Appl Math Optim
DOI 10.1007/s00245-017-9446-9

An Existence Result for Discontinuous Second-Order Nonconvex State-Dependent Sweeping Processes

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Abstract In this paper, we study the existence of solutions for a time and state-dependent discontinuous nonconvex second order sweeping process with a multivalued perturbation. The moving set is assumed to be prox-regular, relatively ball-compact with a bounded variation. The perturbation of the normal cone is a scalarly upper semi-continuous convex valued multimapping satisfying a linear growth condition possibly time-dependent. As an application of the theoretical results, we investigate the theory of evolution quasi-variational inequalities.

Keywords Variational analysis · Measure differential inclusions · Second-order sweeping process · Prox-regular sets · Bounded variation · Variational inequalities

Mathematics Subject Classification 34A60 · 49J52 · 49J53

1 Introduction

In 1988, Castaing [9] introduced in the famous "Travaux du Séminaire d'Analyse Convexe de Montpellier" the concept of second order sweeping process. Given a real $T > 0$, a Hilbert space $\mathcal{H}$ and a multimapping $C : [0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ closed-valued, consider the second order sweeping process written in the form

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Published online: 22 August 2017

$$\begin{cases} -\ddot{u}(t) \in N(C(t, u(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) & \lambda\text{-a.e. } t \in [0, T], \\ \dot{u}(t) \in C(t, u(t)) \quad \text{for all } t \in [0, T], \\ u(0) = b, \dot{u}(0) \in C(0, b), \end{cases} \tag{1.1}$$

where $N(\cdot, \cdot)$ is a general notion of normal cone in $\mathcal{H}$ and $F : [0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a multimapping. The term "sweeping process" refers to the first order evolution problem initiated by Moreau in 1971 [25] with $C(t, u(t)) = C(t)$ (state independent) namely

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) + F(t, u(t)) & \lambda\text{-a.e. } t \in [0, T], \\ u(t) \in C(t) \quad \text{for all } t \in [0, T], \\ u(0) \in C(0). \end{cases} \tag{1.2}$$

It has been well-recognized that (1.2) plays an important role in eslastoplasticity, quasistatics and dynamics (see, e.g., [24,28]) while the second order differential inclusion has applications in dry friction [29].

The unperturbed case of (1.1) (i.e., $F \equiv 0$) has been first developed by Castaing in [9] (see also the book [22]) for an anti-monotone continuous bounded convex moving set independent of the time $C(\cdot)$ of an infinite dimensional Hilbert space. Replacing the assumption of an anti-monotone multimapping with a fixed open ball contained in any $C(t)$ for each $t \in [0, T]$, Castaing et al. in [10] show in the finite dimensional setting the existence of solutions for (1.1) based on a technique of Lipschitz approximations of multimappings. To the best of our knowledge, the first study with $F \not\equiv 0$ is due to Duc Ha and Monteiro Marques [16]. The authors deal with a memory problem involving a compact convex Lipchitz moving set $C(\cdot)$ in $\mathbb{R}^n$ with a continuous compact-valued perturbation $F(\cdot, \cdot)$ of the normal cone which does not depend on the velocity $\dot{u}(t)$. Using the ideas of M. Valadier for the first order, they also investigate the case of the complement of a convex moving set.

It seems that prox-regular sets [31] are a suitable class of moving sets to handle first order nonconvex sweeping process in the infinite dimensional framework (see, e.g., [2,18] and the references therein). Such sets are also of great interest in the second order theory. This is the case in [6], where the authors developed an existence result for a compact prox-regular moving set of $\mathbb{R}^n$ with an upper semicontinuous convex-valued perturbation depending both on time and velocity. For a perturbation F depending on time, state and velocity, we refer to the work [5]. In [11], Castaing et al. studied the problem (1.1) in a separable Hilbert space when $C(t, x)$ is prox-regular and relatively ball-compact. The authors established in [11] the existence of a solution of (1.1) provided that the multimapping $C(\cdot, \cdot)$ is Lipschitz, that is, for some real $L > 0$

$$|d(u, C(t_1, x_1)) - d(v, C(t_2, x_2))| \leq \|u - v\| + L(|t_1 - t_2| + \|x_1 - x_2\|), \tag{1.3}$$

for any $u, v, x_1, x_2 \in \mathcal{H}$ and any $t_1, t_2 \in [0, T]$, and provided that the closed convex-valued multimapping F is scalarly upper semicontinuous and satisfies

$$F(t, u, v) \subset (1 + \|u\| + \|v\|)\mathbb{B} \quad \text{for all } t \in [0, T], (u, v) \in \mathcal{H}^2.$$

As pointed out by Tolstonogov [33], it is difficult for an unbounded moving set $C(\cdot,\cdot)$ to satisfy the inequality (1.3) because the Pompeiu-Hausdorff distance may take the value ∞ for unbounded sets. To avoid such a difficulty, he introduced the concept of multimappings uniformly lower semicontinuous from the right and provided an existence result for first order sweeping process without the classical assumption (1.3). In [1], the authors assume a control on the Pompeiu-Hausdorff distance, not on the whole moving set $C(\cdot,\cdot)$ but only on a bounded truncation of the form $C(\cdot,\cdot)\cap\rho\mathbb{B}$ for some real $\rho>0$ depending on the initial data. Thanks to a careful adaptation of the catching-up algorithm of J.J. Moreau for the first order sweeping process, they showed the existence of solution for the problem (1.1) with a convex moving set of a Hilbert space satisfying a slightly weaker compactness condition than the relative ball-compacity.

In [9], Castaing also considered the discontinuous order sweeping process

$$\begin{cases} -\frac{d\dot{u}}{|d\dot{u}|}(t)\in N(C(u(t));\dot{u}(t)) & |d\dot{u}|\text{-a.e. } t\in[0,T],\\ u(0)=b,\,\dot{u}(0)\in C(b). \end{cases} \tag{1.4}$$

Unlike the problem (1.1), very few studies have been achieved for such differential inclusions [9,10,21,22]. In [9] (see also the book [22]), the existence of solutions is ensured in infinite dimensional setting whenever the moving set $C(\cdot)$ is closed convex with nonempty interior, continuous for the Pompeiu-Hausdorff and anti-monotone. For another result without the latter assumption but in the finite dimensional setting, we refer the reader to [10]. Still in the finite dimensional framework, Kunze and Monteiro Marques in [21] proved that (1.4) has solutions whenever $C(\cdot)$ is convex with nonempty interior and Holder continuous with exponent $\alpha\in[\frac{1}{2},1]$.

The aim of the present paper is to analyze the existence of solutions for the differential inclusion

$$\begin{cases} -d\dot{u}(t)\in N(C(t,u(t));\dot{u}(t))+F(t,u(t),\dot{u}(t)),\\ u(0)=b,\,\dot{u}(0)\in C(0,b). \end{cases} \tag{1.5}$$

Following the work of Edmond and Thibault [18] for the first order sweeping process, we begin here with the developement of a general concept of solutions for such a differential inclusion which justifies the writing (1.5) without the variation measure $|d\dot{u}|$. Then, we give sufficient conditions to ensure that such a problem has at least one solution. Doing so, we assume that $C(\cdot,\cdot)$ is prox-regular, relatively ball-compact and moves in a bounded variation way with respect to the time and in a Lipschitz way with respect to the state, that is for any $u,x_1,x_2\in\mathcal{H}$ and $t_1,t_2\in[0,T]$ with $t_1<t_2$,

$$|d(u,C(t_1,x_1))-d(u,C(t_2,x_2))|\le\mu(]t_1,t_2])+L\,\|x_1-x_2\|\,,$$

where $L>0$ and where μ is a Radon measure on $[0,T]$. As in [1,11], the perturbation F is scalarly upper semicontinuous with closed convex values but with a weaker growth condition

$$F(t, u, v) \subset \alpha(t)(1 + \|u\| + \|v\|)\mathbb{B} \quad \text{for all } t \in [0, T], (u, v) \in \mathcal{H}^2,$$

with $\alpha(\cdot) \in L^1([0, T], \mathbb{R}_+, \lambda)$. Such a time dependance will play a crucial role in order to apply our existence result to the theory of quasi-variational evolution inequalities.

The paper is organized as follows. In Sect. 2, we introduce notation and we give the preliminaries needed throughout the paper. In the next section, we develop the concept of solutions for the discontinuous perturbed second order sweeping process (1.5). Various basic properties of the solutions of such a differential inclusion are also given. Section 4 is devoted to the study of the existence of solutions for (1.5). In the last section, we provide an application of our results to the theory of evolution quasi-variational inequalities.

2 Preliminaries

In the whole paper, $\mathbb{N}$ is the set of positive integers $(1, \ldots)$, $\mathbb{R}_+ := [0, +\infty[$ is the set of nonnegative reals, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real-line, T_0, T are two reals with $T_0 < T$ and λ is the Lebesgue measure on $I := [T_0, T]$. Throughout, $\mathcal{H}$ is a real Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, the associated norm by $\|\cdot\|$ and the closed unit ball by $\mathbb{B}$. For a set $A \subset I$, $\mathbf{1}_A$ stands for the *characteristic function* in the sense of measure theory of A relative to I, i.e., for all $x \in \mathbb{R}$,

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Let ν be a positive measure on I, $p \geq 1$ a real. We denote by $L^p(I, \mathcal{H}, \nu)$ the real space of (classes of) Bochner measurable mappings from I to $\mathcal{H}$ for which the p-th power of their norm value is integrable with respect to the measure ν.
Let S be a subset of $\mathcal{H}$. The *distance function* to S is denoted by $d_S(\cdot)$ or $d(\cdot, S)$ and is defined by

$$d_S(x) =: d(x, S) := \inf_{s \in S} \|x - s\| \quad \text{for all } x \in \mathcal{H}.$$

One denotes by co S (resp., $\overline{\text{co}}\, S$) the *convex* (resp., *closed convex*) *hull* of S. For any $x \in \mathcal{H}$, the possibly empty set of all *nearest points* of x in S is defined by

$$\text{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

When $\text{Proj}_S(x)$ contains one and only one point $\bar{y}$, we will denote by $\text{proj}_S(x)$ the unique element, that is, $\text{proj}_S(x) := \bar{y}$. The set S is said to be *ball-compact* (resp., *relatively ball-compact*) if the intersection of S with any closed ball of $\mathcal{H}$ is compact (resp., relatively compact). If S is nonempty and ball-compact, it is obviously closed and it satisfies

$$\text{Proj}_S(x) \neq \emptyset \quad \text{for all } x \in \mathcal{H}.$$

2.1 Proximal and Clarke Normal Cones and Subdifferentials

In this subsection, S is a nonempty closed subset of the real Hilbert space $\mathcal{H}$ and $f : U \to \overline{\mathbb{R}}$ is a function defined on a nonempty open subset U of $\mathcal{H}$.

