[\frac{1}{r}\big(\gamma_n(t)+2\beta\big)+1\right](2\gamma_{m,n}(t))\\
&+\left[\frac{1}{r}\big(\gamma_m(t)+2\beta\big)+1\right](2\gamma_{m,n}(t))\Bigg\}d\nu(t)
\end{aligned}
$$

we see that $A_{n,m} \to 0$ as $n, m \to +\infty$. On the other hand, Proposition 3.1 says that

$$d(\|u_n(\cdot) - u_m(\cdot)\|^2) \le 2\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(\cdot), u_n(\cdot) - u_m(\cdot)\right\rangle \mathrm{d}\nu$$

$$\text{for all } n, m \in \mathbb{N}. \tag{74}$$

Fix for a moment $n, m \in \mathbb{N}$ with $n, m \ge N$. Putting for all $t \in I$, $\psi_{n,m}(t) = \|u_n(t) - u_m(t)\|^2$ and noting that $u_n(T_0) = u_m(T_0)$, we deduce from (72) that, for all $t \in I$

$$\psi_{n,m}(t) \le \int_{]T_0,t]} 2\Big(l_B(s)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) + \frac{1}{r}\Big)\psi_{n,m}(s)\,\mathrm{d}\nu(s) + A_{n,m}.$$

According to (11), we have $l_B(s)(\mathrm{d}\lambda/\mathrm{d}\nu)(s)\nu(\{s\}) = 0$ for ν-almost every $s \in I$. It follows that, for ν-almost every $t \in]T_0, T]$

$$2\Big(l_B(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) + \frac{1}{r}\Big)\nu(\{t\}) = \frac{2}{r}\nu(\{t\}) = \frac{2}{r}\mu(\{t\}) \le \frac{2}{r}\sup_{s\in]T_0,T]}\mu(\{s\}) < 1,$$

where the last inequality is due to the assumption $\sup_{s\in]T_0,T]}\mu(\{s\}) < r/2$. We can apply Lemma 3.1, and this yields, for all $t \in]T_0, T]$

$$\begin{aligned}\psi_{n,m}(t) &\le A_{n,m}\exp\left(\frac{1}{1-\theta}\int_{]T_0,t]} 2\left(l_B(s)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) + \frac{1}{r}\right)\mathrm{d}\nu(s)\right)\\ &\le A_{n,m}\exp\left(\frac{1}{1-\theta}\left(\int_{]T_0,T]} 2l_B(s)\,\mathrm{d}\lambda(s) + \frac{2}{r}\nu(]T_0,T])\right)\right)\end{aligned}$$

where $\theta = (2/r)\sup_{s\in]T_0,T]}\mu(\{s\})$. Hence, the sequence $(u_n(\cdot))_{n\ge N}$ satisfies the Cauchy property with respect to the norm of uniform convergence on the real Banach space of all bounded mappings from I into $\mathcal{H}$. Consequently, the sequence $(u_n(\cdot))_{n\ge N}$ converges uniformly on I to some mapping $u(\cdot)$. By virtue of (62), extracting a subsequence if necessary, we assume without loss of generality that $((\mathrm{d}u_n/\mathrm{d}\nu)(\cdot))_{n\ge N}$ converges weakly in $L^2(I,\mathcal{H},\nu)$ to some mapping $h(\cdot) \in L^2(I,\mathcal{H},\nu)$, so, for every $t \in I$,

$$\int_{]T_0,t]} \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(s)\,\mathrm{d}\nu(s) \underset{n\to+\infty}{\longrightarrow} \int_{]T_0,t]} h(s)\,\mathrm{d}\nu(s) \text{ weakly in } \mathcal{H}.$$

Since $(\mathrm{d}u_n/\mathrm{d}\nu)(\cdot)$ is a density of du_n relative to ν for all $n \in \mathbb{N}$, we have for all $n \in \mathbb{N}$, for all $t \in I$, $u_n(t) = u_0 + \int_{]T_0,t]}(\mathrm{d}u_n/\mathrm{d}\nu)(s)\,\mathrm{d}\nu(s)$. Thus, for all $t \in I$, $u(t) = u_0 + \int_{]T_0,t]} h(s)\,\mathrm{d}\nu(s)$ and this ensures that $u(\cdot)$ is right-continuous with bounded variation on I and the vector measure $\mathrm{d}u$ has $h(\cdot) \in L^2(I,\mathcal{H},\nu)$ as a density relative to ν and $(\mathrm{d}u/\mathrm{d}\nu)(\cdot) = h(\cdot)$ ν-almost everywhere. We also obtain that

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot) \underset{n\to+\infty}{\longrightarrow} \frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) \text{ weakly in } L^2(I,\mathcal{H},\nu).$$

Step 5. The mapping $u(\cdot)$ satisfies (44) and (45).