For any $x \in S$, the set

$$N^P(S; x) := \left\{\zeta \in \mathcal{H} : \exists r > 0, x \in \mathrm{Proj}_S(x + r\zeta)\right\},$$

which is obviously a convex (not necessarily closed) cone containing 0 is called the *proximal normal cone* of S at x. By convention, for any $x \in \mathcal{H}\backslash S$, one sets $N^P(S; x) := \emptyset$. For $v \in \mathcal{H}$ such that $\mathrm{Proj}_S(v) \neq \emptyset$, it is readily seen that for all $w \in \mathrm{Proj}_S(v)$, one has

$$v - w \in N^P(S; w). \tag{2.1}$$

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal subgradient* of f at $x \in U$ with $|f(x)| < +\infty$ provided there are a real $\sigma \geq 0$ and a real $\eta > 0$ such that

$$\langle \zeta, y - x\rangle \leq f(y) - f(x) + \sigma \|y - x\|^2 \quad \text{for all } y \in B(x, \eta).$$

The set $\partial_P f(x)$ of all such proximal subgradients is called the *proximal subdifferential* of f at x. If f is not finite at $x \in U$, one sets $\partial_P f(x) := \emptyset$.

Before defining Clarke normal cone and Clarke subdifferential, we have to introduce the concept of sequential limit for multimappings. The *sequential limit superior* (or *sequential outer limit*) of a multimapping $M : X \rightrightarrows Y$ between two topological spaces X and Y relative to a subset $X_0 \subset X$ at $\overline{x} \in \mathrm{cl}\, X_0$ is defined as the set

$$^{\mathrm{seq}}\operatorname*{Lim\,sup}_{X_0 \ni x \to \overline{x}} M(x) := \{y \in Y : \exists X_0 \ni x_n \to \overline{x}, y_n \to y, y_n \in M(x_n)\ \forall n \in \mathbb{N}\}.$$

In other words, for any $y \in Y$, one has $y \in {}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{X_0 \ni x \to \overline{x}} M(x)$ if and only if there exist a sequence $(x_n)_{n\in\mathbb{N}}$ of X_0 converging to $\overline{x}$ and a sequence $(y_n)_{n\in\mathbb{N}}$ of Y converging to y with $y_n \in M(x_n)$ for all $n \in \mathbb{N}$.

With the above concept at hand, one defines the *Clarke normal cone* of S at x by

$$N^C(S; x) := \overline{\mathrm{co}}^w\big({}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{S \ni u \to x} N^P(S; u)\big).$$

Here and below, $\overline{\mathrm{co}}^w$ stands for the weakly closed convex hull. It is readily seen that this set is a closed convex cone containing 0 which satisfies $N^C(S; x) = \emptyset$ for any $x \in \mathcal{H} \setminus S$ and

$$N^P(S; x) \subset N^C(S; x) \quad \text{for all } x \in \mathcal{H}.$$

If S is a convex set, it is known (and not difficult to check) that

$$N^P(S;x) = N^C(S;x) = \{\zeta \in \mathcal{H} : \langle \zeta, a - x\rangle \le 0, \forall a \in S\} \quad \text{for all } x \in \mathcal{H}. \tag{2.2}$$

Similarly, if f is Lipschitz continuous near $x \in U$, one defines the *Clarke subdifferential* of f at $x \in U$ with $f(x) \in \mathbb{R}$ as the set

$$\partial_C f(x) := \overline{\mathrm{co}}^w\big({}^{\mathrm{seq}} \operatorname*{Lim\,sup}_{u \to x} \partial_P f(u)\big).$$

For any $x \in U$ with $|f(x)| = +\infty$, one can see that $\partial_C f(x) = \emptyset$, so the following inclusion holds true

$$\partial_P f(x) \subset \partial_C f(x) \quad \text{for all } x \in \mathcal{H}.$$

For any real $\gamma \ge 0$ such that f is γ-Lipschitz near $x \in U$, it is well-known that $\partial_C f(x) \subset \gamma\mathbb{B}$. It is worth pointing out that the following relations hold true for all $x \in S$:

$$\partial_P d_S(x) = N^P(S;x) \cap \mathbb{B} \quad \text{and} \quad \partial_C d_S(x) \subset N^C(S;x) \cap \mathbb{B}. \tag{2.3}$$

For more details on those concepts, we refer to the books [12,23,32].

2.2 Uniformly Prox-Regular Sets

The concept of uniformly prox-regular sets [31] in the Hilbert setting is fundamental in the paper. In this subsection, r is an extended real of $]0, +\infty]$ and $U_r(S)$ denotes the r-open enlargement of S, that is

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\}.$$

Note that we will use the classical convention $\frac{1}{r} := 0$ whenever $r = +\infty$.

We start with the definition of uniformly prox-regular sets.

Definition 2.1 Let S be a nonempty closed subset of $\mathcal{H}$. One says that S is r-prox-regular whenever, for all $x \in S$, for all $\zeta \in N^P(S;x)$ (or $N^C(S;x)$) with $\|\zeta\| \le 1$ and for all $t \in]0, r[$, one has $x \in \mathrm{Proj}_S(x + t\zeta)$.

The following theorem provides fundamental facts about prox-regular sets. For more details, we refer for instance to the survey of Colombo and Thibault [13].

Theorem 2.1 *Let S be a nonempty closed subset of $\mathcal{H}$. Consider the following assertions.*

(a) *The set S is r-prox-regular.*
(b) *For all $x_1, x_2 \in S$, for all $\zeta \in N^P(S;x_1)$ (or $N^C(S;x_1)$), one has*

$$\langle \zeta, x_2 - x_1\rangle \le \frac{1}{2r} \|\zeta\| \, \|x_1 - x_2\|^2 .$$

(c) *For all* $x_1, x_2 \in S$, *for all* $i \in \{1, 2\}$, *for all* $\zeta_i \in N^P(S; x_i)$ *(or* $N^C(S; x_i)$*), one has*

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq -\frac{1}{2}\left(\frac{\|\zeta_1\|}{r} + \frac{\|\zeta_2\|}{r}\right)\|x_1 - x_2\|^2 .$$

(d) *For any* $x \in S$*, one has*

$$N^P(S; x) = N^C(S; x) \quad \text{and} \quad \partial_P d_S(x) = \partial_C d_S(x).$$

(e) *For any* $x \in U_r(S)$, $\mathrm{Proj}_S(x)$ *is a singleton;*
(*f*) *The mapping* $\mathrm{P}_S : U_r(S) \to S$ *defined by*

$$\mathrm{P}_S(x) := \mathrm{proj}_S(x) \quad \text{for all } x \in U_r(S)$$

is locally Lipschitz on $U_r(S)$*.*

Then, the assertions (*a*), (*b*) *and* (*c*) *are pairwise equivalent, each one implies* (*d*), (*e*) *and* (*f*).

As in [2], according to (*d*) of Theorem 2.1, we put

$$N(S; x) := N^P(S; x) = N^C(S; x) \quad \text{for all } x \in S,$$

whenever S is a uniform prox-regular set of the real Hilbert space $\mathcal{H}$.

The next result deals with nearest points of prox-regular sets. We refer to [2] for the proof.

Proposition 2.1 *Let* S *be an* r*-prox-regular set of* $\mathcal{H}$ *and let* $x, x' \in \mathcal{H}$*. If* $x - x' \in N(S; x')$ *and* $\|x - x'\| \leq r$ *(resp.,* $\|x - x'\| < r$*) then* $x' \in \mathrm{Proj}_S(x)$ *(resp.,* $\{x'\} = \mathrm{Proj}_S(x)$*).*

2.3 Scalar Upper Semicontinuity

For any nonempty subset S of the real Hilbert space $\mathcal{H}$, its *support function* $\sigma(\cdot, S) : \mathcal{H} \to \overline{\mathbb{R}}$ is defined by

$$\sigma(\zeta, S) := \sup_{x \in S} \langle \zeta, x \rangle \quad \text{for all } \zeta \in \mathcal{H}.$$

A classical consequence of Hahn–Banach separation theorem is that for any two nonempty closed convex subsets S_1, S_2 of $\mathcal{H}$, one has

$$S_1 \subset S_2 \Leftrightarrow \sigma(\cdot, S_1) \leq \sigma(\cdot, S_2). \tag{2.4}$$

A multimapping $F : X \rightrightarrows \mathcal{H}$ from a topological space X to the real Hilbert space $\mathcal{H}$ is said to be *scalarly upper semicontinuous* whenever, for any $\xi \in \mathcal{H}$, the extended real-valued function $\sigma(\xi, F(\cdot)) : X \to \overline{\mathbb{R}}$ is upper semicontinuous.

Adaptating the proof of [2, Proposition 3.2], it is not difficult to show the following result.

Proposition 2.2 *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping satisfying:*

(i) *there exists an extended real $r \in]0, +\infty]$ such that for all $(t, x) \in I \times \mathcal{H}$, $C(t, x)$ is r-prox-regular;*
(ii) *there exist a positive measure μ on I and a real $L > 0$ such that for all $s_1, s_2 \in I$ with $s_1 \leq s_2$, for all $x_1, x_2 \in \mathcal{H}$, for all $u \in \mathcal{H}$,*

$$d(u, C(s_1, x_1)) - d(u, C(s_2, x_2)) \leq L \|x_1 - x_2\| + \mu(]s_1, s_2]).$$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of I converging to some $t \in I$ with $t_n \geq t$ for all $n \in \mathbb{N}$, $(v_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}$ converging to $v \in \mathcal{H}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}$ converging to some $x \in C(t, v)$ with $x_n \in C(t_n, v_n)$ for all $n \in \mathbb{N}$.

If there exists $N \in \mathbb{N}$ with $\mu(]t, t_N]) < +\infty$, then for any $z \in \mathcal{H}$, one has

$$\limsup_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n,v_n)}(x_n)) \leq \sigma(z, \partial_P d_{C(t,v)}(x)).$$

2.4 Vector Measures and BV Mappings

This section is devoted to recall some preliminaries about vector measure theory. For more details, one can see [15].

Let us start with mappings of bounded variation. Consider a mapping $u : I \to \mathcal{H}$. A *subdivision* σ of $I = [T_0, T]$ being a finite sequence $(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that $T_0 = t_0 < \ldots < t_k = T$. One associates with σ, the real $S_\sigma := \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$. One calls the *variation* of u on I, the extended real $V(u; I) := \sup_{\sigma\in\mathcal{S}} S_\sigma$, where $\mathcal{S}$ is the set of all subdivisions of I. The mapping u is said to be with *bounded variation* on I provided $V(u; I) < +\infty$. It is known (see, e.g., [15]) that a mapping u has one-sided limit whenever it is of bounded variation; in such a case one sets

$$u(\tau^-) := \lim_{t\uparrow\tau} u(t).$$

Before introducing the concept of differential measure, we give the following result which states that a sequence of uniformly bounded in norm and in variation mappings has a pointwise weakly convergent subsequence. We refer to the book of Monteiro Marques [22] for the proof.

Theorem 2.2 *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of mappings from I to $\mathcal{H}$. Assume that:*

(a) *there exists a real $M > 0$ such that*

$$\|g_n(t)\| \leq M \quad \textit{for all } n \in \mathbb{N},\ t \in I;$$

(b) *there exists a real* $L > 0$ *such that*

$$V(g_n; I) \leq L \quad \textit{for all } n \in \mathbb{N}.$$

Then, there exist a mapping $g : I \to \mathcal{H}$ *with bounded variation on* I *and a subsequence* $(g_{s(n)})_{n\in\mathbb{N}}$ *of* $(g_n)_{n\in\mathbb{N}}$ *such that*

$$g_{s(n)}(t) \overset{w}{\to} g(t) \quad \textit{for all } t \in I.$$

Let $u(\cdot) : I \to \mathcal{H}$ be a mapping of bounded variation on I. Then, there exists a vector measure du on I, called *differential measure* (or *Stieltjes measure*) with values in $\mathcal{H}$ associated with $u(\cdot)$ (see, e.g., [15]). If, in addition, $u(\cdot)$ is assumed to be right continuous on I, its differential measure du satisfies

$$u(t) = u(s) + \int_{]s,t]} du \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

Let ν be a positive Radon measure on I, $u(\cdot) : I \to \mathcal{H}$ a mapping and $\tilde{u}(\cdot) \in L^1(I, \mathcal{H}, \nu)$. If the following equality

$$u(t) = u(T_0) + \int_{]T_0,t]} \tilde{u}\, d\nu,$$

holds true for all $t \in I$, then the mapping $u(\cdot)$ is right continuous with bounded variation on I and

$$du = \tilde{u} d\nu.$$

In such a case, one says that $\tilde{u}(\cdot)$ is a *density of the measure* du *relative to* ν. Then, putting $I(t, r) := I \cap [t - r, t + r]$, $I^-(t, r) := [t - r, t] \cap I$ and $I^+(t, r) := [t, t + r] \cap I$ for each real $r > 0$, by Moreau and Valadier [30] for ν-almost every $t \in I$, the strong limits below exist in $\mathcal{H}$ and

$$\tilde{u}(t) = \frac{du}{d\nu}(t) := \lim_{r\downarrow 0} \frac{du(I(t,r))}{\nu(I(t,r))} = \lim_{r\downarrow 0} \frac{du(I^+(t,r))}{\nu(I^+(t,r))} = \lim_{r\downarrow 0} \frac{du(I^-(t,r))}{\nu(I^-(t,r))}. \tag{2.5}$$

Besides the differential measure, the concept of solutions above will involve the notion of derivative of a measure relative to another one. Let ν and $\hat{\nu}$ be two positive Radon measures on I. We recall that the limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r\downarrow 0} \frac{\hat{\nu}(I(t,r))}{\nu(I(t,r))} \tag{2.6}$$

(with the convention $\frac{0}{0} = 0$) exists for ν-almost every $t \in I$. The nonnegative (class of) function $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is called *derivative of the measure* $\hat{\nu}$ *with respect to* ν. It is worth pointing out that $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a Borel function. Further, the measure $\hat{\nu}$ is *absolutely continuous with* *respect to* ν if and only if $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a density relative to ν, i.e., if and only if the equality $\nu = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$ holds. Under such an absolute continuity assumption, a mapping $u(\cdot) : I \to \mathcal{H}$ is $\hat{\nu}$-integrable on I if and only if the mapping $u(\cdot)\frac{d\hat{\nu}}{d\nu}(\cdot)$ is ν-integrable on I. In such a case, one has

$$\int_I u(t)d\hat{\nu}(t) = \int_I u(t)\frac{d\hat{\nu}}{d\nu}(t)d\nu(t).$$

When the two Radon measures ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other one, we will say that they are *absolutely continuously equivalent.*

On the other hand, according to (2.6),

$$\frac{d\lambda}{d\nu}(t) = \frac{\lambda(\{t\})}{\nu(\{t\})} = 0 \quad \text{for all } t \in I \text{ with } \nu(\{t\}) > 0,$$

so for ν-almost every $t \in I$,

$$\frac{d\lambda}{d\nu}(t)\nu(\{t\}) = 0. \tag{2.7}$$

3 Concept of Solutions

Our concept of solutions follows the work [18] for a first order discontinuous perturbed sweeping process. For the absolutely continuous case in the second order framework, one can see [11].

Definition 3.1 Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a uniformly prox-regular valued multimapping and $F : I \times \mathcal{H} \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping. Assume that there exist a positive Radon measure μ on I and a real $L > 0$ such that for all $x_1, x_2 \in \mathcal{H}$, for all $u \in \mathcal{H}$, for all $t_1, t_2 \in I$,

$$|d(u, C(t_1, x_1)) - d(u, C(t_2, x_2))| \leq \mu(]t_1, t_2]) + L\,\|x_1 - x_2\|\,. \tag{3.1}$$

Given $a \in \mathcal{H}$ and $b \in C(T_0, a)$, a mapping $u : [T_0, T] \to \mathcal{H}$ is a solution of the following discontinuous second order sweeping process (associated to μ)

$$(\mathcal{P}) \begin{cases} -d\dot{u}(t) \in N(C(t, u(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b \end{cases}$$

whenever:

(a) u is absolutely continuous on $[T_0, T]$ and $u(T_0) = a$;
(b) there exists a mapping $v : [T_0, T] \to \mathcal{H}$ right continuous with bounded variation such that $v(T_0) = b$, $v(t) \in C(t, u(t))$ for all $t \in [T_0, T]$ and $v(t) = \dot{u}(t)$ λ-a.e.$t \in [T_0, T]$;

(c) there exists a positive Radon measure ν on I absolutely continuously equivalent to $\mu+\lambda$ with respect to which dv admits a density in $L^1(I, \mathcal{H}, \nu)$ and there exists a mapping $z : [T_0, T] \to \mathcal{H}$ λ-integrable on $[T_0, T]$ such that

$$z(t) \in F(t, u(t), v(t)) \quad \lambda\text{-a.e. } t \in I$$

and

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t, u(t)); v(t)) \quad \nu\text{-a.e. } t \in I.$$

Sometimes, it will be convenient for us to say that the mapping $v(\cdot)$ is a *derivative of* $u(\cdot)$ *for* $(\mathcal{P})$.

As in [18], the concept of solution does not depend on the Radon measure ν absolutely continuously equivalent to $\mu+\lambda$ given by (c). Indeed, let $u(\cdot) : I \to \mathcal{H}$ be a solution of $(\mathcal{P})$. Let v (resp., ν and z) given by (b) (resp., (c)) above. In particular, we have

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t, u(t)); v(t)) \quad \nu\text{-a.e. } t \in I. \tag{3.2}$$

Fix any other Radon measure $\hat{\nu}$ absolutely continuously equivalent to $\lambda+\mu$. Then, the measures ν and $\hat{\nu}$ are absolutely continuously equivalent. Consequently, $\frac{d\nu}{d\hat{\nu}}(\cdot)$ and $\frac{d\hat{\nu}}{d\nu}(\cdot)$ exist as densities and for $\frac{dv}{d\hat{\nu}}(\cdot)$ and the derivative $\frac{d\lambda}{d\hat{\nu}}(\cdot)$ the following equalities hold

$$\frac{dv}{d\hat{\nu}}(t) = \frac{dv}{d\nu}(t)\frac{d\nu}{d\hat{\nu}}(t), \quad \frac{d\lambda}{d\hat{\nu}}(t) = \frac{d\lambda}{d\nu}(t)\frac{d\nu}{d\hat{\nu}}(t) \quad \hat{\nu}\text{-a.e. } t \in I.$$

According to (3.2), this yields (keeping in mind that $N(\cdot;\cdot)$ is a cone)

$$\frac{dv}{d\hat{\nu}}(t) + z(t)\frac{d\lambda}{d\hat{\nu}}(t) \in -N(C(t, u(t)); v(t)) \quad \hat{\nu}\text{-a.e. } t \in I.$$

It is of great interest to consider the case where F and C are independent of the state.

Proposition 3.1 *Assume that there exist $\tilde{C} : I \rightrightarrows \mathcal{H}$ and $\tilde{F} : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ such that*

$$C(t, u) = \tilde{C}(t) \quad \textit{and} \quad F(t, u, v) = \tilde{F}(t, v),$$

for all $t \in I$, for all $(u, v) \in \mathcal{H}^2$. If $u(\cdot) : I \to \mathcal{H}$ is a solution of the sweeping process $(\mathcal{P})$ above, then there exists a mapping $v : I \to \mathcal{H}$ satisfying the first order discontinuous sweeping process associated to μ

$$\begin{cases} -dv(t) \in N(\tilde{C}(t); v(t)) + \tilde{F}(t, v(t)) \\ v(T_0) = b. \end{cases}$$

in the sense of [18]. *Moreover, one has*

$$u(t) = a + \int_{[T_0,T]} v(s)d\lambda(s) \quad \textit{for all } t \in I.$$

Proof According to (3.1), we have

$$\left| d_{\tilde{C}(t_1)}(y) - d_{\tilde{C}(t_2)}(y) \right| \le \mu(]t_1, t_2]),$$

for all $y \in \mathcal{H}$, for all $t_1, t_2 \in I$ with $t_1 < t_2$. Fix $u : I \to \mathcal{H}$ a solution of $(\mathcal{P})$. Let v, z and ν given by Definition 3.1. It is readily seen that

$$z(t) \in \tilde{F}(t, v(t)) \quad \lambda\text{-a.e. } t \in I$$

$$v(t) \in \tilde{C}(t) \quad \text{for all } t \in I$$

and

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(\tilde{C}(t); v(t)) \quad \nu\text{-a.e. } t \in I.$$

Since $v(T_0) = b$, the mapping v obviously satisfies

$$\begin{cases} -dv(t) \in N(\tilde{C}(t); v(t)) + \tilde{F}(t, v(t)) \\ v(T_0) = b. \end{cases}$$

On the other hand, the equality $v = \dot{u}$ λ-a.e. gives

$$u(t) = a + \int_{[T_0,t]} \dot{u}(s)d\lambda(s) = a + \int_{[T_0,t]} v(s)d\lambda(s).$$

The proof is then complete. $\square$

Now, following [2,34], we focus on the particular case where the measure μ is absolutely continuous relative to λ.

Proposition 3.2 *Let $C : I \rightrightarrows \mathcal{H}$ be a uniformly prox-regular valued multimapping and $F : I \times \mathcal{H} \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a multimapping. Assume that there exist a nondecreasing absolutely continuous function $\zeta : I \to \mathbb{R}$ and a real $L > 0$ such that for all $u, x, y \in \mathcal{H}$, for all $s, t \in I$ with $s \le t$,*

$$|d(u, C(t, x)) - d(u, C(s, y))| \le \zeta(t) - \zeta(s) + L \|x - y\| .$$

Let $a \in \mathcal{H}$, $b \in C(T_0, a)$ and μ be the Radon measure on I such that $\mu(]s, t]) = \zeta(t) - \zeta(s)$ for all $s, t \in I$ with $s < t$.