As in the proof of Theorem 4.1, we show that

$$u(t) \in C(t) \quad \text{for all } t \in I.$$

Now, let us establish that

$$\frac{du}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \nu\text{−a.e.} t \in I.$$

First, from (73) we notice that, for ν-almost every $t \in I$

$$e_n(t) := f\big(t, u_n(\delta_n(t))\big)\frac{d\lambda}{d\nu}(t) \underset{n\to+\infty}{\longrightarrow} f\big(t, u(t)\big)\frac{d\lambda}{d\nu}(t) =: e(t).$$

By this, assumption (ii) and the fact that $(u_n(\cdot))_n$ is uniformly bounded, the Lebesgue dominated convergence theorem yields that $(e_n(\cdot))_{n\geq N}$ converges strongly to $e(\cdot)$ in $L^2(I, \mathcal{H}, \nu)$. It ensues (recalling the definition of $\zeta_n(\cdot)$ in (59)) that

$$\zeta_n(\cdot) = \frac{du_n}{d\nu}(\cdot) + e_n(\cdot) \to \frac{du}{d\nu}(\cdot) + f(\cdot, u(\cdot))\frac{d\lambda}{d\nu}(\cdot) \text{ weakly in } L^2(I, \mathcal{H}, \nu).$$

Since $\zeta_n(t) \in \partial_C d_{C(\theta_n(t))}(u_n(\theta_n(t)))$ for ν-almost every $t \in I$ by (64), applying Castaing's technique as in the proof of Theorem 4.1, we arrive to

$$\frac{du}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) \in -N^C(C(t); u(t)) \quad \nu\text{−a.e.} t \in I,$$

that is, $u(\cdot)$ is a solution of (44). On the other hand, from (63) we see for $s < t$ in I that $\|u(t) - u(s)\| \leq \nu(]s, t])$, hence making $s \uparrow t$ gives

$$\|u(t) - u(t^-)\| \leq \nu(\{t\}) = \mu(\{t\}).$$

Then using the assumption $\sup_{\tau\in]T_0,T]} \mu(\{\tau\}) < r/2$, we arrive to the desired inequality (45). The proof is then complete. ■

The mapping $u(\cdot)$ satisfying (44) and (45) also satisfies Proposition 4.2.

Proposition 5.1: *Under the assumptions of Theorem* 5.1, *the solution* $u(\cdot) : I \to \mathcal{H}$ *of the sweeping process of the theorem satisfies the following properties*

$$\big\|u(t) - u(t^-)\big\| \leq \mu(\{t\}) \quad \text{and} \quad u(t) = \text{proj}_{C(t)}(u(t^-)) \quad \text{for all } t \in]T_0, T].$$

Proof: It is similar to the proof of Proposition 4.2. ■

Finally, under the convexity of sets $C(t)$ we directly derive the following corollary from Theorem 5.1 and Proposition 5.1. It is partially a slight extension of [14, Theorem 5.1].

Corollary 5.1: *Let $C : I \rightrightarrows \mathcal{H}$ be a closed convex valued multimapping and $u_0 \in C(T_0)$. Let also $f : I \times \mathcal{H} \to \mathcal{H}$ be a mapping with $f \not\equiv 0$ and μ be a positive Radon measure on I. Assume*:

(i) *the mapping $f(\cdot, x)$ is measurable for every $x \in \bigcup_{t\in I} C(t)$, and for each bounded subset B of $\mathcal{H}$ the mapping $f(t, \cdot)$ is uniformly continuous on B for every $t \in I$ and there exists a function $l_B \in L^1(I, \mathbb{R}_+, \lambda)$ such that*

$$\langle f(t,x_1) - f(t,x_2), x_1 - x_2 \rangle \geq -l_B(t) \|x_1 - x_2\|^2 \quad \text{for all } t \in I, x_1, x_2 \in B;$$

(ii) *there exists $\alpha(\cdot) \in L^1(I, \mathbb{R}_+, \lambda)$ with $1 - 2\int_{T_0}^{T} \alpha(s)\, d\lambda(s) > 0$ such that*

$$\|f(t,x)\| \leq \alpha(t)(1 + \|x\|) \quad \text{for all } t \in I, x \in \mathcal{H};$$

(iii) *there exist a real $\rho_0 > \|u_0\| + \mu(]T_0, T])$, an extended real $\rho \geq (\rho_0 + 2\int_{T_0}^{T} \alpha(s)\, d\lambda(s))/(1 - 2\int_{T_0}^{T} \alpha(s)\, d\lambda(s))$ and a real $\eta > 0$ such that*

$$\widehat{\mathrm{haus}}_\rho(C(s), C(t)) \leq \mu(]s,t]),$$

for all $s, t \in I$ with $s \leq t$ and $\mu(]s,t]) < \eta$.

Then, there exists one and only one mapping $u : I \to \mathcal{H}$ satisfying

$$\begin{cases} -du \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0. \end{cases}$$

Further, the solution $u(\cdot)$ satisfies the following properties

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \text{and} \quad u(t) = \mathrm{proj}_{C(t)}(u(t^-)) \quad \textit{for all } t \in]T_0, T].$$

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