If a mapping $u : I \to \mathcal{H}$ is a solution of the discontinuous second order sweeping process (associated to μ)

$$(\mathcal{P}) \begin{cases} -d\dot{u}(t) \in N(C(t,u(t)); \dot{u}(t)) + F(t,u(t),\dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b, \end{cases}$$

then u is a solution of the classical second order sweeping process

$$\begin{cases} -\ddot{u}(t) \in N(C(t,u(t)); \dot{u}(t)) + F(t,u(t),\dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b \end{cases}$$

that is,

(a) *u is absolutely continuous on I and $u(T_0) = a$;*
(b) *there exists a mapping $v : I \to \mathcal{H}$ absolutely continuous on I such that $v(T_0) = b$, $v(t) \in C(t,u(t))$ for all $t \in I$ and $v(t) = \dot{u}(t)$ λ-a.e. $t \in I$;*
(c) *there exists a mapping $z : I \to \mathcal{H}$ λ-integrable on I such that*

$$z(t) \in F(t,u(t),v(t)) \quad \lambda\text{-a.e. } t \in I$$

and

$$\dot{v}(t) + z(t) \in -N(C(t,u(t)); v(t)) \quad \lambda\text{-a.e. } t \in I.$$

Proof Let $u : I \to \mathcal{H}$ be a mapping satisfying $(\mathcal{P})$. Set $\nu = \mu + \lambda$ which is a positive Radon measure on I, absolutely continuously equivalent to λ. Let $v : I \to \mathcal{H}$ be a mapping right continuous with bounded variation such that $v(T_0) = b$, $v(t) \in C(t,u(t))$ for all $t \in I$, $v(t) = \dot{u}(t)$ λ-a.e. $t \in I$ and dv has a density $g \in L^1(I,\mathcal{H},\nu)$ relative to ν. Denotes by $z : I \to \mathcal{H}$ a mapping λ-integrable on I such that

$$z(t) \in F(t,u(t),v(t)) \quad \lambda\text{-a.e. } t \in I$$

and

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t,u(t)); v(t)) \quad \nu\text{-a.e. } t \in I. \tag{3.3}$$

According to the equality

$$u(t) = a + \int_{]T_0,t]} \dot{u}(s)d\lambda(s) \quad \text{for all } t \in I,$$

we know that $\dot{u}(t) = \frac{du}{d\lambda}(t)$ λ-a.e. $t \in I$. On the other hand, thanks to the definition of g, we have

$$v(t) = b + \int_{]T_0,t]} g(s)d\nu(s) \quad \text{for all } t \in I$$

and $g(t) = \frac{d\upsilon}{d\nu}(t)$ ν-a.e. $t \in I$. Since $g(\cdot)$ is ν-integrable, $g(\cdot)\frac{d\nu}{d\lambda}(\cdot)$ is λ-integrable and

$$\int_{]T_0,t]} g(s)\frac{d\nu}{d\lambda}(s)d\lambda(s) = \int_{]T_0,t]} g(s)d\nu(s) \quad \text{for all } t \in I,$$

thus $\upsilon(\cdot)$ is absolutely continuous and $\dot{\upsilon}(t) = g(t)\frac{d\nu}{d\lambda}(t)$ λ-a.e. $t \in I$. By (3.3), we get

$$\frac{d\nu}{d\lambda}(t)\frac{d\upsilon}{d\nu}(t) + z(t)\frac{d\nu}{d\lambda}(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t,u(t)); \upsilon(t)) \quad \nu\text{-a.e. } t \in I.$$

It follows that (since λ and ν are absolutely continuously equivalent)

$$-\dot{\upsilon}(t) \in N(C(t,u(t)); \upsilon(t)) + z(t) \quad \nu\text{-a.e. } t \in I.$$

Thanks to the fact that λ is absolutely continuous with respect to ν, the latter inclusion entails that

$$-\dot{\upsilon}(t) \in N(C(t,u(t)); \upsilon(t)) + z(t) \quad \lambda\text{-a.e. } t \in I.$$

As a consequence, the mapping $u : I \to \mathcal{H}$ is a solution of the classical second order sweeping process. □

4 Existence Result

As mentioned above, the existence of solutions for problem $(\mathcal{P})$ in the absolute continuous framework, that is

$$\begin{cases} -\ddot{u}(t) \in N(C(t,u(t)); \dot{u}(t)) + F(t,u(t),\dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b \end{cases}$$

was investigated in [11] for a prox-regular moving set $C(\cdot,\cdot)$ of a separable Hilbert space, controlled in a Lipschitzian way, i.e., for some $L > 0$, the following inequality holds

$$|d(u, C(t_1,x_1)) - d(u, C(t_2,x_2))| \leq |t_1 - t_2| + L\,\|x_1 - x_2\|\,. \tag{4.1}$$

Such a differential inclusion is also considered in [1] for a convex moving set but with a finer control given by

$$\Gamma(t,x,s,y) \leq L(|t-s| + \|x-y\|),$$

for some $L > 0$ and a suitable multimapping Γ depending on the truncated Pompeiu-Hausdorff excess of the form (with $M > 0$)

$$\text{exc}(C(t,x) \cap M\mathbb{B}, C(s,y)) := \sup_{u \in C(t,x)\cap M\mathbb{B}} d(u, C(s,y)).$$

One of the novelty provided in the present paper is to allow the velocity $\dot{u}(t)$ of a trajectory $u(\cdot)$ of $(\mathcal{P})$ to jump since it is only required to be of bounded variation (see Sect. 3) and no longer (absolutely) continuous. At a discontinuity point t, we will see (under the prox-regularity of the moving set $C(\cdot,\cdot)$) that the behavior of $\dot{u}(t)$ is known via the formula

$$\dot{u}(t) = \mathrm{proj}_{C(t,u(t))}(\dot{u}(t^-)).$$

It is worth noting that such an equality can be seen as an extension of a previous result due to Moreau [27] (see also [2]). The classical Arzela-Ascoli theorem, which is a cornerstone of convergence arguments for [1,11], is then no longer applicable in our context and will be replaced by Theorem 2.2. As said above, our moving set will be prox-regular in any Hilbert space, controlled in a Lipschitz way in state but only in a bounded variation way in time, more precisely

$$|d(u, C(t_1, x_1)) - d(u, C(t_2, x_2))| \le \mu(]t_1, t_2]) + L\,\|x_1 - x_2\|\,, \tag{4.2}$$

for a given real $L > 0$ and a positive Radon measure on $I = [T_0, T]$. It is clear that (4.1) is a particular case of (4.2) with $\mu := \lambda$. Further, one of the interest of such a general assumption can be seen through Proposition 3.1, where we link the existence of solutions for $(\mathcal{P})$ to a *discontinuous* perturbed first order sweeping process which is known to play an important role in many applications of mathematics (see, e.g., [18,22] and the references therein).

Concerning the perturbation $F(\cdot,\cdot)$ both mentionned papers [1,11] assume the convexity of its values and a growth condition

$$F(t, u, v) \subset \alpha(1 + \|u\| + \|v\|)\mathbb{B},$$

for some $\alpha > 0$. Leading by the study of state-dependent evolution variational inequalities (see, Sect. 5) we keep the convexity assumption on values of $F(\cdot,\cdot)$ but we weak the latter inclusion through a mapping $\alpha(\cdot) \in L^1(I, \mathbb{R}_+, \lambda)$ and merely assume that

$$F(t, u, v) \subset \alpha(t)(1 + \|u\| + \|v\|)\mathbb{B}.$$

Besides the difference in convergence arguments with papers [1,11], an adaptation of their algorithm scheme is also necessary to take into account this new time dependence.

Let us state and prove the following existence result.

Theorem 4.1 *Let $C : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a r-prox-regular valued multimapping for some extended real $r \in]0, +\infty]$, $F : I \times \mathcal{H} \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a scalarly upper-semicontinuous multimapping with nonempty closed convex values. Assume that:*

(i) *there exist μ a positive Radon measure of I and a real $L > 0$ such that for all $u \in \mathcal{H}$, for all $(x_1, x_2) \in \mathcal{H}^2$, for all $t_1, t_2 \in I$ with $t_1 < t_2$,*

$$|d(u, C(t_1, x_1)) - d(u, C(t_2, x_2))| \le L\,\|x_1 - x_2\| + \mu(]t_1, t_2]); \tag{4.3}$$

(ii) *for each real* $\gamma > 0$, *the set* $C(I \times \gamma\mathbb{B})$ *is relatively ball-compact;*
(iii) *there exists a function* $\alpha : I \to \mathbb{R}_+$ *with* $\alpha \in L^1(I, \mathbb{R}, \lambda)$ *such that for all* $t \in I$, *for all* $u \in \mathcal{H}$, *for all* $v \in \bigcup\limits_{(\tau,x)\in I\times\mathcal{H}} C(\tau, x)$,

$$F(t, u, v) \subset \alpha(t)(1 + \|u\| + \|v\|)\mathbb{B}. \tag{4.4}$$

Let $a \in \mathcal{H}$ *and* $b \in C(T_0, a)$. *Then, there exists a mapping* $u : I \to \mathcal{H}$ *satisfying*

$$(\mathcal{P}) \begin{cases} -d\dot{u}(t) \in N(C(t, u(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b \end{cases}$$

with a derivative $v : I \to \mathcal{H}$ *for* $(\mathcal{P})$ *such that*

$$\left\| v(t) - v(t^-) \right\| \leq 2\mu(\{t\}) \quad \textit{for all } t \in]T_0, T].$$

If in addition $\sup\limits_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$, *then one has*

$$v(t) = \mathrm{proj}_{C(t,u(t))}(v(t^-)) \quad \textit{for all } t \in]T_0, T]. \tag{4.5}$$

Proof Set

$$\sigma := 1 - L(T - T_0) - 2A(T - T_0 + 1), \tag{4.6}$$

with $A := \int_{[T_0,T]} (\alpha(s) + 1) d\lambda(s)$.

Case 1 Assume that $\sigma > 0$.

Let us set

$$\delta := \frac{1}{\sigma}\big(\|b\| + \mu(]T_0, T]) + 2A(1 + \|a\|)\big) \tag{4.7}$$

and

$$\beta := 1 + \|a\| + \delta(1 + T - T_0). \tag{4.8}$$

Consider on I the positive Radon measure

$$\nu := \mu + (\delta L + \beta(\alpha(\cdot) + 1))\lambda. \tag{4.9}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $\varepsilon_n \downarrow 0$. For each $n \in \mathbb{N}$, choose, as in Moreau [27], $0 = M_0^n < M_1^n < \ldots < M_{q_n}^n = M := \nu(]T_0, T])$ (with $q_n \in \mathbb{N}$) such that

(a) $q_n < q_{n+1}$;
(b) for all $j \in \{0, \ldots, q_n - 1\}$, $M_{j+1}^n - M_j^n \leq \varepsilon_n$;
(c) $\left\{M_0^n, \ldots, M_{q_n}^n\right\} \subset \left\{M_0^{n+1}, \ldots, M_{q_{n+1}}^{n+1}\right\}$.

For every $n \in \mathbb{N}$, set $M_{1+q_n}^n := M + \varepsilon_n$. For each $n \in \mathbb{N}$, consider the partition of I associated with the subsets ($j \in \{0, \ldots, q_n\}$)

$$J_j^n := \{t \in [T_0, T] : M_j^n \leq \nu(]T_0, t]) < M_{j+1}^n\}$$

and note that $(J^m_j)_{0\le j\le q_m}$ is a refinement of $(J^n_j)_{0\le j\le q_n}$ for all $m, n \in \mathbb{N}$ with $m \ge n$. Since $\nu(]T_0, \cdot])$ is nondecreasing and right continuous on I, it is easy to see that, for each $n \in \mathbb{N}$, $j \in \{0, \dots, q_n - 1\}$, the set J^n_j is either empty or an interval of the form $[a, b[$ with $a < b$. Furthermore, we have $J^n_{q_n} = \{T\}$ for all $n \in \mathbb{N}$. This produces for each $n \in \mathbb{N}$, an integer $p(n) \in \mathbb{N}$ and a finite sequence

$$T_0 = t^n_0 < \dots < t^n_{p(n)} = T$$

such that for each $i \in \{0, \dots, p(n) - 1\}$, there is some $j_n(i) \in \{0, \dots, q_n - 1\}$ satisfying $J^n_{j_n(i)} = [t^n_i, t^n_{i+1}[$. Observe that $(p(n))_{n\in\mathbb{N}}$ is an increasing sequence. Fix for a moment any $n \in \mathbb{N}$. For all $i \in \{0, \dots, p(n) - 1\}$, put

$$\eta^n_i := t^n_{i+1} - t^n_i \quad \text{and} \quad \alpha^n_i := \int_{[t^n_i, t^n_{i+1}]} (\alpha(s) + 1)\, d\lambda(s).$$

Put also

$$\Delta_n := \max_{0\le i\le p(n)-1} (t^n_{i+1} - t^n_i).$$

For every $i \in \{0, \dots, p(n) - 1\}$ and every $t \in [t^n_i, t^n_{i+1}[$, one has

$$\nu(]t^n_i, t]) = \nu(]T_0, t]) - \nu(]T_0, t^n_i]) \le M^n_{j_n(i)+1} - M^n_{j_n(i)} \le \varepsilon_n.$$

Keeping in mind $\lambda \le \nu$, the inequality above gives

$$\eta^n_i = t^n_{i+1} - t^n_i \le \nu(]t^n_i, t^n_{i+1}[) \le \varepsilon_n \quad \text{for all } i \in \{0, \dots, p(n) - 1\}. \tag{4.10}$$

Then, one observes that

$$\lim_{k\to+\infty} \Delta_k = 0.$$

Now, if $p(n) = 1$, we choose $s^n_0 \in [T_0, T]$ such that

$$\alpha(s^n_0) \le \inf_{s\in[T_0,T]} \alpha(s) + 1$$

and if $p(n) > 1$, we choose for each $i \in \{0, \dots, p(n) - 2\}$, $s^n_i \in [t^n_i, t^n_{i+1}[$ satisfying

$$\alpha(s^n_i) \le \inf_{s\in[t^n_i, t^n_{i+1}[} \alpha(s) + 1,$$

and some $s^n_{p(n)-1} \in [t^n_{p(n)-1}, t^n_{p(n)}]$ such that

$$\alpha(s^n_{p(n)-1}) \le \inf_{s\in[t^n_{p(n)-1}, t^n_{p(n)}]} \alpha(s) + 1.$$

Define also $\kappa_n : I \to I$ by

$$\begin{cases} s_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ with } i \in \{0, \dots, p(n)-1\} \\ s_{p(n)-1}^n & \text{if } t = T. \end{cases}$$

Set $u_0^n := a \in C(T_0, b)$, $v_0^n := b$ and take $z_0^n \in F(\kappa_n(t_0^n), u_0^n, v_0^n) \neq \emptyset$. With

$$u_1^n := u_0^n + (t_1^n - t_0^n)v_0^n = u_0^n + \eta_0^n v_0^n$$

and according to the ball-compacity of $C(t_1^n, u_1^n)$ (thanks to (ii) and the fact that $C(\cdot, \cdot)$ is closed-valued), we can choose

$$v_1^n \in \mathrm{Proj}_{C(t_1^n, u_1^n)}(v_0^n - \eta_0^n z_0^n) \neq \emptyset.$$

By induction, we construct $(z_k)_{0 \le k \le p(n)-1}$, $(v_k)_{0 \le k \le p(n)}$ and $(u_k)_{0 \le k \le p(n)}$ satisfying

$$z_k^n \in F(\kappa_n(t_k^n), u_k^n, v_k^n) \quad \text{for all } k \in \{0, \dots, p(n)-1\}, \tag{4.11}$$

$$u_k^n = u_{k-1}^n + \eta_{k-1}^n v_{k-1}^n \quad \text{for all } k \in \{1, \dots, p(n)\} \tag{4.12}$$

and

$$v_k^n \in \mathrm{Proj}_{C(t_k^n, u_k^n)}(v_{k-1}^n - \eta_{k-1}^n z_{k-1}^n) \quad \text{for all } k \in \{1, \dots, p(n)\}. \tag{4.13}$$

Fix for a moment any $i \in \{1, \dots, p(n)\}$. Using (4.12), we get

$$\left\| u_i^n \right\| \le \left\| u_{i-1}^n \right\| + \eta_{i-1}^n \left\| v_{i-1}^n \right\|$$

and then it is not difficult to see that

$$\begin{aligned} \left\| u_i^n \right\| &\le \left\| u_0^n \right\| + \sum_{k=0}^{i-1} \eta_k^n \left\| v_k^n \right\| \\ &\le \left\| u_0^n \right\| + \max_{0 \le j \le p(n)-1} \left\| v_j^n \right\| \sum_{k=0}^{i-1} \eta_k^n \\ &\le \left\| u_0^n \right\| + (T - T_0) \max_{0 \le j \le p(n)-1} \left\| v_j^n \right\|. \end{aligned} \tag{4.14}$$

On the other hand, from (4.13), (4.3) and (4.12), we have

$$\begin{aligned} \left\| v_i^n - v_{i-1}^n + \eta_{i-1}^n z_{i-1}^n \right\| &= d_{C(t_i^n, u_i^n)}(v_{i-1}^n - \eta_{i-1}^n z_{i-1}^n) \\ &\le d_{C(t_i^n, u_i^n)}(v_{i-1}^n) + \left\| v_{i-1}^n - v_{i-1}^n + \eta_{i-1}^n z_{i-1}^n \right\| \\ &\le d_{C(t_i^n, u_i^n)}(v_{i-1}^n) - d_{C(t_{i-1}^n, u_{i-1}^n)}(v_{i-1}^n) + \eta_{i-1}^n \left\| z_{i-1}^n \right\| \\ &\le \mu(]t_{i-1}^n, t_i^n]) + L \left\| u_{i-1}^n - u_i^n \right\| + \eta_{i-1}^n \left\| z_{i-1}^n \right\| \\ &\le \mu(]t_{i-1}^n, t_i^n]) + L\eta_{i-1}^n \left\| v_{i-1}^n \right\| + \eta_{i-1}^n \left\| z_{i-1}^n \right\|. \end{aligned} \tag{4.15}$$

It follows that

$$\left\|v_i^n\right\| \le \left\|v_{i-1}^n\right\| + \mu(]t_{i-1}^n, t_i^n]) + L\eta_{i-1}^n \left\|v_{i-1}^n\right\| + 2\eta_{i-1}^n \left\|z_{i-1}^n\right\|$$

and then we deduce

$$\begin{aligned}\left\|v_i^n\right\| &\le \left\|v_0^n\right\| + \sum_{k=0}^{i-1}[\mu(]t_k^n, t_{k+1}^n]) + \eta_k^n(L\left\|v_k^n\right\| + 2\left\|z_k^n\right\|)] \\ &\le \left\|v_0^n\right\| + \mu(]T_0, T]) + L(T-T_0)\max_{0\le j\le p(n)-1}\left\|v_j^n\right\| + 2\sum_{k=0}^{i-1}\eta_k^n\left\|z_k^n\right\|.\end{aligned} \tag{4.16}$$

From (4.4), (4.11), the definition of κ_n, we obtain for all $k \in \{0, \ldots, p(n)-1\}$,

$$\begin{aligned}\eta_k^n\left\|z_k^n\right\| &= \int_{[t_k^n, t_{k+1}^n]}\left\|z_k^n\right\| d\lambda(s) \\ &\le \int_{[t_k^n, t_{k+1}^n]}\alpha(\kappa_n(t_k^n))(1+\left\|u_k^n\right\| + \left\|v_k^n\right\|)d\lambda(s) \\ &\le (1+\left\|u_k^n\right\| + \left\|v_k^n\right\|)\alpha_k^n \\ &\le (1+\max_{0\le j\le p(n)-1}\left\|u_j^n\right\| + \max_{0\le j\le p(n)-1}\left\|v_j^n\right\|)\alpha_k^n.\end{aligned} \tag{4.17}$$

Combining (4.16), (4.17) and (4.14), it results that for all $k \in \{0, \ldots, p(n)\}$,

$$\begin{aligned}\left\|v_k^n\right\| \le{}& \left\|v_0^n\right\| + \mu(]T_0, T]) + \max_{0\le j\le p(n)-1}\left\|v_j^n\right\|[L(T-T_0) + 2A(T-T_0+1)] \\ &+2A(1+\left\|u_0^n\right\|).\end{aligned}$$

According to the definition of σ in (4.6), the latter inequality entails

$$\sigma\max_{0\le j\le p(n)}\left\|v_j^n\right\| \le \left\|v_0^n\right\| + \mu(]T_0, T]) + 2A(1+\left\|u_0^n\right\|).$$

From the definition of δ in (4.7) and the equalities $u_0^n = a$ and $v_0^n = b$, we get

$$\max_{0\le j\le p(n)}\left\|v_j^n\right\| \le \delta. \tag{4.18}$$

Keeping in mind (4.14), the latter inequality gives

$$\max_{0\le j\le p(n)}\left\|u_j^n\right\| \le \|a\| + (T-T_0)\delta. \tag{4.19}$$

Thanks to (4.17), (4.18), (4.19) and (4.8), we have

$$\eta_k^n \left\| z_k^n \right\| \leq (1 + \|a\| + \delta(1 + T - T_0))\alpha_k^n = \beta\alpha_k^n \tag{4.20}$$

for all $k \in \{0, \ldots, p(n) - 1\}$. Coming back to (4.15) and using (4.18), (4.20) and the definition of ν in (4.9), we have for all $k \in \{1, \ldots, p(n)\}$,

$$\begin{aligned} \left\| v_k^n - v_{k-1}^n + \eta_{k-1}^n z_{k-1}^n \right\| &\leq \mu(]t_{k-1}^n, t_k^n]) + L\eta_{k-1}^n \left\| v_{k-1}^n \right\| + \eta_{k-1}^n \left\| z_{k-1}^n \right\| \\ &\leq \mu(]t_{k-1}^n, t_k^n]) + \eta_{k-1}^n \delta L + \beta\alpha_{k-1}^n \\ &\leq \nu(]t_{k-1}^n, t_k^n]). \end{aligned} \tag{4.21}$$

On the other hand, using (4.11), the assumption (iii), (4.19) and (4.18), we obtain

$$\begin{aligned} z_k^n \in F(\kappa_n(t_k^n), u_k^n, v_k^n) &\subset \alpha(\kappa_n(t_k^n))(1 + \left\| u_k^n \right\| + \left\| v_k^n \right\|)\mathbb{B} \\ &\subset \alpha(\kappa_n(t_k^n))(1 + \|a\| + (1 + T - T_0)\delta)\mathbb{B} \\ &= \alpha(\kappa_n(t_k^n))\beta\mathbb{B}, \end{aligned} \tag{4.22}$$

for all $k \in \{0, \ldots, p(n) - 1\}$.

Step 1 Construction of the sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$.

Fix any $n \in \mathbb{N}$. Define the mappings $u_n, v_n : I \to \mathcal{H}$ by putting, for all $i \in \{0, \ldots, p(n) - 1\}$, for all $t \in [t_i^n, t_{i+1}^n]$,

$$u_n(t) = u_i^n + (t - t_i^n)v_i^n$$

and

$$v_n(t) = v_i^n + \frac{\nu(]t_i^n, t])}{\nu(]t_i^n, t_{i+1}^n])}(v_{i+1}^n - v_i^n + \eta_i^n z_i^n) - (t - t_i^n)z_i^n.$$

Note that u_n, $v_n(\cdot)$ are well-defined on I and that $v_n(\cdot)$ is right continuous with bounded variation on I. By definition of $v_n(\cdot)$, we have for all $t \in I$,

$$v_n(t) = v_0^n + \int_{]T_0,t]} \Pi_n(s)d\nu(s) - \int_{]T_0,t]} z_n(s)d\lambda(s) \tag{4.23}$$

where for all $t \in I$,

$$\Pi_n(t) = \sum_{i=0}^{p(n)-1} \frac{v_{i+1}^n - v_i^n + \eta_i^n z_i^n}{\nu(]t_i^n, t_{i+1}^n])}\mathbf{1}_{]t_i^n, t_{i+1}^n]}(t)$$

and

$$z_n(t) = \begin{cases} z_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ for some } i \in \{0, \ldots, p(n) - 1\} \\ z_{p(n)-1}^n & \text{if } t = T. \end{cases}$$

Since the measure λ is absolutely continuous with respect to ν, it has $\frac{d\lambda}{d\nu}(\cdot)$ as a density in $L^\infty(I, [0, +\infty[, \nu)$ relative to ν and then by (4.23), for all $t \in I$,

$$v_n(t) = v_0^n + \int_{]T_0,t]} \big(\Pi_n(s) - z_n(s)\frac{d\lambda}{d\nu}(s)\big) d\nu(s).$$

The latter equality says that dv_n has $\Pi_n(\cdot) - z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot)$ as a density in $L^1(I, \mathcal{H}, \nu)$ relative to ν. So, the derivative $\frac{dv_n}{d\nu}(\cdot)$ is a density of dv_n relative to ν and

$$\frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) = \Pi_n(t) \quad \text{for } \nu\text{-a.e. } t \in I. \tag{4.24}$$

Furthermore, thanks to (4.24), the definition of $\Pi_n(\cdot)$ and (4.21), we have

$$\left\| \frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) \right\| \leq 1 \quad \text{for } \nu\text{-a.e. } t \in I. \tag{4.25}$$

Since the measure $\beta(1 + \alpha(\cdot))\lambda$ is absolutely continuous with respect to ν, it has $\frac{d(\beta(1+\alpha(\cdot))\lambda)}{d\nu}$ as a density and this yields that

$$0 \leq \beta(1 + \alpha(t))\frac{d\lambda}{d\nu}(t) = \frac{d(\beta(1 + \alpha(\cdot))\lambda)}{d\nu}(t) \leq 1. \tag{4.26}$$

From (4.22), the definitions of κ_n and of the sequence $(s_i^n)_{0\leq i\leq p(n)-1}$, it is not difficult to check that for all $t \in I$,

$$\|z_n(t)\| \leq \beta(1 + \alpha(t)). \tag{4.27}$$

Combining (4.25), (4.27) and (4.26), we get for ν-almost every $t \in I$,

$$\left\| \frac{dv_n}{d\nu}(t) \right\| \leq 1 + \left\| z_n(t)\frac{d\lambda}{d\nu}(t) \right\| \leq 1 + \beta(1 + \alpha(t))\frac{d\lambda}{d\nu}(t) \leq 2. \tag{4.28}$$

From (4.28) and the fact that $\frac{dv_n}{d\nu}$ is a density of dv_n relative to ν, we deduce

$$\|v_n(\tau_2) - v_n(\tau_1)\| \leq 2\nu(]\tau_1, \tau_2]), \tag{4.29}$$

for all $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$. It follows that for all $t \in I$,

$$\|v_n(t)\| \leq 2\nu(]T_0, t]) + \|v_n(T_0)\| \leq 2\nu(I) + \|b\|. \tag{4.30}$$

According to (2.1) and (4.13), we have for all $i \in \{0, \ldots, p(n) - 1\}$,

$$\frac{v_{i+1}^n - v_i^n + \eta_i^n z_i^n}{\nu(]t_i^n, t_{i+1}^n])} \in -N^P(C(t_{i+1}^n, u_{i+1}^n); v_{i+1}^n). \tag{4.31}$$

Define the function $\theta_n : I \to I$ where for all $t \in I$,

$$\theta_n(t) = \begin{cases} t_{i+1}^n & \text{if } t \in]t_i^n, t_{i+1}^n] \text{ for some } i \in \{0, \dots, p(n) - 1\} \\ t_1^n & \text{if } t = T_0. \end{cases}$$

By construction, it is obvious that

$$v_n(\theta_n(t)) \in C(\theta_n(t), u_n(\theta_n(t))) \quad \text{for all } t \in I. \tag{4.32}$$

Using (4.31), (4.24) and the definitions of $\theta_n(\cdot)$ and $\Pi_n(\cdot)$, we obtain

$$\frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) \in -N^P(C(\theta_n(t), u_n(\theta_n(t))); v_n(\theta_n(t))) \quad \text{for } \nu\text{-a.e. } t \in I.$$

By (2.3), we know that (4.25) and the latter inclusion entail that

$$\frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) \in -\partial_P d_{C(\theta_n(t), u_n(\theta_n(t)))}(v_n(\theta_n(t))) \quad \text{for } \nu\text{-a.e. } t \in I. \tag{4.33}$$

With $\delta_n : I \to I$ the function defined for all $t \in I$ by

$$\delta_n(t) = \begin{cases} t_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[\text{ for some } i \in \{0, \dots, p(n) - 1\} \\ t_{p(n)-1}^n & \text{if } t = T, \end{cases}$$

it is not difficult to check that

$$u_n(t) = a + \int_{[T_0, t]} v_n(\delta_n(s)) d\lambda(s) \quad \text{for all } t \in I \tag{4.34}$$

and

$$z_n(t) \in F\big(\kappa_n(\delta_n(t)), u_n(\delta_n(t)), v_n(\delta_n(t))\big) \quad \text{for all } t \in I. \tag{4.35}$$

Step 2 Convergence of $(u_n(\cdot))_{n\in\mathbb{N}}$ and $(v_n(\cdot))_{n\in\mathbb{N}}$ up to a subsequence.
For each $n \in \mathbb{N}$, set $g_n(\cdot) = v_n(\theta_n(\cdot))$ and note that [thanks to (4.21) and (4.20)]

$$\begin{aligned} V(g_n; I) &= \sum_{i=0}^{p(n)-1} \left\| v_{i+1}^n - v_i^n \right\| \\ &\le \sum_{i=0}^{p(n)-1} (\nu(]t_i^n, t_{i+1}^n]) + \eta_i^n \left\| z_i^n \right\|) \\ &\le \nu(I) + \beta \sum_{i=0}^{p(n)-1} \alpha_i^n \\ &= \nu(I) + \beta A. \end{aligned} \tag{4.36}$$

According to (4.30) and (4.36), we can apply Theorem 2.2. Doing so, we assume without loss of generality that there is a mapping $v : I \to \mathcal{H}$ with bounded variation on I such that

$$g_n(t) \xrightarrow{w} v(t) \quad \text{for all } t \in I. \tag{4.37}$$

Put $\rho = 2\nu(I) + \|b\|$ and observe by (4.32) and (4.30) that

$$\begin{aligned} g_n(t) &\in C\big(\theta_n(t), u_n(\theta_n(t))\big) \cap \rho\mathbb{B} \\ &\subset \left(\bigcup_{k\in\mathbb{N}} C\big(\theta_k(t), u_k(\theta_k(t))\big) \right) \cap \rho\mathbb{B} =: D(t), \end{aligned}$$

for all $n \in \mathbb{N}$, for all $t \in I$. From the definitions of u_n, θ_n $(n \in \mathbb{N})$ and (4.19), it is straightforward to check that

$$\|u_n(\theta_n(t))\| \leq \|a\| + (T - T_0)\delta,$$

for all $n \in \mathbb{N}$, all $t \in I$. On the other hand, from (4.10), we get

$$\theta_n(t) \to t \quad \text{for all } t \in I. \tag{4.38}$$

Note that the assumption (ii) says that $\bigcup_{k\in\mathbb{N}} C(\tau_k, x_k)$ is relatively ball-compact for each bounded sequence $(\tau_k, x_k)_{k\in\mathbb{N}}$ of $[T_0, T] \times \mathcal{H}$. Thus, we can apply the assumption (ii) which entails that the set $D(t)$ is relatively compact for each $t \in I$. Hence, the weakly pointwise convergence in (4.37) holds with respect to the strong topology on $\mathcal{H}$, i.e.,

$$g_n(t) \to v(t) \quad \text{for all } t \in [T_0, T].$$

Thanks to (4.29) and (4.10), we have

$$\|v_n(t) - g_n(t)\| \leq 2\nu(]t, \theta_n(t)]) \quad \text{and} \quad \|v_n(t) - v_n(\delta_n(t))\| \leq 2\nu(]\delta_n(t), t]) \leq 2\varepsilon_n,$$

for all $t \in I$, for all $n \in \mathbb{N}$, so it follows (keeping in mind (4.38)) that

$$v_n(t) \to v(t) \quad \text{and} \quad v_n(\delta_n(t)) \to v(t) \quad \text{for all } t \in I.$$

Combining the convergence above with (4.29), we get

$$\|v(\tau_1) - v(\tau_2)\| \leq 2\nu(]\tau_1, \tau_2]), \tag{4.39}$$

for all $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$. Coming back to (4.34) and using Lebesgue dominated convergence theorem, $v \in L^1(I, \mathcal{H}, \lambda)$ and

$$u_n(t) = a + \int_{[T_0,t]} v_n(\delta_n(s))d\lambda(s) \to a + \int_{[T_0,t]} v(s)d\lambda(s) =: u(t) \quad \text{for all } t \in I.$$

As a consequence, u is absolutely continuous on I and

$$\dot{u}(t) = v(t) \quad \lambda\text{-a.e. } t \in I.$$

On the other hand, according to (4.28), extracting a subsequence if necessary, we may suppose that $(\frac{dv_n}{d\nu}(\cdot))_{n\in\mathbb{N}}$ converges weakly in $L^2(I, \mathcal{H}, \nu)$ to some mapping $\varphi(\cdot) \in L^2(I, \mathcal{H}, \nu)$. So, for any $t \in I$,

$$\int_{]T_0,t]} \frac{dv_n}{d\nu}(s)d\nu(s) \to \int_{]T_0,t]} \varphi(s)d\nu(s) \quad \text{weakly in } \mathcal{H}.$$

As $v_n(t) \to v(t)$ for all $t \in I$, it results that

$$v(t) = b + \int_{]T_0,t]} \varphi(s)d\nu(s) \quad \text{for all } t \in I,$$

hence $v(\cdot)$ is right continuous with bounded variation on I and dv has $\varphi(\cdot) \in L^2(I, \mathcal{H}, \nu)$ as a density relative to ν. As a result, $\varphi(\cdot) = \frac{dv}{d\nu}(\cdot)$ ν-a.e. and

$$\frac{dv_n}{d\nu}(\cdot) \to \frac{dv}{d\nu}(\cdot) \quad \text{weakly in } L^2(I, \mathcal{H}, \nu),$$

and this yields

$$\frac{dv_n}{d\nu}(\cdot) \to \frac{dv}{d\nu}(\cdot) \quad \text{weakly in } L^1(I, \mathcal{H}, \nu).$$

Thanks to the relation (4.39), we have

$$\left\| v(t) - v(t^-) \right\| \le 2\nu(\{t\}) = 2\mu(\{t\}) \quad \text{for all } t \in I. \tag{4.40}$$

Step 3 Let us prove that $u(\cdot)$ is a solution of $(\mathcal{P})$.

According to the definition of $u_n(\cdot)$ and the inequality (4.18), it is readily seen that

$$\|\dot{u}_n(t)\| \le \delta \quad \lambda\text{-a.e. } t \in I.$$

Since $\theta_n(t) \to t$, the latter inequality entails that

$$u_n(\theta_n(t)) - u_n(t) \to 0,$$

so $u_n(\theta_n(t)) \to u(t)$. Fix for a moment any $t \in I$, any $n \in \mathbb{N}$. Using (4.32) and the variation assumption on $C(\cdot, \cdot)$ in (4.3), we get

$$\begin{aligned} d_{C(t,u(t))}(v_n(\theta_n(t))) &= \left| d_{C(t,u(t))}(v_n(\theta_n(t))) - d_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t))) \right| \\ &\le \mu(]t, \theta_n(t)]) + L \, \|u(t) - u_n(\theta_n(t))\| \end{aligned}$$

and this entails that

$$\lim_{k\to+\infty} d_{C(t,u(t))}(v_k(\theta_k(t))) = d_{C(t,u(t))}(v(t)) = 0.$$

Thanks to the closedness of $C(t,u(t))$, we get

$$v(t) \in C(t,u(t)). \tag{4.41}$$

According to (4.27), we may suppose that $(z_n(\cdot))_{n\in\mathbb{N}}$ converges weakly in $L^1(I,\mathcal{H},\lambda)$ to some mapping $z(\cdot) \in L^1(I,\mathcal{H},\lambda)$. Since $\frac{d\lambda}{d\nu}(\cdot) \in L^\infty(I,\mathbb{R},\nu)$, we have

$$z_n(\cdot)\frac{d\lambda}{d\nu}(\cdot) \to z(\cdot)\frac{d\lambda}{d\nu}(\cdot) \quad \text{weakly in } L^1(I,\mathcal{H},\nu).$$

Now, we follow an usual technique due to C. Castaing ([8]). Applying Mazur's lemma, there exists a sequence $(\zeta_n(\cdot))_{n\in\mathbb{N}}$ which converges strongly in $L^1(I,\mathcal{H},\nu)$ to $\frac{dv}{d\nu}(\cdot) + z(\cdot)\frac{d\lambda}{d\nu}(\cdot)$ with

$$\zeta_n(\cdot) \in \operatorname{co}\left\{\frac{dv_k}{d\nu}(\cdot) + z_k(\cdot)\frac{d\lambda}{d\nu}(\cdot) : k \geq n\right\} \quad \text{for all } n \in \mathbb{N}.$$

Extracting a subsequence if necessary, we may suppose that

$$\zeta_n(t) \to \frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \quad \nu\text{-a.e. } t \in I.$$

Then, we have

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{co}}\left\{\frac{dv_k}{d\nu}(t) + z_k(t)\frac{d\lambda}{d\nu}(t) : k \geq n\right\},$$

for ν-almost every $t \in I$. This inclusion yields for ν-almost every $t \in I$, for all $\xi \in \mathcal{H}$,

$$\left\langle \xi, \frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle \leq \inf_{n\in\mathbb{N}}\sup_{k\geq n}\left\langle \xi, \frac{dv_k}{d\nu}(t) + z_k(t)\frac{d\lambda}{d\nu}(t)\right\rangle.$$

Coming back to (4.33), it follows that, for ν-almost every $t \in I$, for all $\xi \in \mathcal{H}$,

$$\left\langle \xi, \frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle \leq \limsup_{n\to+\infty} \sigma(\xi, -\partial_P d_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t)))).$$

Hence, for ν-almost every $t \in I$, for all $\xi \in \mathcal{H}$, according to Proposition 2.2 and (4.41)

$$\left\langle \xi, \frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\rangle \leq \sigma(\xi, -\partial_P d_{C(t,u(t))}(v(t))).$$

From Theorem 2.1(d), we have

$$\partial_P d_{C(t,u(t))}(v(t)) = \partial_C d_{C(t,u(t))}(v(t)) \quad \text{for all } t \in I.$$

In particular, for each $t \in I$, the proximal subdifferential $\partial_P d_{C(t,u(t))}(v(t))$ is here closed and convex. Then, the latter inequality and (2.4) give that

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -\partial_P d_{C(t,u(t))}(v(t)) \quad \nu\text{-a.e. } t \in I,$$

or equivalently [see (2.3)]

$$\frac{dv}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N^P(C(t,u(t)); v(t)) = -N(C(t,u(t)); v(t)) \quad \nu\text{-a.e. } t \in I.$$

Let us show that $z(t) \in F(t, u(t), v(t))$ for λ-almost every $t \in I$. Thanks to the fact that $(z_k(\cdot))_{k\in\mathbb{N}}$ converges to $z(\cdot)$ weakly in $L^1(I, \mathcal{H}, \lambda)$, via Mazur's lemma again, extracting a subsequence if necessary, we may write

$$z(t) \in \bigcap_{n\in\mathbb{N}} \overline{\text{co}}\,\{z_k(t) : k \geq n\} \quad \lambda\text{-a.e. } t \in I.$$

Combining this inclusion with (4.35), we get for λ-almost every $t \in I$, for all $\xi \in \mathcal{H}$,

$$\langle \xi, z(t)\rangle \leq \limsup_{n\to+\infty} \sigma\Big(\xi, F\big(\kappa_n(\delta_n(t)), u_n(\delta_n(t)), v_n(\delta_n(t))\big)\Big).$$

Using the fact that F is scalarly upper-semicontinuous and the convergence $\kappa_n(\delta_n(t)) \to t$ for all $t \in I$, we get for λ-almost every $t \in I$, for all $\xi \in \mathcal{H}$,

$$\langle \xi, z(t)\rangle \leq \sigma(\xi, F(t, u(t), v(t))).$$

Since $F(t, u(t), v(t))$ is closed and convex for all $t \in I$, we have (thanks to (2.4))

$$z(t) \in F(t, u(t), v(t)) \quad \lambda\text{-a.e. } t \in I.$$

As $u(T_0) = a$ and $v(T_0) = b$, $u(\cdot)$ is a solution of $(\mathcal{P})$ with derivative v for $(\mathcal{P})$ satisfying (4.40).

Case 2 Assume that $\sigma \leq 0$.

There are $p \in \mathbb{N}$, $T_1, \ldots, T_p \in \mathbb{R}$ such that

$$T_0 < T_1 < \ldots < T_p = T$$

and

$$1 - L(T_{i+1} - T_i) - 2(T_{i+1} - T_i + 1)\int_{[T_i, T_{i+1}]} (\alpha(s) + 1)\,d\lambda(s) > 0$$
$$\text{for all } i \in \{0, \ldots, p-1\}.$$

For each $i \in \{0, \ldots, p-1\}$, let us denote by μ_i (resp., λ_i) the Radon measure induced on $[T_i, T_{i+1}]$ by μ (resp., λ) and set $\nu_i := \mu_i + \lambda_i$. From the case 1, we get an absolutely continuous mapping $u_1 : [T_0, T_1] \to \mathcal{H}$, a right continuous mapping with bounded variation $v_1 : [T_0, T_1] \to \mathcal{H}$ and a mapping $z_1 : [T_0, T_1] \to \mathcal{H}$ λ_1-integrable on $[T_0, T_1]$ such that

$$u_1(T_0) = a \quad \text{and} \quad v_1(T_0) = b,$$

$$v_1(t) \in C(t, u_1(t)) \quad \text{for all } t \in [T_0, T_1],$$

$$\left\| v_1(t) - v_1(t^-) \right\| \leq 2\mu_1(\{t\}) \quad \text{for all } t \in]T_0, T_1],$$

$$v_1(t) = \dot{u}_1(t) \quad \text{and} \quad z_1(t) \in F(t, u_1(t), v_1(t)) \quad \lambda_1\text{-a.e. } t \in [T_0, T_1],$$

dv_1 has $\frac{dv_1}{d\nu_1}$ in $L^1([T_0, T_1], \mathcal{H}, \nu_1)$ as a density relative to ν_1 and

$$\frac{dv_1}{d\nu_1}(t) + z_1(t)\frac{d\lambda_1}{d\nu_1}(t) \in -N(C(t, u_1(t)); v_1(t)) \quad \nu_1\text{-a.e. } t \in [T_0, T_1].$$

By finite induction, we have for each $i \in \{2, \ldots, p\}$ an absolutely continuous mapping $u_i : [T_{i-1}, T_i] \to \mathcal{H}$, a right continuous mapping with bounded variation $v_i : [T_{i-1}, T_i] \to \mathcal{H}$ and a mapping $z_i : [T_{i-1}, T_i] \to \mathcal{H}$ λ_i-integrable on $[T_{i-1}, T_i]$ such that

$$\begin{gathered}
u_i(T_{i-1}) = u_{i-1}(T_{i-1}) \quad \text{and} \quad v_i(T_{i-1}) = v_{i-1}(T_{i-1}),\\
v_i(t) \in C(t, u_i(t)) \text{ for all } t \in [T_{i-1}, T_i],\\
\left\| v_i(t) - v_i(t^-) \right\| \leq 2\mu_i(\{t\}) \quad \text{for all } t \in]T_{i-1}, T_i],\\
v_i(t) = \dot{u}_i(t) \quad \text{and} \quad z_i(t) \in F(t, u_i(t), v_i(t)) \quad \lambda_i\text{-a.e. } t \in [T_{i-1}, T_i],
\end{gathered}$$

dv_i has $\frac{dv_i}{d\nu_i}$ in $L^1([T_{i-1}, T_i], \mathcal{H}, \nu_i)$ as a density relative to ν_i and

$$\frac{dv_i}{d\nu_i}(t) + z_i(t)\frac{d\lambda_i}{d\nu_i}(t) \in -N(C(t, u_i(t)); v_i(t)) \quad \nu_i\text{-a.e. } t \in [T_{i-1}, T_i].$$

Let us define $u, v : [T_0, T] \to \mathcal{H}$ with $u(t) := u_i(t)$ (resp., $v(t) := v_i(t)$) if $t \in [T_{i-1}, T_i]$ for some $i \in \{1, \ldots, p\}$. Define also $z, h : [T_0, T] \to \mathcal{H}$ by

$$\begin{cases} z_1(t) & \text{if } t \in [T_0, T_1] \\ z_i(t) & \text{if } t \in]T_{i-1}, T_i] \text{ for some } i \in \{2, \ldots, p\} \end{cases}$$

and

$$h(t) := \mathbf{1}_{[T_0,T_1]}(t)\frac{dv_1}{d\nu_1}(t) + \sum_{i=2}^{p} \mathbf{1}_{]T_{i-1},T_i]}(t)\frac{dv_i}{d\nu_i}(t) \quad \text{for all } t \in [T_0, T].$$

With $\nu' := \mu + \lambda$, it is clear that

$$v(t) = b + \int_{]T_0,t]} h(s)d\nu'(s) \quad \text{for all } t \in [T_0, T]$$

and

$$\left\| v(t) - v(t^-) \right\| \leq 2\mu(\{t\}) \quad \text{for all } t \in]T_0, T].$$

Thus, $h(\cdot)$ is an $L^1([T_0, T], \mathcal{H}, \nu')$-density of $v(\cdot)$ relative to ν'. It remains to see that

$$\frac{dv}{d\nu'}(t) + z(t)\frac{d\lambda}{d\nu'}(t) \in -N(C(t, u(t)); v(t)) \quad \nu'\text{-a.e. } t \in [T_0, T].$$

As a consequence, $u(\cdot)$ is a solution of $(\mathcal{P})$. Now, we show the equality claimed by (4.5). Doing so, we follow [2,34]. Assume that $\sup_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$. According to the development there is a solution $u(\cdot)$ of $(\mathcal{P})$ with derivative $v(\cdot)$ for $(\mathcal{P})$ satisfying

$$\left\| v(t) - v(t^-) \right\| \leq 2\mu(\{t\}) \quad \text{for all } t \in]T_0, T].$$

Fix any $t \in]T_0, T]$. If $\mu(\{t\}) = 0$, the latter inequality says in particular that $v(t) = \text{proj}_{C(t,u(t))}(v(t^-))$. Now, assume that $\mu(\{t\}) > 0$. In this second case, we have

$$\left\| u(t) - u(t^-) \right\| \leq 2\mu(\{t\}) \leq 2 \sup_{s\in]T_0,T]} \mu(\{s\}) < r. \tag{4.42}$$

The inequality $\mu(\{t\}) > 0$ entails straightforwardly $\nu'(\{t\}) > 0$. Combining the definition of a solution and the equality $\frac{d\lambda}{d\nu'}(t) = 0$ [thanks to (2.7)], we get

$$\frac{dv}{d\nu'}(t) \in -N(C(t, u(t)); v(t)).$$

This inclusion with (2.5) give us

$$\frac{dv}{d\nu'}(t) = \lim_{s\uparrow t} \frac{dv(]s, t])}{\nu'(]s, t])} = \lim_{s\uparrow t} \frac{v(t) - v(s)}{\nu'(]s, t])} = \frac{v(t) - v(t^-)}{\nu'(\{t\})} \in -N(C(t, u(t)); v(t)),$$

Since $N(C(t, u(t)); v(t))$ is a cone, the latter inclusion is equivalent to

$$v(t^-) - v(t) \in N(C(t, u(t)); v(t)). \tag{4.43}$$

Using (4.42), (4.43) and Proposition 2.1, we get

$$v(t) = \text{proj}_{C(t,u(t))}(v(t^-)).$$

This completes the proof. $\square$

Remark 4.1 Very recently, Haddad et al. proved in [20] an existence result for the problem (1.2) with a subsmooth [4] moving set depending on the state. It is straightforward to check that replacing Proposition 2.2 by a suitable adaptation of [20, Proposition 2.8] allows to assume merely that the family $\{C(t,x) : (t,x) \in [T_0, T] \times \mathcal{H}\}$ is equi-uniformly subsmooth in Theorem 4.1.
Still concerning the moving set $C(\cdot,\cdot)$, let us also pointing out (as noted in [19]) that the need (or not) of compacity to get solution to a state-dependent sweeping process (even of first order) remains an open question in the infinite dimensional setting.
About the perturbation $F(\cdot,\cdot)$ of the normal cone, let us mention that in [3], it is considered with nonconvex values in the framework of a second order sweeping process with a prox-regular moving set of $\mathbb{R}^k$ but depending only in the state, that is, $C(t,x) = C(x)$. In our development, the convexity assumption on the values of $F(\cdot,\cdot)$ is crucial in order to get (through (2.4)) the selection $z(\cdot)$ of $F(\cdot,\cdot)$ satisfying (3.2). Weakening such an hypothese on $F(\cdot,\cdot)$ is definitely an interesting and not easy task and would be the subject of further investigations. $\square$

5 Application to State-Dependent Evolution Variational Inequalities

In this section, we give an application of our existence result to the theory of evolution quasi-variational inequalities.

Let $C : [T_0, T] \rightrightarrows \mathcal{H}$ be a multimapping with nonempty closed convex values, $A : \mathcal{H}^2 \to \mathcal{H}$ and $f : [T_0, T] \to \mathcal{H}$ be mappings. Assume that there exist a positive Radon measure on I and a real $L > 0$ such that for all $x_1, x_2 \in \mathcal{H}$, for all $u_1, u_2 \in \mathcal{H}$, for all $t_1, t_2 \in I$,

$$|d(u_1, C(t_1, x_1)) - d(u_2, C(t_2, x_2))| \leq \mu(]t_1, t_2]) + L\,\|x_1 - x_2\| + \|u_1 - u_2\|\,.$$

One says that a mapping $u : [T_0, T] \to \mathcal{H}$ satisfies the following *second order evolution quasi variational inequality* (EQVI for short) associated to μ

$$\langle d\dot{u}(t) + A(u(t), \dot{u}(t)), z - \dot{u}(t)\rangle \geq \langle f(t), z - \dot{u}(t)\rangle \quad \forall z \in C(t, u(t)),$$

whenever:

(a) u is absolutely continuous on I;
(b) there exists a mapping $v : I \to \mathcal{H}$ right continuous with bounded variation on I such that for all $t \in I$, $v(t) \in C(t, u(t))$ and $v(\cdot) = \dot{u}(\cdot)$ λ-a.e.;
(c) there exist ν a positive Radon measure on I absolutely continuously equivalent to $\mu + \lambda$ with respect to which dv admits a density in $L^1(I, \mathcal{H}, \nu)$ such that

$$\left\langle \frac{dv}{d\nu}(t) + A(u(t), v(t))\frac{d\lambda}{d\nu}(t), z - v(t)\right\rangle \geq \left\langle f(t)\frac{d\lambda}{d\nu}(t), z - v(t)\right\rangle,$$

for ν-a.e. $t \in I$, for all $z \in C(t, u(t))$.

We derive from Theorem 4.1 the existence of solutions for (EQVI).

Proposition 5.1 *Assume that* (ii) *of Theorem* 4.1 *holds and that there exist* $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ *continuous linear mapping such that*

$$A(u, v) = A_1(u) + A_2(v) \quad \textit{for all } (u, v) \in \mathcal{H}^2.$$

Assume also that f *is continuous. Then, there exists at least one solution for (EQVI) associated to* μ.

Proof Let us define $F : I \times \mathcal{H}^2 \rightrightarrows \mathcal{H}$ by

$$F(t, u, v) = \{A(u, v) - f(t)\} \quad \text{for all } (t, u, v) \in I \times \mathcal{H}^2.$$

Define also $\alpha : I \to \mathbb{R}$ by

$$\alpha(t) = \max\{\|f(t)\|, \|A_1\|, \|A_2\|\} \quad \text{for all } t \in I.$$

It is straightforward that $\alpha(\cdot) \in L^1(I, \mathbb{R}, \lambda)$. Moreover, observe that

$$F(t, u, v) \subset \alpha(t)(1 + \|u\| + \|v\|)\mathbb{B},$$

for all $t \in I, u, v \in \mathcal{H}$. Thanks to the continuity of f, F is scalarly upper semicontinuous. Fix any $a \in \mathcal{H}$ and $b \in C(T_0, a)$. According to Theorem 4.1, there is a mapping $u(\cdot) : [T_0, T] \to \mathcal{H}$ satisfying the following discontinuous second order sweeping process

$$\begin{cases} -d\dot{u}(t) \in N(C(t, u(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b. \end{cases}$$

Since $C(\cdot, \cdot)$ is convex valued, it is not difficult to check [keeping in mind (2.2)] that $u(\cdot)$ is also a solution of (EQVI). □

Remark 5.1 For examples of evolution quasi-variational inequalities of second order type models, we refer the reader to [17, Problem 5.5] with a moving set $C(t, u(t)) = \mathcal{U}(t) \subset H^1(\Omega)$ (here, Ω is an open subset of $\mathbb{R}^n$) independent of the state. An other example is given in [21, Example 3] with a moving set $C(u)$ defined by

$$C(u) = \{\varphi \in H_1^0(0, 1) \; : \; |\varphi'(x)| \le \psi(x, u(x)) \text{ for a.e. } x \in (0, 1)\},$$

for $u \in L^2(0, 1)$ and $\psi(\cdot, \cdot)$ is a prescribed given function arising in the modeling of the evolution of sandpiles (see [21] and references therein for more details). □

6 Concluding Remarks

In this paper, we established that the following second order sweeping process

$$(\mathcal{P}) \begin{cases} -d\dot{u}(t) \in N(C(t, u(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) \\ u(T_0) = a, \dot{u}(T_0) = b \end{cases}$$

has at least one solution. An application of this existence result to the theory of variational inequalities is given. Many questions need further investigations like e.g. the study of the optimal control governed by a discontinuous second order sweeping process or to relax the prox-regularity assumption of the moving set, for instance by considering the class of α-far sets. It will be also interesting to investigate the problem of optimal control governed by a differential inclusion (with or without memory) of the form studied in this paper, to establish an existence result and also necessary optimality conditions [7,14]. Such a study is out of the scope of this manuscript and will be the subject of another work.

Acknowledgements The authors would like to thank the two anonymous reviewers for their valuable suggestions on earlier versions of this paper.

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